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WEIGHTED COMPOSITION OPERATORS AND THEIR PRODUCTS ON $L^2(\Sigma)$

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Abstract. In this paper, we study the ascent and descent of weighted composition operators on $L^2(\Sigma)$. In addition, we discuss measure theoretic characterizations of some classical properties for products of these type operators.

1. Introduction and preliminaries

Let (X,Σ,μ) be a complete sigma finite measure space and let $\mathscr{A}$ be a sub-sigma finite algebra of Σ. If $B \subset X$, let $\mathscr{A}_B = \mathscr{A} \cap B$ denote the relative completion of the sigma-algebra generated by $\{A \cap B : A \in \mathscr{A}\}$ and denote the complement of B in X by B^c. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear spaces of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $u \in L^0(\Sigma)$ and let $\varphi : X \to X$ be a measurable transformation on X, that is, $\varphi^{-1}(A) \in \Sigma$ for all $A \in \Sigma$. Denote by $\mu_{\sigma(u)} \circ \varphi^{-1}$ the positive measure on Σ given by $\mu_{\sigma(u)} \circ \varphi^{-1}(A) = \mu(\varphi^{-1}(A) \cap \sigma(u))$ for all $A \in \Sigma$. Put $\mu_X = \mu$. We say that φ is nonsingular, if $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. In this case we write $\mu \circ \varphi^{-1} \ll \mu$, as usual. Let h be the Radon-Nikodym derivative, $h = \frac{d\mu\circ\varphi^{-1}}{d\mu}$.

Let $1 \leqslant p \leqslant \infty$. By a weighted composition operator in $L^p(\Sigma) = L^p(X,\Sigma,\mu) = L^p(\mu)$ we mean a mapping $W = uC_\varphi : L^p(\Sigma) \supseteq \mathscr{D}(W) \to L^p(\Sigma)$ formally defined by

$$Wf(x) = \begin{cases} u(x)f(\varphi(x)) & x \in \sigma(u) \\ 0 & x \notin \sigma(u), \end{cases}$$

for all $f \in \mathscr{D}(W) = \{f \in L^p(\Sigma) : u.(f \circ \varphi) \in L^p(\Sigma)\}$. In general, such operator may not be well-defined. We use the assumption $\mu_{\sigma(u)} \circ \varphi^{-1} = (\mu \circ \varphi^{-1})|_{\sigma(u)} \ll \mu$ to see that W is well-defined on $\mathscr{D}(W)$, for more details, see [1]. Now, set $u = 1$. Then the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$ on $L^p(\Sigma)$ is well-defined if and only if the transformation φ is nonsingular. It is known that $C_\varphi \in \mathscr{B}(L^p(\Sigma))$, the

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algebra of all bounded linear operators on $L^p(\Sigma)$, if and only if $h \in L^\infty(\Sigma)$. In this case $\mathscr{D}(C_\varphi) = L^p(\Sigma)$, $\|C_\varphi\|^p = \|h\|_\infty$ and $W = M_u C_\varphi$, where M_u is a multiplication operator defined by $M_u(f) = uf$ on $\mathscr{D}(M_u) = \{f \in L^p(\Sigma) : u.f \in L^p(\Sigma)\}$. It is known by the closed graph theorem that $\mathscr{D}(M_u) = L^p(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$, or equivalently, $M_u \in \mathscr{B}(L^p(\Sigma))$. In this case, $\|M_u\| = \|u\|_\infty$ (see [22]).

Assume f is a non-negative Σ-measurable function on X. Since $\mathscr{A}$ is sub-sigma finite, by the Radon-Nikodym theorem, there exists a unique $\mathscr{A}$-measurable function $E^{\mathscr{A}}(f)$ such that $\int_A f d\mu = \int_A E^{\mathscr{A}}(f) d\mu$, where A is any $\mathscr{A}$-measurable set for which $\int_A f d\mu$ exists. Note that $E(f)$ depends both on μ and $\mathscr{A}$. A real-valued measurable function $f = f^+ - f^-$ is said to be conditionable if $\mu(\{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}) = 0$. If f is complex-valued, then $f \in \mathscr{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$ if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. One can show that every $L^p(\Sigma)$ function is conditionable. In the setting of L^p-spaces, the conditional expectation operator $E^{\mathscr{A}}$ plays an important role in the study of weighted composition operators. We use the notation $L^p(\mathscr{A})$ for $L^p(X, \mathscr{A}, \mu_{|_{\mathscr{A}}})$ and henceforth we write μ in place $\mu_{|_{\mathscr{A}}}$. The mapping $E^{\mathscr{A}} : L^p(\Sigma) \to L^p(\mathscr{A})$ defined by $f \mapsto E^{\mathscr{A}}(f)$, is called the conditional expectation operator with respect to $\mathscr{A}$. In the case of $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathscr{A})$. For further discussion of the conditional expectation operator see [1, 8, 15, 19].

For each $n \in \mathbb{N}$, let $\Sigma_n := \varphi^{-n}(\Sigma)$ be a sub-sigma finite algebra of Σ and let $\mu \circ \varphi^{-n} \ll \mu$. Set $u_{(n)} = u.(u \circ \varphi) \cdots (u \circ \varphi^{n-1})$, $(h)_n = d\mu \circ \varphi^{-n}/d\mu$, $E^n = E^{\Sigma_n}$ and $(J)_n = (h)_n E^n(|u_{(n)}|^p) \circ \varphi^{-n}$. We use the symbols h, E and $J = hE(|u|^p) \circ \varphi^{-1}$ instead of $(h)_1$, E^1 and $(J)_1$, respectively. Note that if Σ_n is sigma finite so is Σ_k for any $k < n$. Let $f \in \mathscr{D}(E^n)$. Since $E^n(f)$ is a Σ_n-measurable function, there is a $g \in L^0(\Sigma)$ such that $E^n(f) = g \circ \varphi^n$. In general g is not unique. This deficiency can be solved by assuming $\sigma(g) \subseteq \sigma((h)_n)$, because for each $g_1, g_2 \in L^0(\Sigma)$, $g_1 \circ \varphi^n = g_2 \circ \varphi^n$ if and only if $g_1 = g_2 = g$ on $\sigma((h)_n)$. In this case g is a well-defined and unique. As a notation, we then write $g = E^n(f) \circ \varphi^{-n}$. With this setting by the change of variables formula, we obtain $\int_X f d\mu = \int_X (h)_n E^n(f) \circ \varphi^{-n} d\mu$, in the sense that if one of the integrals exists then so does the other and they have the same value (see [2]). For $1 \leqslant p < \infty$, define

$$\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}};$$

$$\|f\|_{p,hd\mu} = (\int_X |f|^p h d\mu)^{\frac{1}{p}};$$

$$\|f\|_{p,Jd\mu} = (\int_X |f|^p J d\mu)^{\frac{1}{p}}.$$

It is easy to check that

$$\|C_\varphi(f)\|_p = \|M_{\sqrt[p]{h}} f\|_p = \|f\|_{p,hd\mu}, \quad f \in \mathscr{D}(C_\varphi) \subseteq L^p(\Sigma);$$

$$\|W(f)\|_p = \|M_{\sqrt[p]{J}} f\|_p = \|f\|_{p,Jd\mu}, \quad f \in \mathscr{D}(W) \subseteq L^p(\Sigma). \tag{1.1}$$

Hence $\mathscr{D}(C_\varphi) = L^p(\Sigma) \cap L^p(hd\mu)$ and $\mathscr{D}(W) = L^p(\Sigma) \cap L^p(Jd\mu)$. Campbell and Hornor in [2] proved that W is a densely defined and closed operator if and only if J is finite valued, that is, $\mu(\{x \in X : J(x) = \infty\}) = 0$. Also, $\overline{\mathscr{R}(W)} = \{u.(f \circ \varphi) : f \in L^p(Jd\mu)\}$. If $J \in L^\infty(\Sigma)$, then $L^p(\Sigma) \subseteq L^p(Jd\mu)$, and so $\mathscr{D}(W) = L^p(\Sigma)$. Moreover, it follows from (1.1) that $W \in \mathscr{B}(L^p(\Sigma))$ if and only if $J \in L^\infty(\Sigma)$ (see also [9]). In particular, in case $u = 1$, $\overline{\mathscr{D}(C_\varphi)} = L^p(\Sigma)$ if and only if $h < \infty$; that is finite valued, and $\mathscr{R}(E) = \overline{\mathscr{R}(C_\varphi)} = L^p(\varphi^{-1}(\Sigma)) = \{f \circ \varphi : f \in L^p(hd\mu)\}$. If $h \in L^\infty(\Sigma)$, then $L^p(\Sigma) \subseteq L^p(hd\mu)$, and so $\mathscr{D}(C_\varphi) = L^p(\Sigma)$. Lambert et al. in [9] shows that the adjoint W^* of $W \in \mathscr{B}(L^2(\Sigma))$ is given by $W^*(f) = hE(\bar{u}f) \circ \varphi^{-1}$, for each $f \in L^2(\Sigma)$. In this case, $W^*W = M_J$ and $WW^* = M_{u.(h\circ\varphi)}EM_{\bar{u}}$.

