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SHARP INEQUALITIES BETWEEN L^p–NORMS FOR THE HIGHER DIMENSIONAL HARDY OPERATOR AND ITS DUAL

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Abstract. We derive the two-sided inequalities between $L^p(X)$-norms $(1<p<\infty)$ of the higher dimensional Hardy operator and its dual, where the underlying space X is the Heisenberg group $\mathbb{H}^n$ or the Euclidean space $\mathbb{R}^n$. The interest of main results is that it relates two-sided inequalities with sharp constants which are dimension free. The methodology is completely depending on the rotation method.

1. Introduction

The classical one-dimensional Hardy operator H was defined by G. H. Hardy in 1925 (see [6]) in the form of

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt,\ x>0,$$

for all nonnegative measurable functions f on $\mathbb{R}^+ = (0,\infty)$. Accompanying the Hardy operator is the well known Hardy inequality

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leqslant \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)},$$

where $1<p<\infty$ and the constant $p/(p-1)$ is sharp. This inequality has been studied by a large number of authors over the past 90 years and has motivated some important lines of study which are currently active. We refer the reader to two books [10, 11].

The dual operator H^* of the Hardy operator is

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}dt,\ x>0,$$

for all nonnegative measurable functions f on $\mathbb{R}^+$.

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Now, let us briefly present some background to the problems we are about to investigate. It is known from [7, (9.9.1) and (9.9.2), p. 244] that, for $1 < p < \infty$, the operators H and H^* satisfy the L^p inequalities

$$\frac{p-1}{p}\|Hf\|_{L^p(\mathbb{R}^+)} \leqslant \|H^*f\|_{L^p(\mathbb{R}^+)} \leqslant p\|Hf\|_{L^p(\mathbb{R}^+)}, \tag{1}$$

(see also [9] for more details). These estimates are not optimal and the constants in (1) are further improved by Kolyada in [9] who arrived at the following sharp conclusion.

THEOREM A. ([9, Theorem 1.1]) *Let f be a nonnegative measurable function on $\mathbb{R}^+$. Then*

$$(p-1)\|Hf\|_{L^p(\mathbb{R}^+)} \leqslant \|H^*f\|_{L^p(\mathbb{R}^+)} \leqslant (p-1)^{1/p}\|Hf\|_{L^p(\mathbb{R}^+)}$$

if $1 < p \leqslant 2$, and

$$(p-1)^{1/p}\|Hf\|_{L^p(\mathbb{R}^+)} \leqslant \|H^*f\|_{L^p(\mathbb{R}^+)} \leqslant (p-1)\|Hf\|_{L^p(\mathbb{R}^+)}$$

if $2 \leqslant p < \infty$. All constants are the best possible.

It is interesting to generalize the one-dimensional theory to the higher dimensional setting. As a counterpart, the theory in higher dimensions is thought to be much more difficult and profound. Indeed, there are significant problems in higher dimensions for which the one-dimensional techniques are not adequate. It is important to realize, however, that many higher dimensional problems are really one-dimensional in nature and may be successfully analyzed using the one-dimensional theory. The higher dimensional results also provide an elegant approach to studying one-dimensional integral operators. Thus, the aim of this work is to prove that Theorem A holds for the Heisenberg group $\mathbb{H}^n$ and the Euclidean space $\mathbb{R}^n$ (instead of $\mathbb{R}^+$).

Since $\mathbb{R}^n$ is a very familiar space, in order to set up notation and state our main results, we begin by recalling some basic preliminaries on the Heisenberg group. More detailed information, as well as proofs, can be found in [2, 4, 14] and the papers referenced therein.

Let $x = (x_1, \ldots, x_{2n}, x_{2n+1})$ and $y = (y_1, \ldots, y_{2n}, y_{2n+1})$ be in $\mathbb{R}^{2n+1}$ with $n \geqslant 1$. The Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ equipped with the group law

$$x \circ y = \Big(x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n}(y_j x_{n+j} - x_j y_{n+j})\Big).$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r x = (rx_1, \ldots, rx_{2n}, r^2 x_{2n+1}),\ r > 0,$$

and the norm

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right]^{1/4}.$$

We denote the open ball of radius r centered at $x \in \mathbb{H}^n$ by $B(x,r) = \{y \in \mathbb{H}^n : d(x,y) < r\}$, and denote its sphere by $S(x,r) = \{y \in \mathbb{H}^n : d(x,y) = r\}$. The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. We have

$$|B(x,r)| = |B(0,r)| = v_Q r^Q,$$

where v_Q is the volume of the unit ball $B(0,1)$ on $\mathbb{H}^n$, and

$$v_Q = \frac{2\pi^{n+1/2}\Gamma(n/2)}{(n+1)\Gamma(n)\Gamma((n+1)/2)}.$$

The area of the unit sphere $S(0,1)$ is $\omega_Q = Qv_Q$, where $Q = 2n+2$ is called the homogeneous dimension of $\mathbb{H}^n$. Similar to the definition on Euclidean space $\mathbb{R}^n$, we say that u is a radial function if there exists $\widetilde{u} : \mathbb{R}^+ \to \mathbb{R}$ such that $u(x) = \widetilde{u}(|x|_h)$ for all $x \in \mathbb{H}^n$. We still use the same notation u to designate the function $\widetilde{u}$ since it will not cause confusion.

