Mathematical
Inequalities
& Applications
Volume 24, Number 2 (2021), 337–350 doi:10.7153/mia-2021-24-24

NEW EQUIAFFINE CHARACTERIZATIONS OF THE ELLIPSOIDS RELATED TO AN EQUIAFFINE INTEGRAL INEQUALITY ON HYPEROVALOIDS

ZEJUN HU AND CHENG XING*

(*Communicated by M. A. Hernández Cifre*)

Abstract. In this paper, we study hyperovaloids from the perspective of the equiaffine differential geometry. As the main result, we establish an optimal integral inequality of the hyperovaloids in terms of the normalized affine scalar curvature and the squared norm of the equiaffine Weingarten form. Since the integral inequality becomes an integral equality if and only if the hyperovaloids are equiaffinely equivalent to the ellipsoids, our results give new equiaffine characterizations of the ellipsoids.

1. Introduction

Let $\mathbf{R}^{n+1}$ be the $(n+1)$-dimensional real unimodular-affine space equipped with its canonical flat connection and a parallel volume form. A *hyperovaloid* as usual is a locally strongly convex hypersurface immersion $x : M^n \to \mathbf{R}^{n+1}$, where M^n is an n-dimensional connected compact smooth manifold without boundary. As a special case of the hyperovaloids, the ellipsoids possess remarkable properties. The study of the hyperovaloids has always been one of the most interesting subjects in convex geometry and analysis, Euclidean differential geometry, and particularly in affine differential geometry. Amongst which various characterizations of the ellipsoids have been established, for some of them see e.g. [2, 16, 18, 22, 24, 25, 32, 31, 33, 36] and most recently [9, 10, 17, 19].

In this paper, we shall study hyperovaloids from the perspective of the equiaffine differential geometry. For a hyperovaloid $x : M^n \to \mathbf{R}^{n+1}$, we denote by G the (Blaschke-Berwald) affine metric and B the equiaffine Weingarten operator. Then the main result of this paper, which is an optimal equiaffine integral inequality on the hyperovaloids, can be stated as the following

Mathematics subject classification (2010): 53A15, 53C20, 53C24.
Keywords and phrases: Ellipsoid, hyperovaloid, integral inequality, equiaffine characterization, rigidity theorem, semi-parallel cubic form.
The authors were supported by NSF of China, Grant Number 11771404.
* Corresponding author.

THEOREM 1.1. *Let $x : M^n \to \mathbf{R}^{n+1}$ $(n \geqslant 2)$ be a hyperovaloid with affine metric G and affine Weingarten operator B. Then the normalized scalar curvature χ of the metric G and the G-norm $\|B\|_G$ of B satisfy the following integral inequality*

$$\int_{M^n} \chi^2 \, dV_G \leqslant \tfrac{1}{n} \int_{M^n} \|B\|_G^2 \, dV_G, \tag{1.1}$$

where dV_G denotes the volume element of the affine metric G of M^n.

Moreover, the equality in (1.1) holds if and only if $x(M^n)$ is an ellipsoid.

As a direct consequence of Theorem 1.1, we have

COROLLARY 1.1. *Let $x : M^n \to \mathbf{R}^{n+1}$ $(n \geqslant 2)$ be a hyperovaloid with affine metric G and affine Weingarten operator B. If χ and $\|B\|_G$ satisfy the inequality*

$$\chi^2 \geqslant \tfrac{1}{n} \|B\|_G^2 \tag{1.2}$$

identically on M^n, then $x(M^n)$ is an ellipsoid.

REMARK 1.1.

(1) Theorem 1.1 and Corollary 1.1 are interesting in that they provide both global and pointwise new characterizations of the ellipsoids, respectively, in terms of the equiaffine invariants of the hyperovaloids in $\mathbf{R}^{n+1}$.

(2) In Section 5, we will prove a little stronger results, Theorem 5.1 and Corollary 5.1, by which Theorem 1.1 and Corollary 1.1 become direct consequences.