Products of operators appear often in the service of the study of other operators. Weighted composition operators and their products have been used to provide examples and illustrations of many operator theoretic properties. In several cases major conjectures in operator theory have been reduced to the weighted composition operators. The purpose of this note is to find some characterizations of properties of weighted composition operators on $L^2(\Sigma)$ and present a relationship between $W = uC_\varphi$ and their products. A good reference for information on the weighted composition operators in L^2-spaces is the monograph [1]. In Section 2, we collect some sufficient facts on products of weighted composition operators. In section 3, we investigate semi-Kato type weighted composition operators. Finally, in section 4, we characterize the weighted composition operators on $L^2(\Sigma)$ whose ascent and descent is finite.

2. On some classic properties of $W = uC_\varphi$ on $L^2(\Sigma)$

For $i = 1,2$ and $n \in \mathbb{N}$, let $\Sigma_n^i := \varphi_i^{-n}(\Sigma)$ be a sub-sigma finite algebra of Σ and let $\mu \circ \varphi_i^{-n} \ll \mu$. Set $u_{i(n)} = u_i.(u_i \circ \varphi_i)\cdots(u_i \circ \varphi_i^{n-1})$, $(h_i)_n = d\mu \circ \varphi_i^{-n}/d\mu$, $E_i^n = E^{\Sigma_n^i}$ and $(J_i)_n = (h_i)_n E_i^n(|u_{i(n)}|^2) \circ \varphi_i^{-n}$. We use the symbols h_i, E_i and $J_i = h_iE_i(|u_i|^2) \circ \varphi_i^{-1}$ instead of $(h_i)_1$, E_i^1 and $(J_i)_1$, respectively. Put $\varphi_3 = \varphi_1 \circ \varphi_2$, $u_3 = u_2.(u_1 \circ \varphi_2)$. Then $\mu \circ \varphi_3^{-n} \ll \mu$.

REMARK 2.1. For a nonsingular measurable transformation φ_i $(i = 1,2)$, let $h_i < \infty$ and $\varphi_3^{-1}(\Sigma)$ be a sub-sigma finite algebra of Σ. Then by [1, Lemma 26] we have

$$h_3 = h_1E_1(h_2) \circ \varphi_1^{-1} \text{ and } \sigma(h_3 \circ \varphi_3) = X. \tag{2.1}$$

Also, if $C^*_{\varphi_3}$ is densely defined then so is $C_{\varphi_3}C^*_{\varphi_3}$ (see [20, Theorem 1.8 and 7.2]). In this case $C_{\varphi_3}C^*_{\varphi_3} = M_{h_3\circ\varphi_3}E_3$ (see [1, Theorem 18]). Moreover, if $h_i \in L^\infty(\Sigma)$ then by [1, Prposition 17], $C^*_{\varphi_3}(f) = h_1E_1(h_2E_2(f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}$ for all $f \in L^2(\Sigma)$.

LEMMA 2.2. *For a nonsingular measurable transformation φ_i $(i = 1,2)$, let $h_i \in L^\infty(\Sigma)$. Then*

$$E_3(f) = \frac{1}{E_1(h_2) \circ \varphi_2} E_1(h_2E_2(f) \circ \varphi_2^{-1}) \circ \varphi_2, \quad f \in L^2(\Sigma).$$

Proof. Let $f \in L^2(\Sigma)$. Then by Remark 2.1 we have

$$\begin{aligned} E_3(f) &= \frac{1}{h_3 \circ \varphi_3} C_{\varphi_3} C^*_{\varphi_3}(f) \\ &= \frac{h_1 \circ \varphi_3}{h_3 \circ \varphi_3} E_1(h_2 E_2(f) \circ \varphi_2^{-1}) \circ \varphi_2 \\ &= \frac{1}{E_1(h_2) \circ \varphi_2} E_1(h_2 E_2(f) \circ \varphi_2^{-1}) \circ \varphi_2. \quad \square \end{aligned}$$

For nonsingular measurable transformation φ_1 and φ_2, let $h_i < \infty$ $(i = 1,2,3)$. Then we have

$$\begin{aligned} \mathscr{D}(C_{\varphi_2} C_{\varphi_1}) &= \{f \in L^2(\Sigma) : f \in \mathscr{D}(C_{\varphi_1}),\ f \circ \varphi_3 \in L^2(\Sigma)\} \\ &= L^2((1+h_1)d\mu) \cap L^2(h_3 d\mu) \\ &= L^2((1+h_1+h_3)d\mu) \\ &= L^2((1+h_1+h_1 E_1(h_2) \circ \varphi_1^{-1})d\mu) \end{aligned}$$

and $\mathscr{D}(C_{\varphi_3}) = L^2((1+h_1 E_1(h_2) \circ \varphi_1^{-1})d\mu)$. Thus, $\mathscr{D}(C_{\varphi_2} C_{\varphi_1}) \subseteq \mathscr{D}(C_{\varphi_3})$. If $E_1(h_2) \circ \varphi_1^{-1} \geqslant k$ on X for some $k > 0$, then for each $f \in \mathscr{D}(C_{\varphi_3})$,

$$\int_X |f \circ \varphi_1|^2 d\mu = \int_X h_1 |f|^2 d\mu \leqslant \frac{1}{k} \int_X h_1 E_1(h_2) \circ \varphi_1^{-1} |f|^2 d\mu < \infty.$$

It follows that $C_{\varphi_3} = C_{\varphi_2} C_{\varphi_1}$.

PROPOSITION 2.3. *The following assertions hold.*

(a) For nonsingular measurable transformation φ_1 and φ_2, if $\{h_1, h_2\} \subseteq L^\infty(\Sigma)$, then $J_3 = h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1}$.

(b) Let $\varphi_3^{-1}(\Sigma)$ be a sub-sigma finite algebra of Σ. Then W_3 is injective if and only if $\sigma(h_1) = \sigma(E_1(|u_1|^2 J_2)) = X$.

Proof. By assumption, $h_3 \in L^\infty(\Sigma)$. Hence E_3 is well-defined. Now, (a) is immediate from by (2.1) and Lemma 2.2.

