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BOUNDS FOR INDICES OF COINCIDENCE AND ENTROPIES

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Abstract. In this paper we consider a parameterized family of discrete probability distributions and investigate the Rényi,Tsallis, and Shannon entropies associated with them. Lower and upper bounds for these entropies are obtained, improving some results from the literature. The proofs are based on several methods from classical analysis, theory of dual cones, and the stochastic majorization theory. The Rényi and Tsallis entropies are naturally expressed in terms of the index of coincidence. Consequently we study in detail the index of coincidence associated to the corresponding discrete probability distributions. The obtained results lead immediately to properties of the entropies.

1. Introduction

Generalized entropies have been objects of study for many researchers. Rényi entropies and Tsallis entropies are well known as one-parameter generalizations of Shannon entropy. Many applications of them can be found in information theory (secure data transmission, speech coding, cryptography, algorithmic complexity theory) and in physics. This paper is concerned with Rényi, Tsallis, and Shannon entropies associated with a parameterized family of discrete probability distributions. We obtain lower and upper bounds for these entropies, improving some results from the literature. These bounds are obtained using several techniques from classical analysis, theory of dual cones, and the stochastic majorization theory.

The Rényi and Tsallis entropies are naturally expressed in terms of the index of coincidence. Therefore we study in detail the index of coincidence associated to the corresponding discrete probability distributions. It is easy to translate the obtained results in order to get properties of the entropies.

The discrete probability distributions involved in our studies are related to the family of the positive linear operators (depending on a real parameter c) introduced by Baskakov in 1957. Let us describe them explicitly.

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Let I be a real interval and $p_k, k = 0,1,\ldots,$ non-negative continuous functions defined on I, such that $\sum_{k=0}^{\infty} p_k(x) = 1, x \in I$. Using the parameterized probability distribution $p(x) = (p_k(x))_{k\geqslant 0}$ one constructs a positive linear operator as follows:

$$Lf(x) = \sum_{k=0}^{\infty} f(x_k)p_k(x), x \in I, \tag{1}$$

where $x_k \in I$, $k \geqslant 0$, and f is a function defined on I. For example, Bernstein operators, Szász-Mirakjan operators, Baskakov operators are associated with the binomial distribution, the Poisson distribution and the negative binomial distribution, respectively.

The index of coincidence associated with the probability distribution $p(x)$ is

$$S(x) = \sum_{k=0}^{\infty} p_k^2(x), x \in I.$$

The Rényi entropy and the Tsallis entropy of order 2 corresponding to $p(x)$ can be expressed in terms of $S(x)$ as follows:

$$R(x) = -\log S(x);\ T(x) = 1 - S(x), x \in I. \tag{2}$$

Moreover, the classical Shannon entropy is

$$H(x) = -\sum_{k=0}^{\infty} p_k(x)\log p_k(x), x \in I.$$

It is not difficult to prove that

$$T(x) \leqslant R(x) \leqslant H(x), x \in I. \tag{3}$$

Let $c, n \in \mathbb{R}$, $n > c$ for $c \geqslant 0$ and $-\frac{n}{c} \in \mathbb{N}$ for $c < 0$. Denote $I_c = [0,\infty)$ for $c \geqslant 0$ and $I_c = \left[0, -\frac{1}{c}\right]$ for $c < 0$. Consider the basis functions (see [2] and the references therein):

$$p_{n,k}^{[c]}(x) = \begin{cases} \dfrac{n^k}{k!} x^k e^{-nx}, & c = 0, \\ \dfrac{n^{c,\overline{k}}}{k!} x^k (1+cx)^{-\left(\frac{n}{c}+k\right)}, & c \neq 0, \end{cases}$$

where $k \in \mathbb{N}_0$, $x \in I_c$ and $n^{c,\overline{k}} = \Pi_{l=0}^{k-1}(n+cl)$, $n^{c,\overline{0}} = 1$.

In the following we shall be concerned especially with the distribution of probability $(p_{n,k}^{[c]}(x))_{k=0,1,\ldots}$. Remark that for $c = -1,0,1$ we obtain the binomial, Poisson, and negative binomial distributions, respectively. The corresponding indices of coincidence and entropies will be denoted by $S_{n,c}(x)$, $R_{n,c}(x)$, $T_{n,c}(x)$, $H_{n,c}(x)$.

In this paper we present several new lower and upper bounds for these functions. Let us start by recalling some existing results in this direction. Some of them are improved by our new results; see Remarks 1 and 5. Another possible improvement is the object of a problem formulated in Remark 3.

Obviously, each inequality for $S(x)$ will produce inequalities for the entropies $R(x)$, $T(x)$ and $H(x)$. For the sake of brevity we omit the details.