Towards the above results, for better understanding of the motivations we collect some background materials below. Historically, since the beginning of equiaffine differential geometry, characterizing ellipsoids from the hyperovaloids in $\mathbf{R}^{n+1}$ has been an interesting problem. For a hyperovaloid $x : M^n \to \mathbf{R}^{n+1}$, let G be the (Blaschke-Berwald) affine metric and B be the equiaffine Weingarten operator. By definition $L_1 := \frac{1}{n}\mathrm{trace}(B)$ is called the affine mean curvature. Then the classical theorem of Blaschke for $n = 2$ and Deicke for $n \geqslant 3$ implies that a hyperovaloid $x : M^n \to \mathbf{R}^{n+1}$ is an ellipsoid if and only if $B = L_1 \cdot \mathrm{Id}$ (cf. [5] and Theorem 3.35 in [20]). In terms of the affine intrinsic invariants, Schneider [30] solved a conjecture made by Blaschke [5], which states that an ovaloid in $\mathbf{R}^3$ with constant (affine) scalar curvature is an ellipsoid. Schneider's result was further generalized by Kozlowski and Simon [18] to be that a hyperovaloid in $\mathbf{R}^{n+1}$ with Einstein affine metric must be an ellipsoid. On the other hand, in terms of the extrinsic affine invariants, by using the affine Minkowski-type integral formulas (cf. [33]), it is proved that a hyperovaloid with constant affine mean curvature L_1 is an ellipsoid (cf. Theorem 4.2 in [20]). This was further generalized, after several partial results of Süss [36], Simon [33] and Hsiung-Shahin [16], finally by A.-M. Li [22] showing that a hyperovaloid with constant higher affine r-th mean curvature L_r $(r \geqslant 2)$ must be an ellipsoid (cf. Theorem 4.5 in [20]). Moreover, it was also shown that a hyperovaloid in $\mathbf{R}^{n+1}$ satisfying $L_r = \sum_{i=1}^{r-1} a_i L_i$ with constant $a_i \geqslant 0$ and $2 \leqslant i \leqslant n$ must be an ellipsoid (cf. Theorem 4.6 in [20] and the final version by Alias and Colares [2]).

Different from the preceding results under pointwise conditions, in following we shall further review some equiaffine integral (global) inequalities on hyperovaloids in $\mathbf{R}^{n+1}$ that having the ellipsoids as the extremum case.

First, for a hyperovaloid $x : M^n \to \mathbf{R}^{n+1}$, let $S(M)$ and $\mathrm{Vol}(M)$ denote the total affine area of M^n with respect to the affine metric and the $(n+1)$-dimensional volume of the convex body in $\mathbf{E}^{n+1}$ bounded by $x(M^n)$ with respect to the Euclidean induced metric, respectively. Then, according to Blaschke [4, 5], Santaló [29] and Deicke [12], we have the classical affine isoperimetric inequality (cf. also Chapter 7 of [20] or [28]) which shows that

$$(S(M))^{n+2} \leqslant (n+1)^{n+2}(\omega_{n+1})^2(\mathrm{Vol}(M))^n, \tag{1.3}$$

where $\omega_{n+1} = \pi^{(n+1)/2}/\Gamma((n+3)/2)$ denotes the $(n+1)$-dimensional volume of the unit ball in Euclidean space $\mathbf{E}^{n+1}$ and Γ is the Gamma function. Moreover, the equality in (1.3) holds if and only if $x(M^n)$ is an ellipsoid.

It is worthwhile to mention that several different affine-geometric isoperimetric inequalities were also proved by B. Andrews (cf. [1]).

Second, for a hyperovaloid $x : M^n \to \mathbf{R}^{n+1}$ another equiaffine global inequality is the so-called *affine Aleksandrov-Fenchel inequality* (cf. [1]), which says that

$$\mathrm{Vol}(M) \int_{M^n} L_1 \, dV_G \leqslant \tfrac{1}{n+1} (S(M))^2. \tag{1.4}$$

Moreover, the equality in (1.4) holds if and only if $x(M^n)$ is an ellipsoid. See also Lutwak [23] for more optimal equiaffine global inequalities on the hyperovaloids in $\mathbf{R}^{n+1}$.