For the proof of the second statement, we know that W_i is injective if and only if $\sigma(J_i) = X$. Now, let $A = \{x \in X : E_1(|u_1|^2 J_2) = 0\}$. So $A = \varphi_1^{-1}(B)$, for some $B \in \Sigma$. If $\mu(A) > 0$, then $\mu(B) > 0$ because $\mu \circ \varphi_1^{-1} \ll \mu$. Hence

$$\int_B h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1} d\mu = \int_A E_1(|u_1|^2 J_2) d\mu = 0,$$

and so $h_1 = 0$ or $E_1(|u_1|^2 J_2) \circ \varphi_1^{-1} = 0$ on B. Therefore, $h_1 > 0$ and $E_1(|u_1|^2 J_2) \circ \varphi_1^{-1} > 0$ implies that $E_1(|u_1|^2 J_2) > 0$. Now, let $E_1(|u_1|^2 J_2) > 0$. Since $E_1(|u_1|^2 J_2)$ is a $\varphi_1^{-1}(\Sigma)$-measurable, then there exists a unique $g \in L^0(\Sigma)$, with $\sigma(g) \subseteq \sigma(h_1)$, such that $E_1(|u_1|^2 J_2) = g \circ \varphi_1$. It follows that $0 < \int_X g \circ \varphi_1 d\mu = \int_X h_1 g d\mu$, and so $E_1(|u_1|^2 J_2) \circ \varphi_1^{-1} = g > 0$ on $\sigma(h_1)$. We conclude that $\sigma(J_3) = X$ if and only if $\sigma(h_1) = \sigma(E_1(|u_1|^2 J_2)) = X$. $\square$

LEMMA 2.4. *Let $W_i \in \mathscr{B}(L^2(\Sigma))$, $\varphi_3 = \varphi_1 \circ \varphi_2$ and $u_3 = u_2.(u_1 \circ \varphi_2)$. Then the following assertions hold.*
(a) $J_3 \circ \varphi_3 = (h_1 \circ \varphi_3)E_1(|u_1|^2 J_2) \circ \varphi_2$.
(b) $W_3^*(f) = h_1 E_1(\bar{u}_1 h_2 E_2(\bar{u}_2 f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}$.
(c) $W_3^* W_3(f) = (h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1}) f$.
(d) $W_3 W_3^*(f) = u_3(h_1 \circ \varphi_3)E_1(\bar{u}_1 h_2 E_2(\bar{u}_2 f) \circ \varphi_2^{-1}) \circ \varphi_2$.
(e) $W_3^* W_3 W_3(f) = (u_3 h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1}) f \circ \varphi_3$.
(f) $W_3 W_3^* W_3(f) = (u_3(h_1 \circ \varphi_3)E_1(|u_1|^2 J_2) \circ \varphi_2) f \circ \varphi_3$.
(g) $W_3 W_3^*(f) = u_3(h_3 \circ \varphi_3)E_3(\bar{u}_3 f)$.

Proof. Part (a) follows from (2.1) and Lemma 2.2. To prove (b), let $f \in L^2(\Sigma)$. Then

$$\begin{aligned} W_3^*(f) = W_1^*(W_2^*(f)) &= W_1^*(h_2 E_2(\bar{u}_2 f) \circ \varphi_2^{-1}) \\ &= h_1 E_1(\bar{u}_1 h_2 E_2(\bar{u}_2 f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}. \end{aligned}$$

The reminder of the proof is left to the reader. □

In [6], Douglas proved that when $A, B \in \mathscr{B}(\mathscr{H})$, then $AA^* \leqslant \lambda BB^*$ for some $\lambda \geqslant 0$; if and only if $A = BC$ for some $C \in \mathscr{B}(\mathscr{H})$.

PROPOSITION 2.5. *Let for $i = 1, 2$, $W_i = u_i C_{\varphi_i} \in \mathscr{B}(L^2(\Sigma))$. Then $J_3 \leqslant \lambda_1 J_1$ a.e. $[\mu]$ and $J_3 \circ \varphi_3 \leqslant \lambda_2 (J_2 \circ \varphi_2)$ a.e. $[\mu\,|_{\varphi_3^{-1}(\Sigma)}]$ on $\sigma(u_3)$ for some $\lambda_i \geqslant 0$.*

Proof. Since $W_3 = W_2 W_1$, by Douglas' theorem, there exists $\lambda_i \geqslant 0$ such that $W_3^* W_3 \leqslant \lambda_1 W_1^* W_1$ and $W_3 W_3^* \leqslant \lambda_2 W_2 W_2^*$. Then for each $f, g \in L^2(\Sigma)$ we have $\langle J_3 f, f\rangle \leqslant \langle \lambda_1 J_1 f, f\rangle$ and $\langle u_3(h_3 \circ \varphi_3)E_3(\bar{u_3} g), g\rangle \leqslant \langle \lambda_2 u_2(h_2 \circ \varphi_2)E_2(\bar{u_2} g), g\rangle$. For $A \in \Sigma$ with $\mu(A) < \infty$, take $f = \chi_A$ and $g = \chi_{\varphi_3^{-1}(A)} u_3$. Since $E_3(\bar{u_3} g) = \chi_{\varphi_3^{-1}(A)} E_3(|u_3|^2)$ and $E_2(\bar{u_2} g) = \chi_{\varphi_3^{-1}(A)}(u_1 \circ \varphi_2)E_2(|u_2|^2)$, we obtain

$$\int_{\varphi_3^{-1}(A)} |u_3|^2 (J_3 \circ \varphi_3) d\mu \leqslant \int_{\varphi_3^{-1}(A)} \lambda_2 |u_3|^2 (J_2 \circ \varphi_2) d\mu.$$

This completes the proof. □

Let $[T, S] = TS - ST$ for T and S in $\mathscr{B}(\mathscr{H})$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be normal if $[T, T^*] = 0$, quasinormal if $[T, T^*T] = 0$ and hyponormal if $[T, T^*] \geqslant 0$. Normal, quasinormal and hyponormal bounded weighted composition operators have been characterized in [2, 14] as follows:

LEMMA 2.6. *Let $W = uC_\varphi \in \mathscr{B}(L^2(\Sigma))$. Then the following assertions hold.*
(a) W is normal if and only if $\left(\varphi^{-1}(\Sigma)\right)_{\sigma(u)} = \Sigma_{\sigma(u)}$ and $J = \chi_{\sigma(u)} J \circ \varphi$.
(b) W is quasinormal if and only if $J = J \circ \varphi$ on $\sigma(u)$.
(c) W is hyponormal if and only if $\sigma(u) = \sigma(J)$ and $(h \circ \varphi)E(\frac{|u|^2}{J}) \leqslant 1$.

PROPOSITION 2.7. *Let $W_i = u_iC_{\varphi_i} \in \mathscr{B}(L^2(\Sigma))$ with $J_1\circ\varphi_2 = J_1$ and $J_2\circ\varphi_1 = J_2$.*

(a) If W_1 and W_2 are normal (quasinormal), then W_3 is a normal (quasinormal) operator.

(b) If W_1 and W_2 are hyponormal and $h_2E_2(\frac{|u_2|^2}{J_2})\circ\varphi_2^{-1}$ is a $\varphi_1^{-1}(\Sigma)$-measurable function, then W_3 is a hyponormal operator.

Proof. (a) Let W_i be normal operator. Then by Lemma 2.6(a), $(\varphi_i^{-1}(\Sigma))_{\sigma(u_i)} = \Sigma_{\sigma(u_i)}$ and $J_i = \chi_{\sigma(u_i)}J_i\circ\varphi_i$. Also, by hypotheses we get that

$$\begin{aligned}\big(\varphi_3^{-1}(\Sigma)\big)_{\sigma(u_3)} &= \varphi_2^{-1}\big(\varphi_1^{-1}(\Sigma)\big)\cap\sigma(u_3) = \varphi_2^{-1}\big(\varphi_1^{-1}(\Sigma)\big)\cap\sigma(u_2)\cap\varphi_2^{-1}(\sigma(u_1))\\ &= \varphi_2^{-1}\big(\varphi_1^{-1}(\Sigma)\cap\sigma(u_1)\big)\cap\sigma(u_2) = \varphi_2^{-1}\big(\Sigma\cap\sigma(u_1)\big)\cap\sigma(u_2)\\ &= \big(\varphi_2^{-1}(\Sigma)\cap\sigma(u_2)\big)\cap\sigma(u_1\circ\varphi_2) = \Sigma\cap\sigma(u_2)\cap\sigma(u_1\circ\varphi_2)\\ &= \Sigma\cap\sigma\big(u_2(u_1\circ\varphi_2)\big) = \Sigma\cap\sigma(u_3) = \Sigma_{\sigma(u_3)},\end{aligned}$$

and $J_3 = h_1E_1(|u_1|^2J_2)\circ\varphi_1^{-1} = h_1E_1(|u_1|^2J_2\circ\varphi_1)\circ\varphi_1^{-1} = J_2J_1$. Since W_1 and W_2 are normal, we have

$$\begin{aligned}\chi_{\sigma(u_3)}J_3\circ\varphi_3 &= \chi_{\sigma(u_2)\cap\sigma(u_1\circ\varphi_2)}J_3\circ\varphi_3\\ &= \{\chi_{\sigma(u_2)}J_2\circ\varphi_1\circ\varphi_2\}\{\chi_{\sigma(u_1\circ\varphi_2)}J_1\circ\varphi_1\circ\varphi_2\}\\ &= \{\chi_{\sigma(u_2)}J_2\circ\varphi_2\}\{\chi_{\sigma(u_1)}J_1\circ\varphi_1\}\circ\varphi_2\\ &= J_2J_1\circ\varphi_2 = J_2J_1 = J_3.\end{aligned}$$

Thus, W_3 is normal.