The inequality

$$S_{n,c}(x) \leqslant (4(n+c)x(1+cx)+1)^{-\frac{n}{2(n+c)}} \tag{4}$$

was obtained in [3]. Combined with (2) and (3), it yields

$$\frac{n}{2(n+c)}\log\left(4(n+c)x(1+cx)+1\right) \leqslant R_{n,c}(x) \leqslant H_{n,c}(x). \tag{5}$$

Using (5) with $c=0$, and the upper bound for $H_{n,0}(x)$ (see [1, (1)]) we have

$$\frac{1}{2}\log(4nx+1) \leqslant H_{n,0}(x) \leqslant \frac{1}{2}\log\left(2\pi enx+\frac{\pi e}{6}\right), x \geqslant 0. \tag{6}$$

From [3, (3.12)] we derive

$$\log S_{n,0}(x) \leqslant \frac{1}{2}\log\frac{2}{1+\sqrt{1+16n^2x^2}} + \frac{\sqrt{1+16n^2x^2}-1-4nx}{2}. \tag{7}$$

Using (2), (3) and (6) it follows that

$$\begin{aligned}\frac{1}{2}\log(4nx+1) &\leqslant \frac{1}{2}\log\frac{1+\sqrt{1+16n^2x^2}}{2} + \frac{1+4nx-\sqrt{1+16n^2x^2}}{2}\\ &\leqslant H_{n,0}(x) \leqslant \frac{1}{2}\log\left(2\pi enx+\frac{\pi e}{6}\right).\end{aligned} \tag{8}$$

It is easy to see that $S_{n,-1}\left(\frac{1}{2}\right) = \frac{1}{4^n}\binom{2n}{n}$. From this, (2) and (3), we get

$$H_{n,-1}\left(\frac{1}{2}\right) \geqslant R_{n,-1}\left(\frac{1}{2}\right) = -\log S_{n,-1}\left(\frac{1}{2}\right) = -\log\frac{1}{4^n}\binom{2n}{n}.$$

Since (see [7, p. 519])

$$\frac{1}{\sqrt{\pi(n+3)}} < \frac{1}{4^n}\binom{2n}{n} < \frac{1}{\sqrt{\pi(n-1)}}, \tag{9}$$

we get

$$H_{n,-1}\left(\frac{1}{2}\right) > \frac{1}{2}\log\pi(n-1). \tag{10}$$

REMARK 1. (10) improves an existing lower bound for $H_{n,-1}\left(\frac{1}{2}\right)$. Indeed, from [11, p. 69] (see also [6, Lemma 2.2, p. 686]) it is known that

$$\frac{1}{2}\log\pi\frac{n}{2} \leqslant H_{n,-1}\left(\frac{1}{2}\right) \leqslant \frac{1}{2}\log\pi e\frac{n}{2}. \tag{11}$$

Since $\pi\frac{n}{2} < \pi(n-1), \forall n > 2$, it follows that the lower bound of $H_{n,-1}\left(\frac{1}{2}\right)$ from (10) is better than the lower bound from (11).

REMARK 2. From (6) it follows

$$\frac{\frac{1}{2}\log(4nx+1)}{\log n} \leqslant \frac{H_{n,0}(x)}{\log n} \leqslant \frac{\frac{1}{2}\log\left(2\pi enx+\frac{\pi e}{6}\right)}{\log n}.$$

Therefore, $\lim\limits_{n\to\infty}\dfrac{H_{n,0}(x)}{\log n}=\dfrac{1}{2}$ and $H_{n,0}(x)\sim\dfrac{1}{2}\log n$ (see also [11, p. 69]).

2. Bounds for entropies using Cauchy-Schwarz inequality

Consider the positive linear operator L from (1) and let f be positive. Using Cauchy-Schwarz inequality we get

$$(Lf(x))^2 \leqslant \sum_{k=0}^{\infty} f^2(x_k)\sum_{k=0}^{\infty} p_k^2(x),$$

and consequently

$$\sum_{k=0}^{\infty} p_k^2(x) \geqslant \frac{(Lf(x))^2}{\sum\limits_{k=0}^{\infty} f^2(x_k)}. \tag{12}$$

We shall be concerned with the Baskakov operator

$$B_n^{[c]}f(x)=\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)f\left(\frac{k}{n}\right).$$

Case 1. $c=0$.
Let $f(x)=e^{\lambda x}$, $x\geqslant 0$, for a given $\lambda<0$. Then

$$B_n^{[c]}f(x)=e^{nx\left(e^{\frac{\lambda}{n}}-1\right)} \text{ and } \sum_{k=0}^{\infty} f^2(x_k)=\frac{1}{1-e^{\frac{2\lambda}{n}}}.$$

From (12) it follows that

$$S_{n,0}(x)\geqslant e^{2nx\left(e^{\frac{\lambda}{n}}-1\right)}\left(1-e^{\frac{2\lambda}{n}}\right).$$