This paper is organized in six sections. In Section 2, we review some basic notions and the structural equations about equiaffine hypersurfaces. In Section 3, we calculate the Laplacian of the Pick invariant for locally strongly convex affine hypersurfaces. In Section 4, we show two special inequalities in equiaffine geometry. In Section 5, we prove the little stronger Theorem 5.1 so that we complete the proof of Theorem 1.1. Finally, in Section 6, we study locally strongly convex affine hypersurfaces with semi-parallel cubic form, where Theorems 6.1 and 6.2 should be of independent meaning.

2. Preliminaries

In this section, we briefly review some basic notions about the theory of equiaffine hypersurfaces. For more details, we would refer the readers to the monographs [20, 27, 37].

Let $\mathbf{R}^{n+1}$ be the $(n+1)$-dimensional equiaffine space equipped with its canonical flat connection D and a parallel volume element given by the determinant Det. For a connected oriented and smooth n-dimensional manifold M^n, let $x : M^n \to \mathbf{R}^{n+1}$ be a locally strongly convex hypersurface immersion such that the (Blaschke-Berwald) affine metric G is positive definite. Associated to $x : M^n \to \mathbf{R}^{n+1}$, we also have the equiaffine normal vector field ξ, the (self-adjoint) equiaffine Weingarten operator B and the totally symmetric cubic (Fubini-Pick) form A. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

of B are called the affine principal curvatures, and the equiaffine mean curvature $L_1 = \frac{1}{n}\mathrm{trace}\,(B) = \frac{1}{n}\sum_{i=1}^n \lambda_i$.

We choose a local equiaffine frame field $\{x; e_1, \ldots, e_n, e_{n+1}\}$ on M^n such that $e_1, e_2, \ldots, e_n \in T_xM^n$, $e_{n+1} = \xi$ and

$$\mathrm{Det}\,[e_1, e_2, \ldots, e_n, e_{n+1}] = 1, \;\; G_{ij} := G(e_i, e_j) = \delta_{ij}. \tag{2.1}$$

Let ∇ be the Levi-Civita connection of the affine metric G; R_{ijkl} and R_{ij} be the components, with respect to $\{e_i\}_{i=1}^n$, of the Riemannian curvature tensor and the Ricci tensor of the affine metric G, respectively, and let $\kappa = n(n-1)\chi := \sum_i R_{ii}$ be the affine scalar curvature. Denote $A_{ijk} = A(e_i, e_j, e_k)$, $B_{ij} = G(Be_i, e_j)$ and the components of their first and second covariant derivatives by $A_{ijk,l}$, $A_{ijk,lm}$, $B_{ij,k}$ and $B_{ij,kl}$, respectively. Then we have the following local integrability conditions (cf. Section 2.5 of [20]):

$$B_{ij,k} - B_{ik,j} = \sum (A_{ijl}B_{kl} - A_{ikl}B_{jl}), \tag{2.2}$$

$$A_{ijk,l} - A_{ijl,k} = \tfrac{1}{2}(\delta_{ik}B_{jl} + \delta_{jk}B_{il} - \delta_{il}B_{jk} - \delta_{jl}B_{ik}), \tag{2.3}$$

$$\begin{aligned} R_{ijkl} = &\sum (A_{iml}A_{jmk} - A_{imk}A_{jml}) \\ &+ \tfrac{1}{2}(\delta_{ik}B_{jl} + \delta_{jl}B_{ik} - \delta_{il}B_{jk} - \delta_{jk}B_{il}), \end{aligned} \tag{2.4}$$

$$R_{ij} := \sum R_{ikjk} = \sum A_{ikl}A_{jkl} + \tfrac{n-2}{2}B_{ij} + \tfrac{n}{2}L_1\delta_{ij}, \tag{2.5}$$

$$\sum A_{iij} = 0, \;\; 1 \leqslant j \leqslant n, \tag{2.6}$$

here, (2.6) is called the apolarity condition. Let J be the Pick invariant that is defined by $n(n-1)J := \sum (A_{ijk})^2$. Then, from the affine Gauss equation (2.4), we have

$$\chi = J + L_1. \tag{2.7}$$

Finally, we need the following Ricci identities for the second covariant derivatives of the Fubini-Pick form A:

$$A_{ijk,lm} - A_{ijk,ml} = \sum_r A_{rjk}R_{rilm} + \sum_r A_{irk}R_{rjlm} + \sum_r A_{ijr}R_{rklm}. \tag{2.8}$$