(b) By hypotheses, $\sigma(u_i) = \sigma(J_i)$ and $(h_i\circ\varphi_i)E_i(\frac{|u_i|^2}{J_i}) \leqslant 1$ for $i = 1,2$. Hence we obtain

$$\sigma(J_3) = \sigma(J_2J_1) = \varphi_2^{-1}(\sigma(J_1))\cap\sigma(J_2) = \sigma(u_3);$$

$$E_2(J_1) = J_1 = E_2(J_1)\circ\varphi_2^{-1}, \quad E_1(J_2) = J_2 = E_1(J_2)\circ\varphi_1^{-1},$$

and

$$E_2\Big(\frac{1}{J_1}\Big)\circ\varphi_2^{-1} = \frac{1}{J_1} \in L^0(\varphi_2^{-1}(\Sigma)).$$

Since $h_3\circ\varphi_3 = (h_1\circ\varphi_3)E_1(h_2)\circ\varphi_2$ and $\sigma(E_i(h_i)\circ\varphi_i) = X$ (see [11]), we have

$$\begin{aligned}(h_3\circ\varphi_3)E_3\Big(\frac{|u_3|^2}{J_3}\Big) &= (h_1\circ\varphi_3)E_1\Big(h_2E_2\Big(\frac{|u_2|^2|u_1|^2\circ\varphi_2}{J_2J_1}\Big)\circ\varphi_2^{-1}\Big)\circ\varphi_2\\ &= (h_1\circ\varphi_3)E_1\Big(|u_1|^2h_2E_2\Big(\frac{|u_2|^2}{J_2J_1}\Big)\circ\varphi_2^{-1}\Big)\circ\varphi_2\\ &= \{(h_1\circ\varphi_3)E_1\Big(\frac{|u_1|^2}{J_1}\Big)\circ\varphi_2\}\{h_2E_2\Big(\frac{|u_2|^2}{J_2}\Big)\circ\varphi_2^{-1}\}\circ\varphi_2\\ &= \Big\{(h_1\circ\varphi_1)E_1\Big(\frac{|u_1|^2}{J_1}\Big)\Big\}\circ\varphi_2\,\Big\{(h_2\circ\varphi_2)E_2\Big(\frac{|u_2|^2}{J_2}\Big)\Big\} \leqslant 1.\end{aligned}$$

This completes the proof. $\square$

An atom of the measure μ is an element $B \in \Sigma$ with $\mu(B) > 0$ such that for each $F \in \Sigma$, if $F \subset B$ then either $\mu(F) = 0$ or $\mu(F) = \mu(B)$. A measure with no atoms is called non-atomic. Write $X = (\cup_{n\in\mathbb{N}} A_n) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic (see [25]). In [4] Chan proved that M_u is compact on $L^2(\Sigma)$ if and only if for any $\varepsilon > 0$, the set $\{x \in X : |u(x)| \geqslant \varepsilon\}$ consists of finitely many atoms. In the following, we give a sufficient condition for the product of a weighted composition operator W_1 with the adjoint of a weighted composition operator W_2^* on $L^2(\Sigma)$ to be compact. The order of the product gives rise to two different cases (see [5, 24]).

PROPOSITION 2.8. *For $i = 1,2$, let $W_i = u_i C_{\varphi_i} \in \mathscr{B}(L^2(\Sigma))$. Then the following assertions hold.*

(a) If for each $\varepsilon > 0$, the set $A = \{x \in X : (|u_2|^2(J_1 \circ \varphi_2)(h_1 \circ \varphi_2))(x) \geqslant \varepsilon\}$ consists of finitely many atoms, then $W_1 W_2^*$ is compact.

(b) If for each $\varepsilon > 0$, the set $B = \{x \in X : h_1(x)\big(E_1(|u_1|^2(h_2 \circ \varphi_2)) \circ \varphi_1^{-1}\big)(x) \geqslant \varepsilon\}$ consists of finitely many atoms and $u_2 \in L^0(\Sigma_2)$, then $W_2^* W_1$ is compact.

Proof. Let $f \in L^2(\Sigma)$. Then

$$W_1 W_2^*(f) = u_1(h_2 \circ \varphi_1)(E_2(\bar{u_2} f) \circ \varphi_2^{-1}) \circ \varphi_1;$$
$$W_2^* W_1(f) = h_2 E_2\big(\bar{u_2} u_1 (f \circ \varphi_1)\big) \circ \varphi_2^{-1}.$$

Using change of variable formula and inequality $|E_2(f)|^2 \leqslant E_2(|f|^2)$, we obtain

$$\begin{aligned}
\|W_1 W_2^*(f)\|^2 &= \int_X |u_1|^2 h_2^2 \circ \varphi_1 |E_2(\bar{u_2} f) \circ \varphi_2^{-1}|^2 \circ \varphi_1 d\mu \\
&= \int_X (h_1 E_1(|u_1|^2) \circ \varphi_1^{-1}) h_2^2 |E_2(\bar{u_2} f) \circ \varphi_2^{-1}|^2 d\mu \\
&= \int_X J_1 h_2^2 |E_2(\bar{u_2} f)|^2 \circ \varphi_2^{-1} d\mu \\
&= \int_X (J_1 \circ \varphi_2)(h_2 \circ \varphi_2) |E_2(\bar{u_2} f)|^2 d\mu \\
&\leqslant \int_X (J_1 \circ \varphi_2)(h_2 \circ \varphi_2) E_2(|u_2|^2 |f|^2) d\mu \\
&= \|M_{\sqrt{|u_2|^2 (J_1 \circ \varphi_2)(h_2 \circ \varphi_2)}} f\|^2 = \|M_{\sqrt{U_1}}\|^2,
\end{aligned}$$

where $U_1 := \sqrt{|u_2|^2 (J_1 \circ \varphi_2)(h_2 \circ \varphi_2)}$. Similarly,

$$\begin{aligned}
\|W_2^* W_1(f)\|^2 &= \int_X h_2^2 |E_2(\bar{u_2} u_1 (f \circ \varphi_1)) \circ \varphi_2^{-1}|^2 d\mu \\
&= \int_X (h_2 \circ \varphi_2) |E_2(\bar{u_2} u_1 (f \circ \varphi_1))|^2 d\mu \\
&\leqslant \int_X (h_2 \circ \varphi_2) E_2(|u_2|^2 |u_1|^2 |f|^2 \circ \varphi_1) d\mu \\
&\leqslant \int_X h_2^2 E_2(|u_2|^2) \circ \varphi_2^{-1} E_2(|u_1|^2 |f|^2 \circ \varphi_1) \circ \varphi_2^{-1} d\mu
\end{aligned}$$

$$= \int_X h_1 E_1(|u_1|^2 J_2 \circ \varphi_2) \circ \varphi_1^{-1} |f|^2 d\mu$$
$$= \|M_{\sqrt{h_1 E_1(|u_1|^2 J_2 \circ \varphi_2) \circ \varphi_1^{-1}}} f\|^2 = \|M_{\sqrt{U_2}}\|^2,$$

where $U_2 := \sqrt{h_1 E_1(|u_1|^2 J_2 \circ \varphi_2) \circ \varphi_1^{-1}}$. Since sets A and B consist of finitely many atoms, hence the corresponding multiplication operators are compact. It follows that $W_1 W_2^*$ and $W_2^* W_1$ are compact operators on $L^2(\Sigma)$. $\square$

PROPOSITION 2.9. *Let $W_i \in \mathscr{B}(L^2(\Sigma))$ for $i = 1,2$. Then the following assertions hold.*

(a) If J_3 is bounded away from zero on $\sigma(J_3)$, $\sigma(E_1(|u_1|^2 J_2) \circ \varphi_1^{-1}) = X$ and $\sigma(E_1(|u_1|^2) \circ \varphi_1^{-1}) = X$, then $\mathscr{R}(W_1)$ is closed.