Since $\max\limits_{\lambda<0} e^{2nx\left(e^{\frac{\lambda}{n}}-1\right)}\left(1-e^{\frac{2\lambda}{n}}\right)=\dfrac{2e^{\sqrt{1+4n^2x^2}-2nx-1}}{\sqrt{1+4n^2x^2}+1}$, we get

$$\frac{2e^{\sqrt{1+4n^2x^2}-2nx-1}}{\sqrt{1+4n^2x^2}+1}\leqslant S_{n,0}(x).$$

Taking into account [3, (3.12)] and [3, (1.3)], we arrive at

THEOREM 1. *For $n \geqslant 1$ and $x \geqslant 0$, the following inequalities are satisfied:*

$$\frac{2e^{\sqrt{1+4n^2x^2}-2nx-1}}{\sqrt{1+4n^2x^2}+1} \leqslant S_{n,0}(x) \leqslant \left(\frac{2e^{\sqrt{1+16n^2x^2}-4nx-1}}{\sqrt{1+16n^2x^2}+1}\right)^{\frac{1}{2}} \leqslant (4nx+1)^{-\frac{1}{2}}.$$

Case 2. $c > 0$.
With the same function and the same method as in the case $c = 0$, we obtain for $c = 1$

$$S_{n,1}(x) \geqslant \frac{2}{r+x+1}\left(\frac{(2n-1)(x+1)+r}{2n(1+2x)}\right)^{2n-1},$$

where $r = \sqrt{(x+1)^2+4n(n-1)x^2}$.

The inequality is sharp for $x = 0$ and for $x \to \infty$.

A similar lower bound for $S_{n,c}(x)$ can be obtained for arbitrary $c > 0$.

Case 3. $c < 0$.
We treat only the case $c = -1$, when $B_n^{[c]}$ becomes the classical Bernstein operator B_n; the general case can be treated similarly.

Using (12) with $f(x) = e^{\lambda x}$, $x \in [0,1]$, $\lambda \in \mathbb{R}$, we get

$$B_n f(x) = \left(1-x+xe^{\frac{\lambda}{n}}\right)^n, \qquad \sum_{k=0}^{n} f^2\left(\frac{k}{n}\right) = \frac{1-e^{\frac{2\lambda}{n}(n+1)}}{1-e^{\frac{2\lambda}{n}}}.$$

Therefore,

$$S_{n,-1}(x) \geqslant \left(1-x+xe^{\frac{\lambda}{n}}\right)^{2n} \frac{1-e^{\frac{2\lambda}{n}}}{1-e^{\frac{2\lambda}{n}(n+1)}}.$$

Denote $t := e^{\frac{\lambda}{n}} \in (0,\infty)$. Then

$$\begin{aligned} S_{n,-1}(x) &\geqslant \sup_{t>0} (1-x+xt)^{2n} \frac{1-t^2}{1-t^{2(n+1)}} \\ &\geqslant \left(1-x+xe^{\frac{\lambda}{n}}\right)^{2n} \frac{1-e^{\frac{2\lambda}{n}}}{1-e^{\frac{2\lambda}{n}(n+1)}}\Bigg|_{t=\frac{x}{1-x}}. \end{aligned}$$

Finally,

$$S_{n,-1} \geqslant (1-2x)\frac{(1-2x(1-x))^{2n}}{(1-x)^{2n+2}-x^{2n+2}}. \tag{13}$$

REMARK 3. Another lower bound for $S_{n,-1}(x)$ will be provided in Theorem 2. Graphical experiments seem to indicate that the lower bound in Theorem 2 is better than that given in (13). It would be useful to have a formal proof of this fact.

3. Bounds for entropies using integral representation

Using the integral representation of $S_{n,c}$ (see [5, Theorem 1], [16, Theorem 3]) we get

$$\begin{aligned} S_{n,c}(x) &= \frac{1}{\pi}\int_0^1 \left[t+(1-t)(1+2cx)^2\right]^{-\frac{n}{c}} \frac{dt}{\sqrt{t(1-t)}} \\ &\geqslant \frac{2}{\pi}\int_0^1 \left[t+(1-t)(1+2cx)^2\right]^{-\frac{n}{c}} dt \\ &= \frac{1}{2\pi}\,\frac{1-\left[(1+2cx)^2\right]^{1-\frac{n}{c}}}{x(cx+1)(n-c)}, \text{ for } c\neq 0, \\ S_{n,0}(x) &= \frac{1}{\pi}\int_{-1}^1 e^{-2nx(1+t)}\frac{dt}{\sqrt{1-t^2}} \geqslant \frac{1-e^{-4xn}}{2\pi xn}. \end{aligned}$$

For $c=-1$ it follows

$$S_{n,-1}(x) \geqslant \frac{1-(1-2x)^{2n+2}}{2\pi(n+1)x(1-x)}. \tag{14}$$

In the next sections the lower bound of $S_{n,-1}$ given in (14) will be improved as follows:

$$S_{n,-1}(x) \geqslant \frac{1-(1-2x)^{2n+2}}{4(n+1)x(1-x)} := A(x;n). \tag{15}$$

REMARK 4. In [3] the following lower and upper bounds for $S_{n,-1}(x)$ were obtained:

$$B(x;n) := [1+(n-3)x(1-x)]^{-\frac{2n}{n-3}} \leqslant S_{n,-1}(x) \leqslant \frac{1}{\sqrt{1+4(n-1)x(1-x)}}. \tag{16}$$