3. The Laplacian of the Pick invariant

In this section, we shall calculate the Laplacian ΔJ of the Pick invariant J associated to the affine metric G of a locally strongly convex affine hypersurface $x : M^n \to \mathbf{R}^{n+1}$ $(n \geqslant 2)$. One can see that such calculations have been carried out in many situations for various purposes, either by assuming that $x : M^n \to \mathbf{R}^{n+1}$ is an affine hypersphere with $B = L_1 \cdot \mathrm{Id}$ (cf. [5, 6, 7, 8, 21, 30]), or just for the general case in [11, 34, 35].

First of all, we prove the following result which is similar to that of Li [21] (see also Simon [35] and Cheng-Yau [11]) where it is given for affine hyperspheres. For readers' convenience, here we will give the proof in detail.

LEMMA 3.1. *Let* $x : M^n \to \mathbf{R}^{n+1}$ $(n \geqslant 2)$ *be a locally strongly convex affine hypersurface. Then, in terms of the local equiaffine frame field* $\{x; e_1, e_2, \ldots, e_n, e_{n+1}\}$ *satisfying* (2.1)*, we have*

$$\begin{aligned}\tfrac{n(n-1)}{2}\Delta J = \sum (A_{ijk,l})^2 + \sum (R_{ijkl})^2 + \sum (R_{ij})^2 - \tfrac{n}{2}\kappa L_1 \\ - \tfrac{n+2}{2}\big(\sum B_{ij}R_{ij} + \sum A_{ijk}B_{ij,k}\big).\end{aligned} \tag{3.1}$$

Proof. By definition, we have

$$n(n-1)\Delta J = \Delta\big(\sum (A_{ijk})^2\big) = 2\sum (A_{ijk,l})^2 + 2\sum A_{ijk}A_{ijk,ll}. \tag{3.2}$$

By using (2.3), (2.6) and the totally symmetricity of A_{ijk}, we can get

$$\begin{aligned}\sum A_{ijk}A_{ijk,ll} =& \sum A_{ijk}A_{ijl,kl} \\ &+ \tfrac{1}{2}\sum A_{ijk}\big(\delta_{ik}B_{jl,l} + \delta_{jk}B_{il,l} - \delta_{il}B_{jk,l} - \delta_{jl}B_{ik,l}\big) \\ =& \sum A_{ijk}A_{ijl,kl} - \sum A_{ijk}B_{ij,k}.\end{aligned} \tag{3.3}$$

Moreover, by using (2.3), (2.6) and the Ricci identity (2.8), we can drive that

$$\begin{aligned}&\sum A_{ijk}A_{ijl,kl} \\ &= \sum A_{ijk}\big(\sum A_{ilj,lk} + \sum A_{rlj}R_{rikl} + \sum A_{irj}R_{rlkl} + \sum A_{ilr}R_{rjkl}\big) \\ &= -\tfrac{n}{2}\sum A_{ijk}B_{ij,k} + \sum A_{ijk}A_{rlj}R_{rikl} + \sum A_{ijk}A_{irj}R_{rlkl} \\ &\qquad + \sum A_{ijk}A_{ilr}R_{rjkl},\end{aligned} \tag{3.4}$$

where we have used, due to (2.3) and (2.6), the equation

$$\sum A_{ilj,l} = \tfrac{n}{2}(L_1\delta_{ij} - B_{ij}). \tag{3.5}$$

To go on from (3.4), we use the fact $\sum A_{ijk}A_{rlj}R_{rikl} = -\sum A_{jil}A_{rki}R_{rjkl}$ and (2.4) to derive that

$$\begin{aligned}&\sum A_{ijk}A_{ilr}R_{rjkl} + \sum A_{ijk}A_{rlj}R_{rikl} \\ &= \sum (A_{ril}A_{jik} - A_{rik}A_{jil})R_{rjkl} \\ &= \sum \big[R_{rjkl} - \tfrac{1}{2}(\delta_{rk}B_{jl} + \delta_{jl}B_{rk} - \delta_{rl}B_{jk} - \delta_{jk}B_{rl})\big]R_{rjkl} \\ &= \sum (R_{rjkl})^2 - 2\sum B_{ij}R_{ij}.\end{aligned} \tag{3.6}$$