As an extension of the one-dimensional Hardy operator, Christ and Grafakos in [1] gave the definition of n-dimensional Hardy operator

$$\mathscr{H} f(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f(y)dy, \quad x \in \mathbb{R}^n \setminus \{0\},$$

for all nonnegative measurable functions f on $\mathbb{R}^n$. Similar to the one-dimensional case, the dual operator of $\mathscr{H}$ is given by

$$\mathscr{H}^* f(x) = \int_{\{y:\ |y|>|x|\}} \frac{f(y)}{|B(0,|y|)|} dy,$$

where f is a nonnegative measurable function on $\mathbb{R}^n$. Here $B(0,|y|)$ is the ball with center at the origin and radius $|y|$ in $\mathbb{R}^n$. Recently, Wu and Fu in [17] introduced the analogue for the Hardy operator defined on the Heisenberg group. Following Wu and Fu in [17], for all nonnegative measurable functions on $\mathbb{H}^n$ function f, one defines the Hardy operator on $\mathbb{H}^n$ by

$$\mathbf{H} f(x) = \frac{1}{|B(0,|x|_h)|} \int_{B(0,|x|_h)} f(y)dy, \quad x \in \mathbb{H}^n \setminus \{0\},$$

and its dual operator

$$\mathbf{H}^* f(x) = \int_{\{y:\ |y|_h>|x|_h\}} \frac{f(y)}{|B(0,|y|_h)|} dy.$$

Let T be the operator $\mathscr{H}$ or $\mathbf{H}$ and let X be the underlying space $\mathbb{R}^n$ or $\mathbb{H}^n$. The Hardy's inequality is

$$\|Tf\|_{L^p(X)} \leqslant \frac{p}{p-1} \|f\|_{L^p(X)}, \ 1 < p < \infty, \tag{2}$$

where the constant $p/(p-1)$ is sharp. In the case of $\mathbb{R}^n$, the result has been obtained by Christ and Grafakos in [1] and Drábek, Heinig and Kufner in [3]. In the case of $\mathbb{H}^n$, it was proved by Wu an Fu in [17]. Inequality (2) has received considerable attention after its appearance and a number of papers have appeared in the literature dealing with alternative proofs, various extensions and generalizations, see [3, 5, 8, 12, 13, 15, 16, 18].

Let T^* be the operator $\mathscr{H}^*$ or $\mathbf{H}^*$. Actually, a linear operator has the same norm as the norm of its adjoint. So T^* maps $L^{p'}$ into $L^{p'}$ with the operator norm $p/(p-1)$ if and only if T maps L^p into L^p with the operator norm $p/(p-1)$ for $1<p<\infty$, where p' is the conjugate index of p. Denote by $|x|_X$ the norm of $x\in X$ and by $|B(0,|x|_X)|$ the volume of the open ball $B(0,|x|_X)$ on X. Applying Fubini's theorem, a straightforward computation then shows that

$$\begin{aligned}Tf(x) &= \frac{1}{|B(0,|x|_X)|}\int_{B(0,|x|_X)}\left(\int_{\{z:\ |y|_X<|z|_X<|x|_X\}}\frac{f(z)}{|B(0,|z|_X)|}dz\right)dy\\ &\leqslant \frac{1}{|B(0,|x|_X)|}\int_{B(0,|x|_X)}T^*f(y)dy,\end{aligned}$$

and

$$\begin{aligned}T^*f(x) &= \int_{\{z:\ |z|_X>|x|_X\}}\left(\frac{1}{|B(0,|z|_X)|^2}\int_{\{y:\ |x|_X<|y|_X<|z|_X\}}f(y)dy\right)dz\\ &\leqslant \int_{\{z:\ |z|_X>|x|_X\}}\frac{Tf(z)}{|B(0,|z|_X)|}dz.\end{aligned}$$

With these estimates, the fairy rough inequalities similar to the one-dimensional case will be established as follows

$$\frac{p-1}{p}\|Tf\|_{L^p(X)}\leqslant\|T^*f\|_{L^p(X)}\leqslant p\|Tf\|_{L^p(X)},$$

for any $1<p<\infty$.

In the present paper, we establish an analogous result of Theorem A which in fact is motivated by the interesting result given by Kolyada in [9]. We are going to give the relation between L^p-norms of the higher dimensional Hardy operator and its dual operator, and to establish best constants in inequalities. It is interesting to find that sharp constants are dimension free, which means that they do not depend on the dimension of underlying space n.

Our results can be stated as follows.

THEOREM 1. *Let f be a nonnegative measurable function on $\mathbb{H}^n$. Then*

$$(p-1)\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}\leqslant\|\mathbf{H}^*f\|_{L^p(\mathbb{H}^n)}\leqslant(p-1)^{1/p}\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}\tag{3}$$

if $1<p\leqslant 2$, and

$$(p-1)^{1/p}\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}\leqslant\|\mathbf{H}^*f\|_{L^p(\mathbb{H}^n)}\leqslant(p-1)\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}\tag{4}$$

if $2\leqslant p<\infty$. All constants in (3) and (4) are sharp.

THEOREM 2. *Let f be a nonnegative measurable function on $\mathbb{R}^n$. Then*

$$(p-1)\|\mathscr{H}f\|_{L^p(\mathbb{R}^n)} \leqslant \|\mathscr{H}^*f\|_{L^p(\mathbb{R}^n)} \leqslant (p-1)^{1/p}\|\mathscr{H}f\|_{L^p(\mathbb{R}^n)} \tag{5}$$

if $1 < p \leqslant 2$, and

$$(p-1)^{1/p}\|\mathscr{H}f\|_{L^p(\mathbb{R}^n)} \leqslant \|\mathscr{H}^*f\|_{L^p(\mathbb{R}^n)} \leqslant (p-1)\|\mathscr{H}f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

if $2 \leqslant p < \infty$. All constants in (5) and (6) are sharp.