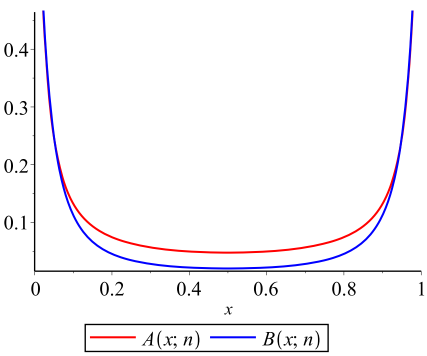

Figure 1: *Lower bound for* $S_{n,-1}$ *;* $n=20$

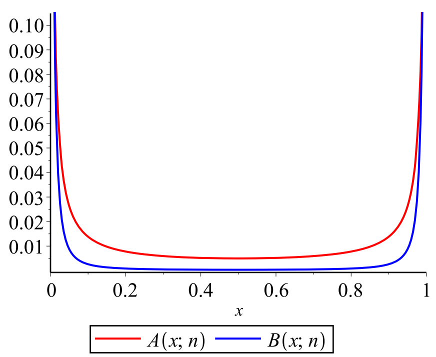

Figure 2: *Lower bound for* $S_{n,-1}$ *;* $n=200$

The lower bound $B(x;n)$ is improved for certain intervals by (15). In Figures 1-2 we illustrated the graphs of the lower bounds $A(x;n)$ and $B(x;n)$ for $n=20$ and $n=200$, respectively.

In the next two sections we present two proofs of (15) using theory of dual cones and the stochastic majorization technique, respectively.

4. Bounds for entropies using theory of dual cones

The purpose of this section is to prove the inequality (15). In order to present this result we recall the following lemma (see [10, Lemma 7.2, Chapter XI]):

LEMMA 1. [10] *Let* $\{a_0, a_1, \ldots, a_n\}$ *be a finite sequence. If*

$$\sum_{k=0}^{n} a_k = 0 \text{ and } \sum_{k=0}^{n} k a_k = 0$$

hold, and the signatures of $a_0, \ldots, a_n$ *are* $+, -, +$ *in the sense that*

$$\begin{aligned} &a_k \geqslant 0 \text{ for } 0 \leqslant k \leqslant k_1, \\ &a_k \leqslant 0 \text{ for } k_1 + 1 \leqslant k \leqslant k_2, \\ &a_k \geqslant 0 \text{ for } k_2 + 1 \leqslant k \leqslant n, \end{aligned}$$

for some k_1, k_2, *with strict inequality holding at least once in each of the three indicated regions, then* $\sum_{k=0}^{n} a_k \varphi(k) \geqslant 0$ *for all convex sequences* $\varphi(k)$, $k = 0, 1, \ldots, n$.

Let $a_{n,j} := 4^{-n}\binom{2j}{j}\binom{2n-2j}{n-j}$, $j = 0, 1, \ldots, n$. Remark that

$$a_{n,n-j} = a_{n,j}, \ j = 0, 1, \ldots, n. \tag{17}$$

Then (see [9, (3.90)],

$$\sum_{j=0}^{n} a_{n,j} = 1, \quad \sum_{j=0}^{n} \left(a_{n,j} - \frac{1}{n+1} \right) = 0, \tag{18}$$

$$\begin{aligned} \sum_{j=0}^{n} j a_{n,j} &= 4^{-n} \sum_{j=0}^{n} j \binom{2j}{j} \binom{2n-2j}{n-j} \\ &= 4^{-n} \sum_{j=0}^{n} (n-j) \binom{2n-2j}{n-j} \binom{2j}{j} = n - \sum_{j=0}^{n} j a_{n,j}. \end{aligned} \tag{19}$$

Therefore, $\sum_{j=0}^{n} j a_{n,j} = \frac{n}{2}$, and now

$$\sum_{j=0}^{n} j \left(a_{n,j} - \frac{1}{n+1} \right) = 0. \tag{20}$$

But $\frac{a_{n,j+1}}{a_{n,j}} = \frac{2j+1}{j+1} \cdot \frac{n-j}{2n-2j-1}$, and consequently

$$a_{n,j+1} \leqslant a_{n,j} \text{ iff } 0 \leqslant j \leqslant \frac{n+1}{2}. \tag{21}$$

So the sequence $\left(a_{n,j}-\frac{1}{n+1}\right)_{j=0,1,\ldots,n}$ is decreasing for $j\in\left\{0,1,\ldots,\frac{n+1}{2}\right\}$ and increasing for $j\in\left\{\frac{n+1}{2},\ldots,n\right\}$. Since $\sum_{j=0}^{n}\left(a_{n,j}-\frac{1}{n+1}\right)=0$, there exist k_1,k_2 such that

$$\begin{aligned}
a_{n,k}-\frac{1}{n+1} &\geqslant 0 \text{ for } 0\leqslant k\leqslant k_1,\\
a_{n,k}-\frac{1}{n+1} &\leqslant 0 \text{ for } k_1+1\leqslant k\leqslant k_2,\\
a_{n,k}-\frac{1}{n+1} &\geqslant 0 \text{ for } k_2+1\leqslant k\leqslant n,
\end{aligned}$$

with strict inequality holding at least once in each of the three indicated regions. Together with (18) and (20), this shows that the assumptions of [10, Lemma 7.2, Chapter XI] are satisfied. Consequently,

$$\sum_{j=0}^{n}\left(a_{n,j}-\frac{1}{n+1}\right)\varphi(j)\geqslant 0, \tag{22}$$

for every convex sequence $\varphi(j)$, $j=0,1,\ldots,n$.