(b) Let W_1 and W_2 have closed range. If $\sigma(J_2) = X$ or $\sigma(J_2)^c$ is contained in union of a finite number of atoms, then W_3 has closed range.

Proof. (a) Let $f \in L^2(\Sigma)$. Then

$$\|M_{\sqrt{J_3}} f\|_2^2 = \int_X h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1} |f|^2 d\mu$$
$$= \int_X |u_1|^2 J_2 |f|^2 \circ \varphi_1 d\mu$$
$$\leqslant \|J_2\|_\infty \int_X |u_1|^2 |f|^2 \circ \varphi_1 d\mu$$
$$= \|W_2\|^2 \ \|M_{\sqrt{J_1}} f\|_2^2.$$

Recall that for $u \in L^\infty(\Sigma)$, $\mathscr{R}(M_u)$ is closed in $L^2(\Sigma)$ if and only if u is bounded away from zero on $\sigma(u)$ (see [21]). Thus there exists $\lambda \geqslant 0$ such that $\lambda\|f\| \leqslant \|M_{\sqrt{J_1}} f\|$ for each $f \in L^2(\sigma(J_3))$. By hypotheses, we have $\sigma(J_3) = \sigma(h_1)$, $\sigma(J_1) = \sigma(h_1)$ and so $\sigma(J_1) = \sigma(J_3)$. It follows that $\mathscr{R}(W_1)$ is closed.

(b) It is a classical fact that W_3 has closed range if and only if $\mathscr{N}(W_2) + \mathscr{R}(W_1)$ is closed (see [18, Corollary 1]). Now, by assumptions, $\mathscr{N}(W_2) = \{0\}$ or $\mathscr{N}(W_2)$ is a finite dimensional subspace of $L^2(\Sigma)$ and hence W_3 has closed range. $\square$

3. Semi-Kato type weighted composition operators

DEFINITION 3.1. We say that $T \in \mathscr{B}(\mathscr{H})$ is an operator of semi-Kato type, if the null space of T is contained in $\bigcap_{n=1}^{\infty} \overline{\mathscr{R}(T^n)}$. $T \in \mathscr{B}(\mathscr{H})$ is called Kato if $\mathscr{R}(T)$ is closed and $\mathscr{N}(T) \subseteq \bigcap_{n=1}^{\infty} \mathscr{R}(T^n)$.

Any bounded operator that is either onto or bounded below is Kato (see [17]). The set of all semi-Kato and Kato type operators will be denoted by $\mathscr{SK}(\mathscr{H})$ and $\mathscr{K}(\mathscr{H})$ respectively. Obviously, $\mathscr{K}(\mathscr{H}) \subseteq \mathscr{SK}(\mathscr{H})$. Also, if $T \in \mathscr{SK}(\mathscr{H})$ and for each

$n \in \mathbb{N}$, T^n has closed range, then $T \in \mathscr{K}(\mathscr{H})$. Now, for $W = uC_\varphi \in \mathscr{B}(L^2(\Sigma))$, Campbell and Hornor in [2] proved that

$$\overline{\mathscr{R}(W^n)} = c.l.s\{c_n\chi_A : A \in (\varphi^{-n}(\Sigma))_{\sigma(c_n)}\}, \tag{3.1}$$

where $c_n = u_{(n)} = \prod_{i=1}^{n-1} u \circ \varphi^i$. This holds even in case W is a densely defined unbounded operator [2, Lemma 6.2]. It is easy to check that $\|W^n f\| = \|M_{\sqrt{(J)_n}} f\|$, for all $f \in L^2(\Sigma)$. This implies that

$$\begin{aligned}\mathscr{N}(W^n) &= c.l.s\{\chi_{X\setminus\sigma((J)_n)}L^2(\Sigma)\} \\ &= c.l.s\{f \in L^2(\Sigma) : f = 0 \text{ on } \sigma((J)_n)\} \\ &:= L^2(\Sigma \cap \sigma((J)_n)^c).\end{aligned} \tag{3.2}$$

Also, we deduce that W^n has closed range if and only if $(J)_n$ is bounded away from zero on $\sigma((J)_n)$ (e.g., see [10]).

THEOREM 3.2. *Let $W = uC_\varphi \in \mathscr{B}(L^2(\Sigma))$, $\Sigma_\infty := \bigcap_{n=1}^{\infty} \big(\varphi^{-n}(\Sigma)\big)_{\sigma(c_n)}$ and let $\|c_n - 1\|_2 \to 0$ as $n \to \infty$. Then the following assertions hold.*

(a) $W \in \mathscr{S}\mathscr{K}(L^2(\Sigma))$ if and only if $\Sigma \cap (\sigma(J))^c \subseteq \Sigma_\infty$.

(b) $W \in \mathscr{K}(L^2(\Sigma))$ if and only if, for each $n \in \mathbb{N}$, $(J)_n$ is bounded away from zero on $\sigma((J)_n)$ and $\Sigma \cap (\sigma(J))^c \subseteq \Sigma_\infty$.

Proof. (a) Using (3.1) and (3.2) we have

$$\overline{\mathscr{R}(W^n)} = c.l.s\{c_n L^2\big((\varphi^{-n}(\Sigma))_{\sigma(c_n)}\big)\}$$

and $\mathscr{N}(W) = L^2\big(\Sigma \cap (\sigma(J))^c\big)$. It follows that $W \in \mathscr{S}\mathscr{K}(L^2(\Sigma))$ whenever $L^2(\Sigma \cap (\sigma(J))^c) \subseteq \bigcap_{n=1}^{\infty} \overline{R(W^n)} = c.l.s\{c_\infty L^2\big(\bigcap_{n=1}^{\infty} \big(\varphi^{-n}(\Sigma)\big)_{\sigma(c_n)}\big)$, where $c_\infty = \prod_{i=1}^{\infty} u \circ \varphi^i$. But by hypothesis $c_\infty = 1$ (a.e.). Hence $L^2\big(\Sigma \cap (\sigma(J))^c\big) \subseteq L^2(\Sigma_\infty)$, and so $\Sigma \cap (\sigma(J))^c \subseteq \Sigma_\infty$. Conversely, if $\Sigma \cap (\sigma(J))^c \subseteq \Sigma_\infty$ then we obtain

$$\begin{aligned}L^2(\Sigma \cap (\sigma(J))^c) &\subseteq L^2(\Sigma_\infty) = c_\infty L^2\big(\bigcap_{n=1}^{\infty} (\varphi^{-n}(\Sigma))_{\sigma(c_n)}\big) \\ &= \bigcap_{n=1}^{\infty}\{c.l.s\{c_n L^2\big((\varphi^{-n}(\Sigma))_{\sigma(c_n)}\big)\}\} = \bigcap_{n=1}^{\infty} \overline{\mathscr{R}(W^n)}.\end{aligned}$$

(b) Let $W \in \mathscr{K}(L^2(\Sigma))$. Then for each $n \in \mathbb{N}$, $\mathscr{R}(W^n)$ is closed. So $(J)_n$ is bounded away from zero on $\sigma((J)_n)$. Moreover, $L^2(\Sigma \cap (\sigma(J))^c) = \mathscr{N}(W) \subseteq \bigcap_{n=1}^{\infty} \overline{\mathscr{R}(W^n)} = L^2(\Sigma_\infty)$. Conversely, if for each $n \in \mathbb{N}$, $(J)_n$ is bounded away from zero on $\sigma((J)_n)$ and $\Sigma \cap (\sigma(J))^c \subseteq \Sigma_\infty$. Then $\overline{\mathscr{R}(W^n)} = \mathscr{R}(W^n)$ and $\mathscr{N}(W) \subseteq \bigcap_{n=1}^{\infty} \mathscr{R}(W^n)$. $\square$

Recall that $\varphi_3 = \varphi_1 \circ \varphi_2$, $u_3 = u_2.(u_1 \circ \varphi_2)$ and $u_{j(n)} = \prod_{i=1}^{n-1} (u_j \circ \varphi_j^i)$. Hence

$$u_{3(n)} = \prod_{i=1}^{n-1} (u_2 \circ \varphi_3^i)(u_1 \circ \varphi_2 \circ \varphi_3^i).$$