In general, the proof on the Heisenberg group is considerably more complicated than the proof in Euclidean space because it does not use symmetrization. However, we find that many techniques and results are similar for some averaging integral operators both in the case of $\mathbb{H}^n$ and $\mathbb{R}^n$. With this fact, we only give the proof of Theorem 1 in details. The method by which our results are obtained is quite elementary by means of the rotation method. As we said, the main results show that the n-dimensional inequalities are equivalent to one-dimensional ones with the identical constants. We must show that there is an optimizer for the inequality, i.e., that there is a function f that actually gives equality in Theorems 1 and 2 with the sharp constant. Indeed, the problem of finding the best constant is reduced to test equality over the class of radial functions.

2. Proof of Theorems 1 and 2

We shall need the following lemma which is analogous to the one-dimensional case on $\mathbb{R}^1$ presented in [9] in which the author did not provide the proof.

LEMMA 1. *Let $1 < p < \infty$. Assume that f is a nonnegative measurable function on $\mathbb{H}^n$. If $\mathbf{H}f \in L^p(\mathbb{H}^n)$, then*

$$|x|_h^{Q/p}\mathbf{H}f(x) \to 0 \text{ as } |x|_h \to 0^+ \text{ or } |x|_h \to \infty.$$

Proof. For any $x \in \mathbb{H}^n\backslash\{0\}$, there exists $j \in \mathbb{Z}$ such that $2^j < |x|_h \leqslant 2^{j+1}$. Since f is nonnegative, and by the definition of Hardy operator $\mathbf{H}$, we have

$$2^{-Q}\mathbf{H}f(2^j) \leqslant \mathbf{H}f(x) \leqslant 2^Q\mathbf{H}f(2^{j+1}). \tag{7}$$

On the one hand, we find that

$$\begin{aligned}
\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)} &\geqslant \int_{\{x:\ |x|_h>1\}} \left(\frac{1}{\nu_Q|x|_h^Q}\int_{B(0,|x|_h)} f(y)dy\right)^p dx \\
&= \sum_{j=0}^{\infty}\int_{\{x:\ 2^j<|x|_h\leqslant 2^{j+1}\}} \left(\frac{1}{\nu_Q|x|_h^Q}\int_{B(0,|x|_h)} f(y)dy\right)^p dx \\
&\geqslant \sum_{j=0}^{\infty}\int_{\{x:\ 2^j<|x|_h\leqslant 2^{j+1}\}} \left(2^{-Q}\mathbf{H}f(2^j)\right)^p dx \\
&= \frac{\nu_Q(2^Q-1)}{2^{Qp}}\sum_{j=0}^{\infty} 2^{jQ}\left(\mathbf{H}f(2^j)\right)^p.
\end{aligned}$$

Noting that $\mathbf{H}f \in L^p(\mathbb{H}^n)$, it implies that

$$\lim_{j\to\infty} 2^{jQ}\left(\mathbf{H}f(2^j)\right)^p = 0.$$

It follows from (7) that for $2^j < |x|_h \leqslant 2^{j+1}$,

$$2^{-Qp}2^{jQ}\left(\mathbf{H}f(2^j)\right)^p \leqslant |x|_h^Q\left(\mathbf{H}f(x)\right)^p \leqslant 2^{Qp}2^{(j+1)Q}\left(\mathbf{H}f(2^{j+1})\right)^p,$$

which clearly yields

$$\lim_{|x|_h\to\infty} |x|_h^Q\left(\mathbf{H}f(x)\right)^p = 0.$$

On the other hand, we have

$$\begin{aligned}
\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)} &\geqslant \int_{\{x:\ 0<|x|_h\leqslant 1\}}\left(\frac{1}{\nu_Q|x|_h^Q}\int_{B(0,|x|_h)} f(y)dy\right)^p dx\\
&= \sum_{j=-\infty}^{0}\int_{\{x:\ 2^{j-1}<|x|_h\leqslant 2^j\}}\left(\frac{1}{\nu_Q|x|_h^Q}\int_{B(0,|x|_h)} f(y)dy\right)^p dx\\
&\geqslant \sum_{j=-\infty}^{0}\int_{\{x:\ 2^{j-1}<|x|_h\leqslant 2^j\}}\left(2^{-Q}\mathbf{H}f(2^{j-1})\right)^p dx\\
&= \frac{\nu_Q(2^Q-1)}{2^{Qp}}\sum_{j=-\infty}^{0} 2^{(j-1)Q}\left(\mathbf{H}f(2^{j-1})\right)^p.
\end{aligned}$$

Using that $\mathbf{H}f \in L^p(\mathbb{H}^n)$, it implies that

$$\lim_{j\to-\infty} 2^{(j-1)Q}\left(\mathbf{H}f(2^{j-1})\right)^p = 0,$$

and then we must have

$$\lim_{|x|_h\to 0^+} |x|_h^Q\left(\mathbf{H}f(x)\right)^p = 0.$$

We complete the proof.

A similar proof as that of Lemma 1 leads easily to the following result.