Next, from (2.5) we can derive that

$$\begin{aligned}\sum A_{ijk}A_{irj}R_{rlkl} &= \sum A_{ijk}A_{irj}R_{rk} \\ &= \sum\big(R_{rk} - \tfrac{n}{2}L_1\delta_{rk} - \tfrac{n-2}{2}B_{rk}\big)R_{rk} \\ &= \sum (R_{rk})^2 - \tfrac{n}{2}\kappa L_1 - \tfrac{n-2}{2}\sum B_{rk}R_{rk}.\end{aligned} \tag{3.7}$$

The combination of (3.2), (3.3), (3.4), (3.6) and (3.7) immediately gives the assertion. $\square$

In particular, if $n=2$, we have

$$R_{ijkl}=\chi(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),\ \ R_{ij}=\chi\delta_{ij}. \tag{3.8}$$

Then, combining with Lemma 3.1, we have the following

COROLLARY 3.1. (cf. [34], or (4.12) in [20]) *Let $x:M^2\to\mathbf{R}^3$ be a locally strongly convex affine surface, then we have*

$$\Delta J=\sum(A_{ijk,l})^2+6\chi J-2\sum A_{ijk}B_{ij,k}. \tag{3.9}$$

4. Auxiliary estimation

The following Lemma, which is partially due to U. Simon [35], is crucial for our proof of Theorem 1.1. We include the proof here for readers' convenience.

LEMMA 4.1. *Let $x:M^n\to\mathbf{R}^{n+1}$ be a locally strongly convex affine hypersurface. Then the following inequality holds*

$$\sum(A_{ijk,l})^2\geqslant\tfrac{3n(n+2)}{4(n+4)}\sum(\tilde B_{ij})^2, \tag{4.1}$$

where $\tilde B_{ij}=B_{ij}-L_1\delta_{ij}$ denotes the trace-free part of the affine Weingarten form.

Moreover, the equality in (4.1) holds if and only if the covariant derivatives of the cubic form A have the expression below:

$$\begin{aligned}A_{ijk,l}=&\tfrac{1}{n+4}(\delta_{ik}\tilde B_{jl}+\delta_{ij}\tilde B_{kl}+\delta_{jk}\tilde B_{il})\\&-\tfrac{n+2}{2(n+4)}(\delta_{jl}\tilde B_{ik}+\delta_{kl}\tilde B_{ij}+\delta_{il}\tilde B_{jk}),\ \ \forall i,j,k,l.\end{aligned} \tag{4.2}$$

Proof. We define a type $(0,4)$-tensor P by

$$\begin{aligned}P_{ijkl}:=&A_{ijk,l}+A_{jkl,i}+A_{kli,j}+A_{lij,k}\\&+\tfrac{n}{n+4}(\tilde B_{jk}\delta_{il}+\tilde B_{il}\delta_{jk}+\tilde B_{ik}\delta_{lj}+\tilde B_{lj}\delta_{ki}+\tilde B_{ij}\delta_{lk}+\tilde B_{kl}\delta_{ij}).\end{aligned} \tag{4.3}$$

By using (2.3), it is easy to check that P is a totally symmetric traceless tensor, i.e.,

$$P_{ijkl}=P_{jikl}=P_{ikjl}=P_{ijlk},\ \ \sum\delta_{ij}P_{ijkl}=0. \tag{4.4}$$

Straightforward computations with the use of (2.3) show that

$$\begin{aligned}\sum(P_{ijkl})^2=&4\sum(A_{ijk,l})^2+12\sum A_{ijk,l}A_{ijl,k}-\tfrac{6n^2}{n+4}\sum(\tilde B_{ij})^2\\=&16\sum(A_{ijk,l})^2-\tfrac{12n(n+2)}{n+4}\sum(\tilde B_{ij})^2.\end{aligned} \tag{4.5}$$