Let $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$. Then $u_{3(n)} = (\prod_{i=1}^{n-1} u_2 \circ \varphi_1^i \circ \varphi_2^i)(\prod_{i=1}^{n-1} u_1 \circ \varphi_2^{i+1} \circ \varphi_1^i)$. In this case if $u_2 \circ \varphi_1 = u_2$, then $\sigma(u_{3(n)}) \subseteq \sigma(\prod_{i=1}^{n-1} u_2 \circ \varphi_2^i) = \sigma(u_{2(n)})$. Moreover, if $u_1 \circ \varphi_2 = u_1$, then $u_{3(n)} = (u_{2(n)}).(u_{1(n)})$ and hence $\sigma(u_{3(n)}) = \sigma(u_{2(n)}) \cap \sigma(u_{1(n)})$. For $i \in \{1,2,3\}$, define $\Sigma_\infty^i := \bigcap_{n=1}^{\infty} (\varphi_i^{-n}(\Sigma))_{\sigma(u_{i(n)})}$. Then

$$\Sigma_\infty^3 = \bigcap_{n=1}^{\infty} \big(\varphi_2^{-n}(\varphi_1^{-n}(\Sigma))\big)_{\sigma(u_{3(n)})} \subseteq \bigcap_{n=1}^{\infty} \big(\varphi_2^{-n}(\Sigma)\big)_{\sigma(u_{2(n)})},$$

also, $\Sigma_\infty^3 \subseteq \Sigma_\infty^1$. So, if $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, $u_1 \circ \varphi_2 = u_1$ and $u_2 \circ \varphi_1 = u_2$, then $\Sigma_\infty^3 \subseteq \Sigma_\infty^1 \cap \Sigma_\infty^2$.

THEOREM 3.3. *For $i \in \{1,2\}$, let $W_i = u_i C_{\varphi_i} \in \mathscr{B}(L^2(\Sigma))$ and let $\|u_{i(n)} - 1\|_2 \to 0$ as $n \to \infty$. Then the following assertions hold.*

(a) If $W_i \in \mathscr{SK}(L^2(\Sigma))$, $\sigma\big(E_1(u_1^2 J_2) \circ \varphi_1^{-1}\big) = \sigma\big(E_1(J_2)\big)$ and $\Sigma_\infty^1 \cup \Sigma_\infty^2 \subseteq \Sigma_\infty^3$, then $W_3 \in \mathscr{SK}(L^2(\Sigma))$.

(b) If $W_3 \in \mathscr{SK}(L^2(\Sigma))$, $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, $u_2 \circ \varphi_1 = u_2$ and $u_1 \circ \varphi_2 = u_1$, then $W_1 \in \mathscr{SK}(L^2(\Sigma))$.

Proof. (a) Recall that $J_3 = h_1 E_1(u_1^2 J_2) \circ \varphi_1^{-1}$ and $\sigma(J_1) \subseteq \sigma(h_1)$. Then by hypothesis and Theorem 3.2(a), we have $\Sigma \cap (\sigma(J_3))^c = \Sigma \cap \{\sigma(h_1) \cap \sigma(E_1(J_2))\}^c = \Sigma \cap \{(\sigma(h_1))^c \cup (\sigma(J_2))^c\} \subseteq \Sigma_\infty^1 \cup \Sigma_\infty^2 \subseteq \Sigma_\infty^3$, and so $W_3 \in \mathscr{SK}(L^2(\Sigma))$.

(b) By our assumptions $\Sigma_\infty^3 \subseteq \Sigma_\infty^1$ and $u_{3(n)} = (u_{2(n)}).(u_{1(n)})$ for all $n \in \mathbb{N}$. It follows that $u_{3(\infty)} = \lim_{n\to\infty} u_{3(n)} = (u_{2(\infty)}).(u_{1(\infty)}) = 1$, and so

$$\begin{aligned}
\mathscr{N}(W_1) \subseteq \mathscr{N}(W_3) &\subseteq \bigcap_{n=1}^{\infty} \overline{\mathscr{R}(W_3^n)} \\
&= \bigcap_{n=1}^{\infty} c.l.s\{u_{3(n)} L^2\big((\varphi_3^{-n}(\Sigma))_{\sigma(u_{3(n)})}\big)\} \\
&= L^2\big(\bigcap_{n=1}^{\infty} (\varphi_3^{-n}(\Sigma))_{\sigma(u_{3(n)})}\big) \\
&= L^2(\Sigma_\infty^3) \subseteq L^2(\Sigma_\infty^1) \\
&= \bigcap_{n=1}^{\infty} c.l.s\{u_{1(n)} L^2\big((\varphi_1^{-n}(\Sigma))_{\sigma(u_{1(n)})}\big)\} \\
&= \bigcap_{n=1}^{\infty} \overline{\mathscr{R}(W_1^n)}.
\end{aligned}$$

Thus, $W_1 \in \mathscr{SK}(L^2(\Sigma))$. $\square$

4. Ascent and descent of weighted composition operators

Let T be a bounded linear operator on a Banach space $\mathscr{B}$. Recall that for each non-negative integer k, $\mathscr{N}(T^k) \subseteq \mathscr{N}(T^{k+1})$ and $\mathscr{R}(T^{k+1}) \subseteq \mathscr{R}(T^k)$. The ascent $\alpha(T)$ of T is the least non-negative integer such that $\mathscr{N}(T^k) = \mathscr{N}(T^{k+1})$, for all $k \geqslant \alpha(T)$ and the descent $d(T)$ of T is the least non-negative integer such that $\mathscr{R}(T^k) =$

$\mathscr{R}(T^{k+1})$, for all $k \geqslant d(T)$. It is a classical fact that if $\alpha(T) < \infty$ and $d(T) < \infty$ then $\alpha(T) = d(T)$. For more comprehensive study, we reefer the reader to [23].

Let $n \in \mathbb{N}$, φ be nonsingular and let $W = uC_\varphi \in \mathscr{B}(L^2(\Sigma))$. For this u and φ, we define the measure $\mu^n_{u,\varphi}$ by

$$\mu^n_{u,\varphi}(A) = \begin{cases} \int_{\varphi^{-1}(A)} |u|^2 d\mu & n = 1; \\ \int_{\varphi^{-1}(A)} |u|^2 d\mu^{n-1}_{u,\varphi} & n \geqslant 2. \end{cases}$$

It is easy to check that

$$\begin{aligned} &\mu^2_{u,\varphi} \ll \mu_{u,\varphi} \circ \varphi^{-1} \ll \mu \circ \varphi^{-2} \ll \mu \circ \varphi^{-1} \ll \mu; \\ &\mu^{n+1}_{u,\varphi} \ll \mu^n_{u,\varphi} \circ \varphi^{-1} \ll \mu^{n-1}_{u,\varphi} \circ \varphi^{-2} \ll \cdots \ll \mu^1_{u,\varphi} \circ \varphi^{-n} \\ &\qquad \ll \mu \circ \varphi^{-(n+1)} \ll \mu \circ \varphi^{-n} \ll \cdots \ll \mu \circ \varphi^{-1} \ll \mu. \end{aligned}$$

We prove by induction that

$$\mu^n_{u,\varphi}(A) = \int_A (J)_n d\mu, \quad n \in \mathbb{N},\ A \in \Sigma.$$

It is clear that $d\mu_{u,\varphi} = Jd\mu$. Suppose $d\mu^k_{u,\varphi} = (J)_k d\mu$ holds for $k = 1, 2, \cdots, n-1$. Then we have

$$\begin{aligned} \mu^n_{u,\varphi}(A) &= \int_{\varphi^{-1}(A)} |u|^2 d\mu^{n-1}_{u,\varphi} \\ &= \int_{\varphi^{-1}(A)} |u|^2 (J)_{n-1} d\mu \\ &= \int_{\varphi^{-1}(A)} |u|^2 E^{n-1}(|u_{(n-1)}|^2) \circ \varphi^{-(n-1)} d\mu \circ \varphi^{-(n-1)} \\ &= \int_{\varphi^{-n}(A)} |u \circ \varphi^{(n-1)}|^2 E^{n-1}(|u_{(n-1)}|^2) d\mu \\ &= \int_{\varphi^{-n}(A)} |u_{(n)}|^2 d\mu \\ &= \int_A (h)_n E^n(|u_{(n)}|^2) \circ \varphi^{-n} d\mu \\ &= \int_A (J)_n d\mu. \end{aligned}$$