Let $x\in[0,1]$ be given. Then the sequence $\varphi(j)=(1-2x)^{2j}$, $j=0,1,\ldots,n$, is convex, and so (22) shows that

$$\sum_{j=0}^{n}\left(a_{n,j}-\frac{1}{n+1}\right)(1-2x)^{2j}\geqslant 0.$$

According to [4, (25)], [8],

$$\sum_{j=0}^{n}a_{n,j}(1-2x)^{2j}=S_{n,-1}(x). \tag{23}$$

So we have proved

THEOREM 2. *For $n\geqslant 1$ and $x\in[0,1]$ one has*

$$S_{n,-1}(x)\geqslant\frac{1}{n+1}\sum_{j=0}^{n}(1-2x)^{2j}=\frac{1}{n+1}\frac{1-(1-2x)^{2n+2}}{4x(1-x)}.$$

5. Bounds for entropies using stochastic majorization technique

In this section we present the proof of the inequality (15) using the stochastic majorization technique. To this aim we recall Ohlin's lemma (see [12], [14], [15]):

LEMMA 2. [12] *Let X, Y be two random variables such that $EX=EY$. If the distribution functions F_X, F_Y cross exactly one time, i.e., for some x_0 holds*

$$F_X(x)\leqslant F_Y(x) \text{ if } x<x_0 \text{ and } F_X(x)\geqslant F_Y(x) \text{ if } x>x_0,$$

then $\mathbb{E}f(X)\leqslant\mathbb{E}f(Y)$, for all convex functions $f:\mathbb{R}\to\mathbb{R}$.

For a given integer $n \geqslant 1$ let X and Y be random variables such that

$$\mathscr{P}(X=j)=\frac{1}{n+1},\ \mathscr{P}(Y=j)=a_{n,j},\ j=0,1,\ldots,n.$$

Then, according to (20),

$$\mathbb{E}X=\mathbb{E}Y. \tag{24}$$

Moreover, (17) and (21) show that the decreasing rearrangement of $(a_{n,0},a_{n,1},\ldots,a_{n,n})$ is

$$y:=(a_{n,0},a_{n,0},a_{n,1},a_{n,1},\ldots).$$

(For the terminology see [13]).

Let P be the bistochastic matrix $P:=(p_{ij})_{i,j=0,1,\ldots,n}$, $p_{i,j}=\dfrac{1}{n+1}$. According to [13, Th. A.4, p. 31], $yP \prec y$, where $\prec$ is the majorization ordering. Since

$$yP=\left(\frac{1}{n+1},\ldots,\frac{1}{n+1}\right)\in\mathbb{R}^{n+1},$$

we have

$$\left(\frac{1}{n+1},\frac{1}{n+1},\ldots,\frac{1}{n+1}\right)\prec(a_{n,0},a_{n,0},a_{n,1},a_{n,1},\ldots). \tag{25}$$

In the sequel we investigate the case $n=2m+1$; the proofs are similar when $n=2m$. From (25) we deduce

$$\frac{1}{2m+2}\leqslant a_{2m+1,0}, \tag{26}$$

$$\frac{2}{2m+2}\leqslant a_{2m+1,0}+a_{2m+1,1}, \tag{27}$$

$$\ldots$$

$$\frac{m+1}{2m+2}\leqslant a_{2m+1,0}+\cdots+a_{2m+1,m}. \tag{28}$$

In fact, the precise form of (28) is

$$\frac{m+1}{2m+2}=a_{2m+1,0}+a_{2m+1,1}+\ldots+a_{2m+1,m}. \tag{29}$$

Using increasing rearrangements we get similarly

$$\left(\frac{1}{2m+2},\ldots,\frac{1}{2m+2}\right)\succ(a_{2m+1,m},a_{2m+1,m},\ldots,a_{2m+1,0},a_{2m+1,0}). \tag{30}$$

The relation (30) yields successively

$$\frac{1}{2m+2}\geqslant a_{2m+1,m};\ \frac{2}{2m+2}\geqslant a_{2m+1,m}+a_{2m+1,m-1}$$

$$\frac{m+1}{2m+2}\geqslant a_{2m+1,m}+\ldots+a_{2m+1,0}.$$

Using (17) we get

$$\frac{1}{2m+2} \geqslant a_{2m+1,m+1}, \tag{31}$$

$$\frac{2}{2m+2} \geqslant a_{2m+1,m+1} + a_{2m+1,m+2}, \tag{32}$$

$$\dots$$

$$\frac{m+1}{2m+2} \geqslant a_{2m+1,m+1} + \dots + a_{2m+1,2m+1}. \tag{33}$$

Adding (29) to (31)–(33) one obtains

$$\frac{k}{2m+2} \geqslant \sum_{i=0}^{k-1} a_{2m+1,i},\ k = m+2,\dots,2m+2. \tag{34}$$