LEMMA 2. *Let $1 < p < \infty$. Assume that f is a nonnegative measurable function on $\mathbb{R}^n$. If $\mathscr{H}f \in L^p(\mathbb{R}^n)$, then*

$$|x|^{n/p}\mathscr{H}f(x) \to 0 \text{ as } |x| \to 0^+ \text{ or } |x| \to \infty.$$

Proof of Theorem 1. Assume that $\mathbf{H}f$ and $\mathbf{H}^*f$ belong to $L^p(\mathbb{H}^n)$. From the definitions of the Hardy operator $\mathbf{H}$ and its dual operator $\mathbf{H}^*$ on $\mathbb{H}^n$, these two operators are radial. Write

$$F_Q(s) = \int_{S(0,1)} s^{Q-1} f(\delta_s y')d\sigma(y').$$

The norm of $\|\mathbf{H}f\|^p_{L^p(\mathbb{H}^n)}$ can be written in the polar coordinates form as follows:

$$\begin{aligned}\|\mathbf{H}f\|^p_{L^p(\mathbb{H}^n)} &= \int_{\mathbb{H}^n} |\mathbf{H}f(x)|^p dx \\ &= \frac{\omega_Q}{v_Q^p}\int_0^\infty t^{Q-1-Qp}\left(\int_0^t \int_{S(0,1)} s^{Q-1} f(\delta_s y')d\sigma(y')ds\right)^p dt \\ &= \frac{\omega_Q}{v_Q^p}\int_0^\infty t^{Q-1-Qp}\left(\int_0^t F_Q(s)ds\right)^p dt.\end{aligned} \tag{8}$$

Similarly, we may rewrite $\|\mathbf{H}^*f\|^p_{L^p(\mathbb{H}^n)}$ as

$$\|\mathbf{H}^*f\|^p_{L^p(\mathbb{H}^n)} = \int_{\mathbb{H}^n} |\mathbf{H}^*f(x)|^p dx = \frac{\omega_Q}{v_Q^p}\int_0^\infty s^{Q-1}\left(\int_s^\infty \frac{F_Q(t)}{t^Q}dt\right)^p ds. \tag{9}$$

Applying integration by parts and using Lemma 1, (8) can be estimated as follows

$$\begin{aligned}\|\mathbf{H}f\|^p_{L^p(\mathbb{H}^n)} &= \frac{\omega_Q}{v_Q^p}\left(\frac{1}{Q-Qp}t^{Q-Qp}\left(\int_0^t F_Q(s)ds\right)^p\Bigg|_{t=0}^{\infty}\right. \\ &\qquad \left.-\frac{p}{Q-Qp}\int_0^\infty t^{Q-Qp}F_Q(t)\left(\int_0^t F_Q(s)ds\right)^{p-1}dt\right) \\ &= \frac{1}{v_Q^{p-1}}\cdot\frac{p}{p-1}\int_0^\infty t^{Q-Qp}F_Q(t)\left(\int_0^t F_Q(s)ds\right)^{p-1}dt.\end{aligned} \tag{10}$$

Next we shall prove the theorem in three steps.

Step 1. We shall show that

$$\|\mathbf{H}^*f\|_{L^p(\mathbb{H}^n)} \leqslant (p-1)^{1/p}\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}$$

if $1 < p \leqslant 2$, and

$$(p-1)^{1/p}\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)} \leqslant \|\mathbf{H}^*f\|_{L^p(\mathbb{H}^n)}$$

if $2 \leqslant p < \infty$.

For $t > 0$, we let

$$\Phi_Q(s,t) = \int_s^t \frac{F_Q(u)}{u^Q}du, \quad 0 < s \leqslant t < \infty.$$

An easy calculation show that

$$Q\int_0^t s^{Q-1}\Phi_Q(s,t)ds = \int_0^t F_Q(u)du, \tag{11}$$

and also for any $q > 0$

$$\left(\int_s^\infty \frac{F_Q(t)}{t^Q}dt\right)^q = \int_s^\infty \frac{\partial}{\partial t}\left(\int_s^t \frac{F_Q(u)}{u^Q}du\right)^q dt = q\int_s^\infty \frac{F_Q(t)}{t^Q}\Phi_Q(s,t)^{q-1}dt. \tag{12}$$

It follows from (12) with $q = p$ and the Fubini theorem that (9) can be written as

$$\|\mathbf{H}^* f\|_{L^p(\mathbb{H}^n)}^p = \frac{\omega_Q}{v_Q^p} \cdot p \int_0^\infty s^{Q-1} \int_s^\infty \frac{F_Q(t)}{t^Q} \Phi_Q(s,t)^{p-1} dt ds \tag{13}$$
$$= \frac{\omega_Q}{v_Q^p} \cdot p \int_0^\infty \frac{F_Q(t)}{t^Q} \int_0^t s^{Q-1} \Phi_Q(s,t)^{p-1} ds dt.$$

From (10) and (11), we have the identity

$$\|\mathbf{H} f\|_{L^p(\mathbb{H}^n)}^p = \left(\frac{Q}{v_Q}\right)^{p-1} \cdot \frac{p}{p-1} \int_0^\infty t^{Q-Qp} F_Q(t) \left(\int_0^t s^{Q-1} \Phi_Q(s,t) ds\right)^{p-1} dt. \tag{14}$$

Obviously, $\|\mathbf{H}^* f\|_{L^2(\mathbb{H}^n)}^2 = \|\mathbf{H} f\|_{L^2(\mathbb{H}^n)}^2$. Thus we shall only deal with the case $p \neq 2$. We now divide p into two cases: $1 < p < 2$ and $2 < p < \infty$.