Hence, $d\mu^n_{u,\varphi}/d\mu = (J)_n$. Now, set $Q_0 = J_0 = 1$ and $Q_n = hE(Q_{n-1}|u|^2) \circ \varphi^{-1}$. Then $Q_1 = J = (J)_1$, and so $d\mu_{u,\varphi} = Q_1 d\mu$. Suppose $d\mu^k_{u,\varphi} = Q_k d\mu$ holds for $k = 1, 2, \cdots, n-1$. Then for each $A \in \Sigma$ we have

$$\begin{aligned} \mu^n_{u,\varphi}(A) &= \int_{\varphi^{-1}(A)} |u|^2 d\mu^{n-1}_{u,\varphi} = \int_{\varphi^{-1}(A)} |u|^2 Q_{n-1} d\mu \\ &= \int_A hE(Q_{n-1}|u|^2) \circ \varphi^{-1} d\mu = \int_A Q_n d\mu. \end{aligned}$$

So, $d\mu^n_{u,\varphi}/d\mu = Q_n$. Thus, we conclude that

$$(J)_n = (h)_n E^n(|u_{(n)}|^2)\circ\varphi^{-n} = hE(Q_{n-1}|u|^2)\circ\varphi^{-1} = Q_n.$$

The measure ν and μ are called equivalent on Σ if $\mu \ll \nu \ll \mu$ and denoted by $\mu \simeq \nu$. In [13] Kumar has characterized the weighted composition operators on $L^2(\Sigma)$ whose ascent and descent is 1. The following theorem characterizes weighted composition operators with finite ascent.

THEOREM 4.1. *$W \in \mathscr{B}(L^2(\Sigma))$. Then $\alpha(W) = n_0$ if and only if $\mu_{u,\varphi^{n_0+1}} \simeq \mu_{u,\varphi^{n_0}}$.*

Proof. Recall that $\mathscr{N}(W^n) = \chi_{X_n}L^2(\Sigma) = L^2(X_n)$, where $X_n = \{x \in X : (J)_n(x) = 0\}$. Now, suppose $\alpha(W) = n_0$. Thus $\mathscr{N}(W^{n_0}) = \mathscr{N}(W^{n_0+1})$, by definition. Then we have

$$\begin{aligned}
\alpha(W) = n_0 &\Longleftrightarrow \mathscr{N}(W^{n_0}) = \mathscr{N}(W^{n_0+1})\\
&\Longleftrightarrow \quad L^2(X_{n_0}) = L^2(X_{n_0+1})\\
&\Longleftrightarrow \quad X_{n_0} = X_{n_0+1}\\
&\Longleftrightarrow ((J)_{n_0}\,|_A = 0 \Leftrightarrow (J)_{n_0+1}\,|_A = 0,\ \forall A \in \Sigma)\\
&\Longleftrightarrow (\mu^{n_0}_{u,\varphi}(A) = \int_A (J)_{n_0}d\mu = 0 \Leftrightarrow \mu^{n_0+1}_{u,\varphi}(A) = \int_A (J)_{n_0+1}d\mu = 0,\ \forall A \in \Sigma)\\
&\Longleftrightarrow \mu^{n_0+1}_{u,\varphi} \simeq \mu^{n_0}_{u,\varphi}.
\end{aligned}$$

This completes the proof. $\square$

THEOREM 4.2. *Let $W \in \mathscr{B}(L^2(\Sigma))$, $A_n := \sigma(u_{(n)})$ and let, for all $n \in \mathbb{N}$, $\Sigma_n = \varphi^{-n}(\Sigma)$ be a sub-sigma finite algebra of Σ. Then $d(W) = n_0 < \infty$ if and only if the following assertions hold.*

(a) $\Sigma_{n_0+1} \cap A_{n_0+1} = \Sigma_{n_0} \cap A_{n_0}$, and

(b) $L^2(\Sigma_{n_0} \cap A_{n_0})$ is an invariant subspace for $M_{\frac{\chi_{A_{n_0}}}{u\circ\varphi^{n_0}}}$.

Proof. Let $d(W) = n_0$. Since $\Sigma_{n_0+1} \subseteq \Sigma_{n_0}$ and $A_{n_0+1} \subseteq A_{n_0}$, so $\Sigma_{n_0+1} \cap A_{n_0+1} \subseteq \Sigma_{n_0} \cap A_{n_0}$. Let $A \in \Sigma$ and take $u_{(n_0)} = u(u\circ\varphi)(u\circ\varphi^2)\cdots(u\circ\varphi^{n_0-1})$. We shall show that $A_{n_0} \cap \varphi^{-n_0}(A) \in \Sigma_{n_0+1} \cap A_{n_0+1}$. By hypothesis $R(W^{n_0}) = R(W^{n_0+1})$ and n_0 is finite. Hence W has closed range. Thus

$$\mathscr{R}(W^{n_0}) = \{u_{(n_0)}f : \quad f \in L^2(\Sigma_{n_0} \cap A_{n_0})\};$$
$$\mathscr{R}(W^{n_0+1}) = \{u_{(n_0+1)}g : \quad g \in L^2(\Sigma_{n_0+1} \cap A_{n_0+1})\}.$$

Take $B = \varphi^{-n_0}(A) \cap A_{n_0}$ and choose $f = \chi_B \in L^2(\Sigma_{n_0} \cap A_{n_0})$. Hence there exists $g \in L^2(\Sigma_{n_0+1} \cap A_{n_0+1})$ such that $u_{(n_0)}f = u_{(n_0+1)}g$. Since $A_{n_0} = \sigma(u_{(n_0)})$, $B = \sigma(u_{(n_0)}\chi_B) = \sigma(g) \cap A_{n_0+1}$. But $\sigma(g) \cap A_{n_0+1} \in (\Sigma_{n_0+1} \cap A_{n_0+1})$. Consequently, $\Sigma_{n_0} \cap A_{n_0} \subseteq \Sigma_{n_0+1} \cap A_{n_0+1}$. This proves (a).

To prove (b), suppose $f \in L^2(\Sigma_{n_0} \cap A_{n_0})$. Then, by hypothesis, $u_{(n_0)} f = u_{(n_0+1)} g$ for some $g \in L^2(\Sigma_{n_0+1} \cap A_{n_0+1})$.

It follows that $f\chi_{A_{n_0}} = (u \circ \varphi^{n_0})\chi_{A_{n_0}} g$, and so

$$\chi_{A_{n_0+1}} g = \frac{f}{u \circ \varphi^{n_0}} \chi_{A_{n_0+1}} \in L^2(\Sigma_{n_0+1} \cap A_{n_0+1}).$$

This implies $\frac{f}{u\circ\varphi^{n_0}} \chi_{A_{n_0+1}} \in L^2((\Sigma_{n_0} \cap A_{n_0})$, by part (a).

Conversely, assume that (a) and (b) hold. From (a) we see that $L^2(\Sigma_{n_0} \cap A_{n_0}) = L^2(\Sigma_{n_0+1} \cap A_{n_0+1})$, and so $A_{n_0} = A_{n_0+1}$. Since $\mathscr{R}(W^{n_0+1}) \subseteq \mathscr{R}(W^{n_0})$, it will thus be sufficient to prove $\mathscr{R}(W^{n_0}) \subseteq \mathscr{R}(W^{n_0+1})$. Let $u_{(n_0)} f \in \mathscr{R}(W^{n_0})$ for some $f \in L^2(\Sigma_{n_0} \cap A_{n_0})$. Then $g = (f/(u \circ \varphi^{n_0}))\chi_{A_{n_0}} \in L^2(\Sigma_{n_0} \cap A_{n_0})$, by part (b). But, this implies that $u_{(n_0+1)} g = u_{(n_0)} f \in \mathscr{R}(W^{n_0+1})$. This the completes proof. $\square$

Let $W \in \mathscr{B}(L^2(\Sigma))$ and $\alpha(W) = d(W) = n_0 < \infty$. Then by [12, Theorem 1.12], $L^2(\Sigma) = L^2(X_{n_0}) \oplus \mathscr{R}(W^{n_0})$ and the restriction of W to $L^2(X_{n_0})$ is nilpotent and W, when restricted to $\mathscr{R}(W^{n_0})$, is bijection. Note that, in the proof of surjectivity of W on $\mathscr{R}(W^{n_0})$ we need not to have $\alpha(W) = n_0 < \infty$. Moreover, since $\frac{L^2(\Sigma)}{\mathscr{N}(W^{n_0})}$ algebraically isomorphic with $\chi_{\sigma((J)_{n_0})} L^2(\Sigma) := L^2(X^c_{n_0})$, so $L^2(X^c_{n_0})$ isomorphic with $\mathscr{R}(W^{n_0}) = \{u_{(n_0)}.(f \circ \varphi^{n_0}) : f \in L^2((J)_{n_0} d\mu)\}$.