Let $F_X(x)$ be the distribution function of X. The relations (26)–(28) and (34) show that

$$F_X(x) \leqslant F_Y(x),\ x \leqslant \frac{n}{2};\ F_X(x) \geqslant F_Y(x),\ x > \frac{n}{2}. \tag{35}$$

As mentioned above, the inequalities (35) can be proved similarly when $n = 2m$.

Now (24) and (35) show that Ohlin's lemma can be applied, and we obtain

$$\mathbb{E}\varphi(X) \leqslant \mathbb{E}\varphi(Y),\ \varphi : \mathbb{R} \to \mathbb{R} \text{ convex.}$$

Explicitly this means that

$$\frac{1}{n+1}\sum_{j=0}^{n} \varphi(j) \leqslant \sum_{j=0}^{n} a_{n,j}\varphi(j),\ \varphi \text{ convex.}$$

For a given $x \in [0,1]$, let $\varphi(t) := (1-2x)^{2t}$. Then

$$\frac{1}{n+1}\sum_{j=0}^{n}(1-2x)^{2j} \leqslant \sum_{j=0}^{n} a_{n,j}(1-2x)^{2j}.$$

Now (23) shows that

$$\frac{1}{n+1}\sum_{j=0}^{n}(1-2x)^{2j} \leqslant S_{n,-1}(x),\ x \in [0,1],$$

and the proof of (15) is finished.

6. The Shannon entropy

In this section we are concerned with the Shannon entropy corresponding to $p_{n,k}^{[c]}$.

Case 1. For $c \geqslant 0$ the Shannon entropy can be expressed as follows:

$$H_{n,c}(x) := -\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x)\log p_{n,k}^{[c]}(x),\ x \geqslant 0,\ H_{n,c}(0) = 0.$$

THEOREM 3. *The following inequalities hold for all $c \geqslant 0,\ x \geqslant 0$:*

$$\frac{n}{2(n+c)}\log\left[4(n+c)x(1+cx)+1\right] \leqslant H_{n,c}(x) \leqslant (nx+1)\log(nx+1) - nx\log(nx).$$

Proof. Since $\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1$, we have

$$H'_{n,c}(x) = -\sum_{k=0}^{\infty}\left(p_{n,k}^{[c]}\right)'(x)\log p_{n,k}^{[c]}(x).$$

On the other hand (see [2, (5)]),

$$\left(p_{n,k}^{[c]}\right)'(x) = n\left(p_{n+c,k-1}^{[c]}(x) - p_{n+c,k}^{[c]}(x)\right),\ p_{n+c,k-1}^{[c]}(x) \equiv 0 \text{ for } k=0,$$

and so

$$H'_{n,c}(x) = n\left[\sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x)\log p_{n,k}^{[c]}(x) - \sum_{k=1}^{\infty} p_{n+c,k-1}^{[c]}(x)\log p_{n,k}^{[c]}(x)\right] \tag{36}$$

$$= n\sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x)\log\frac{p_{n,k}^{[c]}(x)}{p_{n,k+1}^{[c]}(x)}. \tag{37}$$

Finally,

$$H'_{n,c}(x) = n\left(\log\frac{1+cx}{x} + \sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x)\log\frac{k+1}{n+ck}\right).$$

By Jensen's inequality for the concave function $\log$ we have

$$\sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x)\log\frac{k+1}{n+ck} \leqslant \log\sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x)\frac{k+1}{n+ck} = \log\frac{1+nx}{n(1+cx)}.$$

In [17] it was proved that $H'_{n,c}(x) \geqslant 0$. Then

$$0 \leqslant H'_{n,c}(x) \leqslant n\log\frac{x+\frac{1}{n}}{1+cx} - n\log\frac{x}{1+cx},$$

and

$$0 \leqslant \int_0^t H'_{n,c}(x)dx \leqslant (nt+1)\log(nt+1) - nt\log(nt).$$

Therefore,

$$0 \leqslant H_{n,c}(t) \leqslant (nt+1)\log(nt+1) - nt\log(nt). \tag{38}$$

Now (5) and (38) complete the proof. □

REMARK 5. According to [1, (1)], the best known upper bound on $H_{n,0}$ is perhaps that one presented in (6). Theorem 3 gives a partial improvement of this bound. Indeed, let $K_1(x;n) = \frac{1}{2}\log\left(2\pi enx + \frac{\pi e}{6}\right)$ and $K_2(x;n) = (nx+1)\log(nx+1) - nx\log(nx)$ be the upper bounds of Shannon entropy $H_{n,0}$ given in (6) and (38). In Figure 3 we compare these two upper bounds of Shannon entropy. Note that on the first interval the upper bound of $H_{n,0}$ obtained in this section improves the result presented in [1, (1)].