Case I. Let $1 < p < 2$. By Hölder's inequality with exponents $(p-1)^{-1}$ and $(2-p)^{-1}$, we have that

$$\int_0^t s^{Q-1} \Phi_Q(s,t)^{p-1} ds \leqslant \left(\int_0^t s^{Q-1} \Phi_Q(s,t) ds\right)^{p-1} \left(\int_0^t s^{Q-1} ds\right)^{2-p}$$
$$= Q^{p-2} t^{Q(2-p)} \left(\int_0^t s^{Q-1} \Phi_Q(s,t) ds\right)^{p-1}.$$

The above estimate together with (13) and (14) gives that

$$\|\mathbf{H}^* f\|_{L^p(\mathbb{H}^n)}^p \leqslant \left(\frac{Q}{v_Q}\right)^{p-1} \cdot p \int_0^\infty F_Q(t) t^{Q-Qp} \left(\int_0^t s^{Q-1} \Phi_n(s,t) ds\right)^{p-1} dt$$
$$= (p-1) \|\mathbf{H} f\|_{L^p(\mathbb{H}^n)}^p.$$

Case II. Let $p > 2$. Using Hölder's inequality with exponents $p-1$ and $(p-1)/(p-2)$, we have

$$\left(\int_0^t s^{Q-1} \Phi_Q(s,t) ds\right)^{p-1} \leqslant \int_0^t s^{Q-1} \Phi_Q(s,t)^{p-1} ds \left(\int_0^t s^{Q-1} ds\right)^{p-2}$$
$$= Q^{2-p} t^{Q(p-2)} \int_0^t s^{Q-1} \Phi_Q(s,t)^{p-1} ds.$$

With the help of (13) and (14), we have, as a consequence, that

$$\|\mathbf{H} f\|_{L^p(\mathbb{H}^n)}^p \leqslant \frac{\omega_Q}{v_Q^p} \cdot \frac{p}{p-1} \int_0^\infty \frac{F_Q(t)}{t^Q} \int_0^t s^{Q-1} \Phi_Q(s,t)^{p-1} ds dt$$
$$= \frac{1}{p-1} \|\mathbf{H}^* f\|_{L^p(\mathbb{H}^n)}^p.$$

Thus, we finish the proof of Step 1.

Step 2. We shall prove that

$$(p-1)\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)} \leqslant \|\mathbf{H}^*f\|_{L^p(\mathbb{H}^n)}, \text{ if } 1<p<2,$$

and

$$\|\mathbf{H}^*f\|_{L^p(\mathbb{H}^n)} \leqslant (p-1)\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}, \text{ if } 2<p<\infty.$$

Noting that for any $q>0$, we have

$$\left(\int_s^\infty \frac{F_Q(t)}{t^Q}dt\right)^q = q\int_s^\infty \frac{F_Q(t)}{t^Q}\left(\int_t^\infty \frac{F_Q(u)}{u^Q}du\right)^{q-1} dt. \tag{15}$$

Applying (15) with the exponent $q=p$, (9) can be written as

$$\begin{aligned}\|\mathbf{H}^*f\|^p_{L^p(\mathbb{H}^n)} &= \frac{\omega_Q}{\nu_Q^p}\int_0^\infty s^{Q-1}\left(\int_s^\infty \frac{F_Q(t)}{t^Q}dt\right)^p ds \\ &= \frac{\omega_Q}{\nu_Q^p}\cdot p\int_0^\infty s^{Q-1}\int_s^\infty \frac{F_Q(t)}{t^Q}\left(\int_t^\infty \frac{F_Q(u)}{u^Q}du\right)^{p-1} dtds \\ &= p\nu_Q^{1-p}\int_0^\infty F_Q(t)\left(\int_t^\infty \frac{F_Q(u)}{u^Q}du\right)^{p-1} dt.\end{aligned} \tag{16}$$

Once again, we use (15) with the exponent $q=p-1$ and switch the order of integration. This gives

$$\begin{aligned}\|\mathbf{H}^*f\|^p_{L^p(\mathbb{H}^n)} &= \nu_Q^{1-p}\cdot p(p-1)\int_0^\infty F_Q(t)\left(\int_t^\infty \frac{F_Q(u)}{u^Q}\left(\int_u^\infty \frac{F_Q(v)}{v^Q}dv\right)^{p-2} du\right)dt \\ &= \nu_Q^{1-p}\cdot p(p-1)\int_0^\infty \frac{F_Q(u)}{u^Q}\left(\int_u^\infty \frac{F_Q(v)}{v^Q}dv\right)^{p-2}\left(\int_0^u F_Q(t)dt\right)du.\end{aligned}$$

Define now the following functions

$$\varphi_Q(u) = \frac{F_Q^{1/(p-1)}(u)}{u^Q}\int_0^u F_Q(t)dt,$$

and

$$\psi_Q(u) = F_Q(u)^{(p-2)/(p-1)}\left(\int_u^\infty \frac{F_Q(v)}{v^Q}dv\right)^{p-2}.$$

Then, obviously

$$\|\mathbf{H}^*f\|^p_{L^p(\mathbb{H}^n)} = \nu_Q^{1-p}\cdot p(p-1)\int_0^\infty \varphi_Q(u)\psi_Q(u)du. \tag{17}$$

With the help of expressions of $\|\mathbf{H}f\|^p_{L^p(\mathbb{H}^n)}$ in (10) and $\|\mathbf{H}^*f\|^p_{L^p(\mathbb{H}^n)}$ in (16), we have that

$$\int_0^\infty \varphi_Q(u)^{p-1}du = \nu_Q^{p-1}\cdot\frac{p-1}{p}\,\|\mathbf{H}f\|^p_{L^p(\mathbb{H}^n)}, \tag{18}$$

and

$$\int_0^\infty \psi_Q(u)^{(p-1)/(p-2)} du = v_Q^{p-1} \cdot \frac{1}{p} \, \|\mathbf{H}^* f\|^p_{L^p(\mathbb{H}^n)} \tag{19}$$

for any $p > 1$. We now also divide p into two cases: $1 < p < 2$ and $2 < p < \infty$.