PROPOSITION 4.3. *Let $W \in \mathscr{B}(L^2(\Sigma))$. Then $d(W) < \infty$ if and only if W, when restricted to $\mathscr{R}(W^{n_0})$, is onto mapping of $\mathscr{R}(W^{n_0})$ to all of itself for some $n_0 \in \mathbb{N}$.*

Proof. Let $d(W) = n_0 < \infty$ and choose $f \in \mathscr{R}(W^{n_0})$. Since $\mathscr{R}(W^{n_0}) = \mathscr{R}(W^{n_0+1})$, there exists $g \in L^2(\Sigma)$ such that $W(W^{n_0}(g)) = f$ and $W^{n_0}(g) \in \mathscr{R}(W^{n_0})$. This implies that $W : \mathscr{R}(W^{n_0}) \to \mathscr{R}(W^{n_0})$ is onto. Conversely, if for some non-negative integer n_0, $W : \mathscr{R}(W^{n_0}) \to \mathscr{R}(W^{n_0})$ is onto, then $\mathscr{R}(W^{n_0+1}) = W(\mathscr{R}(W^{n_0})) = \mathscr{R}(W^{n_0})$, and thus $d(W) \leqslant n_0 < \infty$. $\square$

In [16] Morrel and Muhly introduced the concept of a centered operator. Let H be the infinite dimensional complex Hilbert space. An operator T on a Hilbert space H is said to be centered if the doubly infinite sequence $\{T^n T^{*n}, T^{*m} T^m : n, m \geqslant 0\}$ consists of mutually commuting operators. Let C_{φ_1} and C_{φ_2} be normal operators and let $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$. By Fuglede-Putnam theorem we have $C_{\varphi_j} C^*_{\varphi_i} = C^*_{\varphi_i} C_{\varphi_j}$. Since normal operators are centered, it follows that $C_{\varphi_3} = C_{\varphi_2} C_{\varphi_1}$ is centered. In [7], Embry-Wardrop and Lambert proved that the composition operator $C_\varphi \in B(L^2(\Sigma))$ is centered if and only if h is Σ_∞-measurable, where $\Sigma_\infty = \cap_{n=1}^{\infty} \Sigma_n$.

PROPOSITION 4.4. *Let $C_\varphi \in \mathscr{B}(L^2(\Sigma))$ and for all $n \in \mathbb{N}$, Σ_n is a sub-sigma finite algebra of Σ. If $d(C_\varphi) = k$ and h is Σ_k-measurable, then C_φ is centered.*

Proof. By hypothesis, $L^2(\Sigma_k) = L^2(\Sigma_n)$ for all $n \geqslant k$. Thus $\Sigma_\infty = \Sigma_k$. Now, the desired conclusion follows from [7, Theorem 5]. $\square$

Let $w = \{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that for all $n \in \mathbb{N}$, $0 < \alpha \leqslant m_n \leqslant \beta$. Set $l^2(w) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. For $\varphi : \mathbb{N} \to \mathbb{N}$, suppose $C_\varphi \in \mathscr{B}(l^2(w))$. In the following we give a characterization of C_φ on $l^2(w)$ whose ascent and descent are infinite.

PROPOSITION 4.5. *Let $C_\varphi \in \mathscr{B}(l^2(w))$. Then the following assertions are hold.*

(a) $\alpha(C_\varphi) = \infty$ if and only if for all $k \in \mathbb{N}$, there exists a sequence of distinct integers $\{n_k\}$ such that $n_k \in \mathscr{R}(\varphi^k)$ but $n_k \notin \mathscr{R}(\varphi^{k+1})$.

(b) $d(C_\varphi) = \infty$ if and only if φ, when restricted to $\mathscr{R}(\varphi^k)$, is not injective for all $k \in \mathbb{N}$.

Proof. (a) Set $X_k = \{n \in \mathbb{N} : (h)_k(n) = 0\}$. Because $(h)_{k+1} = (h)_k(E_k(h)) \circ \varphi^{-1}$, $X_k \subseteq X_{k+1}$ for each $k \in \mathbb{N}$. Since $(h)_k(n) = \frac{1}{m_n} \sum_{j \in \varphi^{-k}(n)} m_j$, $(h)_k(n) = 0$ if and only if $n \notin \mathscr{R}(\varphi^k)$. Thus, $X_k = \{n \in \mathbb{N} : n \notin \mathscr{R}(\varphi^k)\}$. Therefore,

$$\begin{aligned}
\alpha(C_\varphi) = \infty &\Longleftrightarrow L^2(X_k) \subset L^2(X_{k+1}),\ \forall k \in \mathbb{N} \\
&\Longleftrightarrow X_k \subset X_{k+1},\ \forall k \in \mathbb{N} \\
&\Longleftrightarrow \forall k \in \mathbb{N}\ \ \exists n_k \in \mathbb{N} :\ n_k \in \mathscr{R}(\varphi^k) \setminus \mathscr{R}(\varphi^{k+1}).
\end{aligned}$$

Note that $(\mathscr{R}(\varphi^k) \setminus \mathscr{R}(\varphi^{k+1})) \cap (\mathscr{R}(\varphi^{k-1}) \setminus \mathscr{R}(\varphi^k)) = \emptyset$, for all $k \in \mathbb{N}$.

(b) Let $n_0 \in \mathbb{N}$ and $n \in \mathscr{R}(\varphi^{n_0})$. Then $\varphi^{n_0}(p) = n$, for some $p \in \mathbb{N}$. Then

$$(h)_{n_0}(n) = \frac{1}{m_n} \sum_{j \in \varphi^{-n_0}(n)} m_j \geqslant \frac{m_p}{m_n}.$$

Thus $(h)_{n_0} \geqslant \frac{\alpha}{\beta}$ on $\sigma((h)_{n_0}) = \mathscr{R}(\varphi^{n_0})$, and so $C_{\varphi^{n_0}}$ has closed range. First, we show that $\mathscr{R}(C_{\varphi^{n_0}}) = L^2(X_{n_0}^c)$, where $X_{n_0}^c = \sigma((h)_{n_0})$. For this, let $f \in L^2(\Sigma)$. Then $\|C_{\varphi^{n_0}}(f)\|^2 = \|\chi_{X_{n_0}^c} f\|_{(h)_{n_0} d\mu}$. This implies that the mapping $\Lambda(\chi_{X_{n_0}^c} f) = f \circ \varphi^{n_0}$ from $L^2(X_{n_0}^c)$ onto $\mathscr{R}(C_{\varphi^{n_0}}) = \{f \circ \varphi^{n_0} : f \in L^2((h)_{n_0} d\mu)\}$ is an isometry isomorphism. Now, by Proposition 4.3, $d(C_\varphi) = n_0 < \infty$ if and only if $C_\varphi : L^2(X_{n_0}^c) \to L^2(X_{n_0}^c)$ is onto. But, it is a classical fact that $C_\varphi \in \mathscr{B}(L^2(X_{n_0}^c))$ is surjective if and only if $\varphi^{n_0} : X_{n_0}^c \to X_{n_0}^c$ is injective (see [22]). But, $X_{n_0}^c = \{k \in \mathbb{N} : (h)_{n_0}(k) > 0\} = \{k \in \mathbb{N} : k \in \mathscr{R}(\varphi^{n_0})\}$. This completes the proof. $\square$

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