This proves the inequality (4.1). Moreover, the equality in (4.1) holds if and only if $P=0$. Finally, by using (2.3) again, we can easily show that $P=0$ is equivalent to that the covariant derivatives of the cubic form A have the expression as stated by (4.2). □

In order to prove Theorem 1.1, a technical trick we making use is the following estimate of the invariant $\sum(R_{ijkl})^2$.

We begin with recalling the Weyl conformal curvature tensor for $n \geqslant 3$,

$$\begin{aligned} W_{ijkl} := R_{ijkl} - \tfrac{1}{n-2}(\delta_{ik}R_{jl} + \delta_{jl}R_{ik} - \delta_{il}R_{jk} - \delta_{jk}R_{il}) \\ + \tfrac{\kappa}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \end{aligned} \tag{4.6}$$

and the introduction of another $(0,4)$-tensor Q defined by

$$Q_{ijkl} := \sum(A_{imj}A_{kml} - A_{iml}A_{jmk}). \tag{4.7}$$

LEMMA 4.2. *Let $x : M^n \to \mathbf{R}^{n+1}$ $(n \geqslant 3)$ be a locally strongly convex affine hypersurface. Then, for any real number $a \in (0,1)$, it holds that*

$$\begin{aligned} \sum(R_{ijkl})^2 \geqslant a\left[4\sum B_{ij}R_{ij} - (n-2)\sum(B_{ij})^2 - (nL_1)^2\right] \\ + (1-a)\left[\sum(W_{ijkl})^2 + \tfrac{4}{n-2}\sum(R_{ij})^2 - \tfrac{2\kappa^2}{(n-1)(n-2)}\right] \end{aligned} \tag{4.8}$$

and the equality holds if $Q = 0$.

Proof. According to (2.4) and (4.7), a straightforward calculation shows that

$$\begin{aligned} \sum(R_{ijkl})^2 &= \sum(Q_{ijkl})^2 + 4\sum B_{ij}R_{ij} - (n-2)\sum(B_{ij})^2 - (nL_1)^2 \\ &\geqslant 4\sum B_{ij}R_{ij} - (n-2)\sum(B_{ij})^2 - (nL_1)^2, \end{aligned} \tag{4.9}$$

and the equality holds if $Q = 0$.

On the other hand, the expression (4.6) implies that

$$\sum(R_{ijkl})^2 = \sum(W_{ijkl})^2 + \tfrac{4}{n-2}\sum(R_{ij})^2 - \tfrac{2\kappa^2}{(n-1)(n-2)}. \tag{4.10}$$

Now, for any real number $a \in (0,1)$, we write

$$\sum(R_{ijkl})^2 = a\sum(R_{ijkl})^2 + (1-a)\sum(R_{ijkl})^2. \tag{4.11}$$

Then, the assertion of Lemma 4.2 immediately follows. $\square$

5. Proofs of the main results

We first prove the following generalized theorem by which Theorem 1.1 is a direct consequence.

THEOREM 5.1. *Let $x : M^n \to \mathbf{R}^{n+1}$ be a hyperovaloid with affine metric G and affine Weingarten operator B. Then the following integral inequality holds:*

$$\int_{M^n} \chi^2\, dV_G + \varepsilon_n \int_{M^n} \|\tilde{B}\|_G^2 dV_G \leqslant \tfrac{1}{n}\int_{M^n} \|B\|_G^2 dV_G, \tag{5.1}$$

where

$$\varepsilon_n = \begin{cases} \frac{5n+8}{4(n+4)(n^2-1)}, & \text{if } n \geqslant 3, \\ \frac{1}{6}, & \text{if } n = 2. \end{cases} \tag{5.2}$$

Moreover, the equality in (5.1) *holds if and only if* $x(M^n)$ *is an ellipsoid.*