Case I. Let $1 < p < 2$. Using in (17) the Hölder inequality with exponents $p-1$ and $(p-1)(p-2)^{-1}$ (cf. [7, Theorem 189, p. 140]), and by equalities (18) and (19), we have

$$\begin{aligned}
\|\mathbf{H}^* f\|^p_{L^p(\mathbb{H}^n)} &\geqslant v_Q^{1-p} \cdot p(p-1) \cdot \left(v_Q^{p-1} \cdot \frac{p-1}{p} \, \|\mathbf{H} f\|^p_{L^p(\mathbb{H}^n)} \right)^{1/(p-1)} \\
&\quad \cdot \left(v_Q^{p-1} \cdot \frac{1}{p} \, \|\mathbf{H}^* f\|^p_{L^p(\mathbb{H}^n)} \right)^{(p-2)/(p-1)} \\
&= (p-1)^{\frac{p}{p-1}} \, \|\mathbf{H} f\|^{\frac{p}{p-1}}_{L^p(\mathbb{H}^n)} \|\mathbf{H}^* f\|^{\frac{p(p-2)}{p-1}}_{L^p(\mathbb{H}^n)}.
\end{aligned}$$

Hence

$$\|\mathbf{H}^* f\|_{L^p(\mathbb{H}^n)} \geqslant (p-1) \|\mathbf{H} f\|_{L^p(\mathbb{H}^n)}$$

in the case $1 < p < 2$.

Case II. Let $p > 2$. One has using in (17) the Hölder inequality with exponents $p-1$ and $(p-1)(p-2)^{-1}$ together with equalities (18) and (19),

$$\|\mathbf{H}^* f\|^p_{L^p(\mathbb{H}^n)} \leqslant (p-1)^{\frac{p}{p-1}} \cdot \|\mathbf{H} f\|^{\frac{p}{p-1}}_{L^p(\mathbb{H}^n)} \|\mathbf{H}^* f\|^{\frac{p(p-2)}{p-1}}_{L^p(\mathbb{H}^n)}.$$

Hence

$$\|\mathbf{H}^* f\|_{L^p(\mathbb{H}^n)} \leqslant (p-1) \|\mathbf{H} f\|_{L^p(\mathbb{H}^n)}$$

in the case $p > 2$.

Step 3. In order to prove that constants in (3) and (4) are sharp, we are going now to construct some suitable functions. We shall prove it by choosing three classes of functions.

Case I. We shall prove that the constant $(1/(p-1))^{1/p}$ in the right-hand side of (3) and the left-hand side of (4) is the sharp one.

For any $\varepsilon > 0$, take $f_\varepsilon(x) = \chi_{\{x:\ 1 \leqslant |x|_h \leqslant 1+\varepsilon\}}(x)$. On the one hand, we have

$$\begin{aligned}
\|\mathbf{H} f_\varepsilon\|^p_{L^p(\mathbb{H}^n)} &= \int_{\mathbb{H}^n} \left(\frac{1}{|B(0,|x|_h)|} \int_{B(0,|x|_h)} \chi_{\{y:\ 1 \leqslant |y|_h \leqslant 1+\varepsilon\}}(y) dy \right)^p dx \\
&= \int_{\{x:\ 1 < |x|_h < 1+\varepsilon\}} \left(\frac{1}{|B(0,|x|_h)|} \int_{\{y:\ 1 \leqslant |y|_h < |x|_h\}} dy \right)^p dx \\
&\quad + \int_{\{x:\ |x|_h \geqslant 1+\varepsilon\}} \left(\frac{1}{|B(0,|x|_h)|} \int_{\{y:\ 1 \leqslant |y|_h \leqslant 1+\varepsilon\}} dy \right)^p dx \\
&=: I_1 + I_2.
\end{aligned}$$

By simple calculations, we conclude that

$$I_1=\omega_Q\int_1^{1+\varepsilon}t^{Q-1-Qp}(t^Q-1)^p dt\leqslant \nu_Q\cdot\frac{((1+\varepsilon)^Q-1)^p}{p-1}\cdot\left(1-(1+\varepsilon)^{Q(1-p)}\right),$$

and

$$I_2=\nu_Q\cdot\frac{((1+\varepsilon)^Q-1)^p}{p-1}\cdot(1+\varepsilon)^{Q(1-p)}.$$

With above estimates, we yield that

$$\frac{((1+\varepsilon)^Q-1)^p(1+\varepsilon)^{Q(1-p)}}{p-1}\leqslant \nu_Q^{-1}\|\mathbf{H}f_\varepsilon\|_p^p\leqslant\frac{((1+\varepsilon)^Q-1)^p}{p-1}.$$