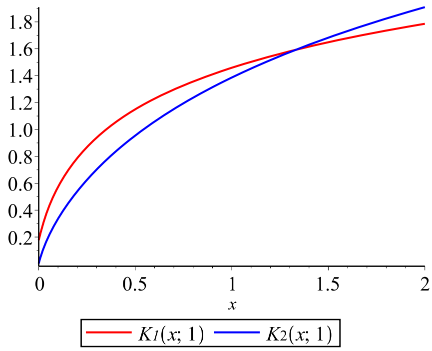

Figure 3: *Upper bound of entropies $H_{n,0}$*

Case 2. For $c = -1$ the Shannon entropy can be expressed as follows:

$$\begin{aligned} H_{n,-1}(x) &:= -\sum_{k=0}^{n} b_{n,k}(x)\log b_{n,k}(x) \\ &= -n\left(x\log x + (1-x)\log(1-x)\right) - \sum_{k=0}^{n} b_{n,k}(x)\log\binom{n}{k}. \end{aligned}$$

THEOREM 4. *The Shannon entropy $H_{n,-1}(x)$ admits the following bounds*

$$\begin{aligned} &\frac{n}{2(n-1)}\log\left(4(n-1)x(1-x)+1\right) \leqslant H_{n,-1}(x) \\ &\leqslant \begin{cases} (nt+1)\log(nt+1) - nt\log(nt), \ 0 < t \leqslant \frac{1}{2} \\ (n(1-t)+1)\log(n(1-t)+1) - n(1-t)\log n(1-t), \ \frac{1}{2} \leqslant t < 1. \end{cases} \end{aligned}$$

Proof. Since $H'_{n,-1}(x) = -n\log\frac{x}{1-x} + n\sum_{k=0}^{n-1} b_{n-1,k}(x)\log\frac{k+1}{n-k}$ and

$$\begin{aligned} \sum_{k=0}^{n-1} b_{n-1,k}(x)\log\frac{k+1}{n-k} &< \log\left(\frac{x}{1-x} + \frac{1-(n+1)x^n}{n(1-x)}\right) \\ &< \log\left(\frac{x}{1-x} + \frac{1}{n(1-x)}\right) = \log\frac{nx+1}{n(1-x)} \end{aligned}$$

we get

$$H'_{n,-1}(x) < n\left(\log\frac{nx+1}{n(1-x)} - \log\frac{x}{1-x}\right) = n\left(\log(nx+1) - \log n - \log x\right),$$

and

$$\int_0^t H'_{n,-1}(x)dx \leqslant (nt+1)\log(nt+1) - nt\log(nt),\ 0<t<\frac{1}{2}.$$

But $H_{n,-1}(t) = H_{n,-1}(1-t)$. Combined with (5), this concludes the proof. □

7. The integral of $S_{n,c}$

Case 1. For $c=0$ we have the following representation of $S_{n,0}$ (see [16, Theorem 3]):

$$S_{n,0}(x) = e^{-2nx}\sum_{k=0}^{\infty}\frac{(nx)^{2k}}{(k!)^2},\ x\in[0,\infty).$$

Since $\int_0^\infty e^{-2nx}x^{2k}dx = \frac{(2k)!}{(2n)^{2k+1}}$, we get

$$\int_0^\infty S_{n,0}(x)dx = \sum_{k=0}^{\infty}\frac{n^{2k}}{(k!)^2}\frac{(2k)!}{2^{2k+1}n^{2k+1}} = \frac{1}{2n}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{4^k}.$$

Using (9) and the fact that $\sum_{k=0}^{\infty}\frac{1}{\sqrt{\pi(k+3)}}$ is divergent, it follows that $\int_0^\infty S_{n,0}(x)dx$ is divergent.

Case 2. For $c>0$, $S_{n,c}$ can be represented as (see [16, Theorem 3]):

$$S_{n,c}(x) = (1+cx)^{-\frac{2n}{c}}\sum_{k=0}^{\infty}\left(\frac{n(n+c)\cdots(n+(k-1)c)}{k!}\right)^2\left(\frac{x}{1+cx}\right)^{2k},\ x\in[0,\infty).$$

Therefore,

$$\int_0^\infty S_{n,c}(x)dx = \sum_{k=0}^{\infty}\left(\frac{n(n+c)\cdots(n+(k-1)c)}{k!}\right)^2\int_0^\infty(1+cx)^{-\frac{2n}{c}-2k}x^{2k}dx = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{4^k}E_n(k,c),$$

where

$$E_n(k,c) = \frac{(2n)^{2c,\overline{k}}}{(2n-c)^{2c,\overline{k+1}}}$$

It is elementary to prove that

$$(E_n(k,c))^2 > \frac{1}{2n-c}\cdot\frac{1}{2n+(2k-1)c}.$$

Consequently,

$$\int_0^\infty S_{n,c}(x)dx > \sum_{k=0}^\infty \frac{1}{2\sqrt{k+1}} \cdot \frac{1}{\sqrt{2n-c}} \cdot \frac{1}{\sqrt{2n+(2k-1)c}}$$

and $\int_0^\infty S_{n,c}(x)dx$ is a divergent integral.