On the other hand, for $\mathscr{H}^*f_\varepsilon$, we have

$$\begin{aligned}\|\mathbf{H}^*f_\varepsilon\|_{L^p(\mathbb{H}^n)}^p&=\int_{\mathbb{H}^n}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{\chi_{\{y:\ 1\leqslant|y|_h\leqslant1+\varepsilon\}}(y)}{\nu_Q|y|_h^Q}dy\right)^p dx\\&=\int_{\{x:\ |x|_h\leqslant1\}}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{\chi_{\{y:\ 1\leqslant|y|_h\leqslant1+\varepsilon\}}(y)}{\nu_Q|y|_h^Q}dy\right)^p dx\\&\quad+\int_{\{x:\ |x|_h>1\}}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{\chi_{\{y:\ 1\leqslant|y|_h\leqslant1+\varepsilon\}}(y)}{\nu_Q|y|_h^Q}dy\right)^p dx\\&=:I_3+I_4.\end{aligned}$$

It is easy to calculate that

$$I_3=\nu_Q\cdot Q^p\cdot(\ln(1+\varepsilon))^p,$$

and

$$I_4=Q^p\cdot\omega_Q\int_1^{1+\varepsilon}t^{Q-1}\left(\ln\left(\frac{1+\varepsilon}{t}\right)\right)^p dt\leqslant\nu_Q\cdot Q^p\cdot(\ln(1+\varepsilon))^p\left((1+\varepsilon)^Q-1\right).$$

Consequently, above estimates of I_3 and I_4 force

$$Q^p(\ln(1+\varepsilon))^p\leqslant\nu_Q^{-1}\|\mathbf{H}^*f_\varepsilon\|_{L^p(\mathbb{H}^n)}^p\leqslant Q^p(1+\varepsilon)^Q(\ln(1+\varepsilon))^p.$$

Applying asymptotic relations $(1+x)^\alpha-1\sim\alpha x$ and $\ln(1+x)\sim x$ as $x\to0$ for any $\alpha\in\mathbb{R}$, we gain that

$$\lim_{\varepsilon\to0^+}\frac{\|\mathbf{H}f_\varepsilon\|_{L^p(\mathbb{H}^n)}}{\|\mathbf{H}^*f_\varepsilon\|_{L^p(\mathbb{H}^n)}}=\left(\frac{1}{p-1}\right)^{1/p}.$$

Case II. We shall prove that the constant $p-1$ in the left-hand side of (3) and the right-hand side of (4) is the sharp one. To this end, we need to choose different functions according to the range of p.

Let $1<p<2$. For any ε such that $0<\varepsilon<Q/p$, define

$$f_\varepsilon(x)=|x|_h^{\varepsilon-Q/p}\chi_{\{x:\ 0<|x|_h\leqslant 1\}}(x).$$

Then we have

$$\begin{aligned}
\|\mathbf{H}f_\varepsilon\|_{L^p(\mathbb{H}^n)}^p&=\frac{1}{\nu_Q^p}\int_{\mathbb{H}^n}\left(\frac{1}{|x|_h^Q}\int_{B(0,|x|_h)}|y|_h^{\varepsilon-Q/p}\chi_{\{y:\ 0<|y|_h\leqslant 1\}}(y)dy\right)^p dx\\
&\geqslant\frac{1}{\nu_Q^p}\int_{\{x:\ |x|_h<1\}}\frac{1}{|x|_h^{Qp}}\left(\int_{\{y:\ 0<|y|_h<|x|_h\}}|y|_h^{\varepsilon-Q/p}dy\right)^p dx\\
&=Q^p\cdot\omega_Q\cdot\frac{1}{p\varepsilon}\cdot\left(\frac{1}{\varepsilon+Q(1-1/p)}\right)^p.
\end{aligned}$$

To estimate the $L^p(\mathbb{H}^n)$-norm of $\mathbf{H}^*f_\varepsilon$, by using polar coordinates, we conclude that

$$\begin{aligned}
\|\mathbf{H}^*f_\varepsilon\|_{L^p(\mathbb{H}^n)}^p&=\int_{\mathbb{H}^n}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{|y|_h^{\varepsilon-Q/p}\chi_{\{y:\ 0<|y|_h\leqslant 1\}}(y)}{\nu_Q|y|_h^Q}dy\right)^p dx\\
&=\int_{\{x:\ |x|_h<1\}}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{|y|_h^{\varepsilon-Q/p}\chi_{\{y:\ 0<|y|_h\leqslant 1\}}(y)}{\nu_Q|y|_h^Q}dy\right)^p dx\\
&=\frac{\omega_Q^{p+1}}{\nu_Q^p}\int_0^1\left(\int_t^1 r^{\varepsilon-Q/p-1}dr\right)^p t^{Q-1}dt\\
&=\frac{\omega_Q^{p+1}}{\nu_Q^p}\cdot\frac{1}{(Q/p-\varepsilon)^p}\int_0^1\left(t^{\varepsilon-Q/p}-1\right)^p t^{Q-1}dt\\
&\leqslant\frac{\omega_Q^{p+1}}{\nu_Q^p}\cdot\frac{1}{(Q/p-\varepsilon)^p}\int_0^1 t^{p\varepsilon-1}dt\\
&=Q^p\cdot\omega_Q\cdot\frac{1}{p\varepsilon}\cdot\frac{1}{(Q/p-\varepsilon)^p}.
\end{aligned}$$

Therefore,

$$\varliminf_{\varepsilon\to 0^+}\frac{\|\mathbf{H}f_\varepsilon\|_{L^p(\mathbb{H}^n)}}{\|\mathbf{H}^*f_\varepsilon\|_{L^p(\mathbb{H}^n)}}\geqslant\frac{1}{p-1}.$$

It implies the constant $1/(p-1)$ in the left side of (3) is the best possible.

At last we now deal with the case of $p > 2$. For any ε satisfying $0 < \varepsilon < Q(1 - 1/p)$, take $f_\varepsilon(x) = |x|_h^{-\varepsilon - Q/p}\chi_{\{x:\ |x|_h \geqslant 1\}}(x)$. Then we have

$$\begin{aligned}
\|\mathbf{H}f_\varepsilon\|_{L^p(\mathbb{H}^n)}^p &= \frac{1}{\nu_Q^p}\int_{\mathbb{H}^n}\left(\frac{1}{|x|_h^Q}\int_{B(0,|x|_h)} |y|_h^{-\varepsilon-Q/p}\chi_{\{y:\ |y|_h\geqslant 1\}}(y)dy\right)^p dx \\
&= \frac{\omega_Q^{p+1}}{\nu_Q^p}\int_1^\infty\left(\int_1^t r^{-\varepsilon+Q(1-1/p)-1}dr\right)^p t^{Q(1-p)-1}dt \\
&= \frac{\omega_Q^{p+1}}{\nu_Q^p}\cdot\frac{1}{(Q(1-1/p)-\varepsilon)^p}\int_1^\infty\left(t^{Q(1-1/p)-\varepsilon}-1\right)^p t^{Q(1-p)-1}dt \\
&\leqslant \frac{\omega_Q^{p+1}}{\nu_Q^p}\cdot\frac{1}{(Q(1-1/p)-\varepsilon)^p}\int_1^\infty t^{-p\varepsilon-1}dt \\
&= Q^p\cdot\omega_Q\cdot\frac{1}{p\varepsilon}\cdot\frac{1}{(Q(1-1/p)-\varepsilon)^p}.
\end{aligned}$$

In the same way as above, we have

$$\begin{aligned}
\|\mathbf{H}^* f_\varepsilon\|_{L^p(\mathbb{H}^n)}^p &= \int_{\mathbb{H}^n}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{|y|_h^{-\varepsilon-Q/p}\chi_{\{y:\ |y|_h\geqslant 1\}}(y)}{\nu_Q|y|_h^Q}dy\right)^p dx \\
&\geqslant \int_{\{x:\ |x|_h\geqslant 1\}}\left(\int_{\{y:\ |y|_h>|x|_h\}}\frac{|y|_h^{-\varepsilon-Q/p}}{\nu_Q|y|_h^Q}dy\right)^p dx \\
&= Q^p\cdot\omega_Q\cdot\frac{1}{p\varepsilon}\cdot\frac{1}{(Q/p+\varepsilon)^p}.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain that

$$\varliminf_{\varepsilon\to 0^+}\frac{\|\mathbf{H}^* f_\varepsilon\|_{L^p(\mathbb{H}^n)}}{\|\mathbf{H}f_\varepsilon\|_{L^p(\mathbb{H}^n)}} \geqslant p-1.$$

The sharpness assertion in the left side of (4) is thereby established. Combining all estimates together, we finish the proof.

REMARK 1. It is worth noting that we do not need to assume $f \in L^p(\mathbb{H}^n)$ in Theorem 1. We can easily see that the condition $\mathbf{H}f \in L^p(\mathbb{H}^n)$ does not imply that $f \in L^p(\mathbb{H}^n)$. Indeed, set $f(x) = (|x|_h - 1)^{-1/p}\chi_{\{x:\ 1\leqslant|x|_h\leqslant 2\}}(x),\ p > 1$. Obviously,

$$\mathbf{H}f(x) = 0 \quad \text{for} \quad 0 < |x|_h \leqslant 1.$$

So we have

$$
\begin{aligned}
\|\mathbf{H}f\|_{L^p(\mathbb{H}^n)}^p &= \int_{\{x:\ |x|_h \geqslant 1\}} (\mathbf{H}f(x))^p\,dx \\
&= \int_{\{x:\ |x|_h \geqslant 1\}} \left(\frac{1}{|B(0,|x|_h)|} \int_{B(0,|x|_h)} (|y|_h - 1)^{-1/p} \chi_{\{y:\ 1 \leqslant |y|_h \leqslant 2\}}(y)dy \right)^p dx \\
&\leqslant \int_{\{x:\ |x|_h \geqslant 1\}} \left(\frac{2^{Q-1}\omega_Q}{\nu_Q} \cdot \frac{p}{p-1} \cdot \frac{1}{|x|_h^Q} \right)^p dx \\
&= \left(\frac{2^{Q-1}\omega_Q}{\nu_Q} \cdot \frac{p}{p-1} \right)^p \cdot \frac{\nu_Q}{p-1}.
\end{aligned}
$$

Hence, $\mathbf{H}f \in L^p(\mathbb{H}^n)$. However, a simple estimate shows that

$$
\begin{aligned}
\|f\|_{L^p(\mathbb{H}^n)}^p &= \int_{\mathbb{H}^n} ((|x|_h - 1)^{-1/p} \chi_{\{x:\ 1 \leqslant |x|_h \leqslant 2\}}(x))^p dx \\
&= \omega_Q \int_1^2 t^{Q-1}(t-1)^{-1} dt \\
&\geqslant \omega_Q \int_1^2 (t-1)^{-1} dt \\
&= \infty,
\end{aligned}
$$

which implies $f \notin L^p(\mathbb{H}^n)$.

Proof of Theorem 2. With the help of Lemma 2, the proof of Theorem 2 will be essentially similar to that of Theorem 1 and we omit the details.

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