Case 3. For $c<0$ we have the following representation of $S_{n,c}$ (see [16, Theorem 3]):

$$S_{n,c}(x) = (1+cx)^{-\frac{2n}{c}} \sum_{k=0}^{-\frac{n}{c}} \left(\frac{n(n+c)\cdots(n+(k-1)c)}{k!} \right)^2 \left(\frac{x}{1+cx} \right)^{2k}, \, x \in \left[0, -\frac{1}{c}\right],$$

$n=-cl, l \in \mathbb{N} \setminus \{0\}$.

It is not difficult to prove that

$$\int_0^{-\frac{1}{c}} S_{n,c}(x)dx = \frac{l}{n} \cdot \frac{1}{2l+1} \cdot \frac{1}{\binom{2l}{l}} \sum_{k=0}^{l} \binom{2k}{k}\binom{2l-2k}{l-k}.$$

From [9, (3.90)] (see also (18)) we have $\sum_{k=0}^{l} \binom{2k}{k}\binom{2l-2k}{l-k} = 4^l$. Using relation (9) we get

$$\int_0^{-\frac{1}{c}} S_{n,c}(x)dx > \frac{l}{n} \cdot \frac{1}{\sqrt{2l+1}}.$$

Also, using relation (9) we get

$$\int_0^{-\frac{1}{c}} S_{n,c}(x)dx < \frac{l}{n} \cdot \frac{1}{2l+1} \cdot 2\sqrt{l}.$$

Choosing $l = -\frac{n}{c}$ we get the following bounds for the integral of $S_{n,c}$:

$$\frac{1}{\sqrt{c^2-2nc}} < \int_0^{-\frac{1}{c}} S_{n,c}(x)dx < \frac{2}{2n-c}\sqrt{-\frac{n}{c}}.$$

REFERENCES

[1] J. A. ADELL, A. LEKUONA, Y. YU, *Sharp Bounds on the Entropy of the Poisson Law and Related Quantities*, IEEE Transactions on Information Theory **56**, 5 (2010), 2299–2306.

[2] K. BAUMANN, M. HEILMANN, I. RASA, *Further Results for kth order Kantorovich Modification of Linking Baskakov type operators*, Results. Math. **69** (2016), 297–315.

[3] A. BĂRAR, G. R. MOCANU, I. RAŞA, *Bounds for some entropies and special functions*, Carpathian J. Math. **34**, 1 (2018), 9–15.

[4] A. BĂRAR, G. MOCANU, I. RASA, *Heun functions related to entropies*, RACSAM **113** (2019), 819–830.

[5] E. BERDYSHEVA, *Studying Baskakov-Durrmeyer operators and quasi-interpolants via special functions*, J. Approx. Theory **149** (2007), 131–150.

[6] S. CHANG AND E. WELDON, *Coding for T-user multiple-access channels*, IEEE Trans. Information Theory **25** (1979), 684–691.
[7] C. ELSNER, M. PRÉVOST, *Expansion of Euler's constant in terms of Zeta numbers*, J. Math. Anal. Appl. **398** (2013), 508–526.
[8] I. GAVREA, M. IVAN, *On a conjecture concerning the sum of the squared Bernstein polynomials*, Appl. Math. Comput. **241** (2014), 70–74.
[9] H. W. GOULD, *Combinatorical Identities – A standardized set of tables listing 500 binomial coefficient summations*, Morgantown, 1972.
[10] S. KARLIN, W. J. STUDDEN, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Interscience, New York, 1966.
[11] C. KNESSEL, *Integral Representations and Asymptotic Expansions for Shannon and Renyi Entropies*, Appl. Math. Lett. **11**, 2 (1998), 69–74.
[12] J. OHLIN, *On a class of measures of dispersion with application to optimal reinsurance*, Astin Bull. **5**, 2 (1969), 249–266.
[13] A. W. MARSHALL, I. OLKIN, C. A. BARRY, *Inequalities: Theory of Majorization and Its Applications*, Springer, New York, 2011.
[14] J. MROWIEC, T. RAJBA, S. WASOWICZ, *A solution to the problem of Raşa connected with Bernstein polynomials*, J. Math. Anal. Appl. **446** (2017), 864–878.
[15] T. RAJBA, *On some recent applications of stochastic convex ordering theorems to some functional inequalities for convex functions: A Survey Developments in Functional Equations and Related Topics*, (J. Brzdek, K. Cieplinski, and T. Rassias (Eds.)), Springer Optim. Appl. **124** (2017), 231–274.
[16] I. RAŞA, *Entropies and Heun functions associated with positive linear operators*, Applied Mathematics and Computation **268** (2015), 422–431.
[17] I. RAŞA, *Complete monotonicity of some entropies*, Period Math Hung **17** (2017), 159–166.

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