Proof. From (3.5) we can derive that

$$\sum B_{ij}A_{ijk,k} = -\tfrac{n}{2}\sum(\tilde{B}_{ij})^2. \tag{5.3}$$

This combining with (3.1) yields

$$\begin{aligned} \tfrac{1}{2}n(n-1)\Delta J = &\sum(A_{ijk,l})^2 + \sum(R_{ijkl})^2 + \sum(R_{ij})^2 - \tfrac{n(n+2)}{4}\sum(\tilde{B}_{ij})^2 \\ &- \tfrac{n}{2}\kappa L_1 - \tfrac{n+2}{2}\sum B_{ij}R_{ij} - \tfrac{n+2}{2}\sum(B_{ij}A_{ijk})_{,k}, \end{aligned} \tag{5.4}$$

where $_{,k}$ denotes the covariant derivative with respect to the affine metric G.

Now, we consider two cases depending on the dimension n.

Case (1) If $n \geqslant 3$, then from (5.4) and Lemma 4.2 we obtain

$$\begin{aligned} \tfrac{1}{2}n(n-1)\Delta J \geqslant &\sum(A_{ijk,l})^2 - \tfrac{n+2-8a}{2}\sum B_{ij}R_{ij} + \tfrac{n+2-4a}{n-2}\sum(R_{ij})^2 \\ &- a(n-2)\sum(B_{ij})^2 - \tfrac{n(n+2)}{4}\sum(\tilde{B}_{ij})^2 - \tfrac{n}{2}\kappa L_1 \\ &- a(nL_1)^2 - \tfrac{2(1-a)}{(n-1)(n-2)}\kappa^2 - \tfrac{n+2}{2}\sum(B_{ij}A_{ijk})_{,k}, \end{aligned} \tag{5.5}$$

where the equality holds if and only if $W = 0$.

We now fix $a = \frac{5}{8}$ so that $n+2-8a \geqslant 0$ for $n \geqslant 3$.

By using the fact

$$\sum(B_{ij})^2 = \sum(\tilde{B}_{ij})^2 + n(L_1)^2, \tag{5.6}$$

and the following two inequalities, i.e.,

$$\sum B_{ij}R_{ij} \leqslant \tfrac{1}{2(n-1)}\sum(R_{ij})^2 + \tfrac{n-1}{2}\sum(B_{ij})^2 \tag{5.7}$$

with equality sign holding if and only if $R_{ij} = (n-1)B_{ij}$ for $1 \leqslant i,j \leqslant n$, and

$$\kappa L_1 \leqslant \tfrac{n(n-1)}{2}(L_1)^2 + \tfrac{1}{2n(n-1)}\kappa^2 \tag{5.8}$$

with equality sign holding if and only if $\kappa = n(n-1)L_1$ or equivalently $\chi = L_1$ and $J = 0$, we can derive from (5.5) that

$$\begin{aligned} \tfrac{1}{2}n(n-1)\Delta J \geqslant &\sum(A_{ijk,l})^2 - \tfrac{4n^2+n-4}{8}\sum(\tilde{B}_{ij})^2 - \tfrac{n+2}{2}\sum(B_{ij}A_{ijk})_{,k} \\ &+ \tfrac{n+1}{4(n-1)(n-2)}\left[(3n-4)\sum(R_{ij})^2 - \kappa^2\right] - \tfrac{n(n^2-1)}{2}(L_1)^2. \end{aligned} \tag{5.9}$$

This, combining with (2.7), (4.1) and the inequality

$$\sum(R_{ij})^2 \geqslant \tfrac{\kappa^2}{n}, \tag{5.10}$$

gives

$$\begin{aligned}\tfrac{1}{2}n(n-1)\Delta J \geqslant{}& \left[\tfrac{3n(n+2)}{4(n+4)} - \tfrac{4n^2+n-4}{8}\right]\sum(\tilde{B}_{ij})^2 \\ &+ \tfrac{n(n^2-1)}{2}\left[\chi^2 - (L_1)^2\right] - \tfrac{n+2}{2}\sum(B_{ij}A_{ijk})_{,k}\,.\end{aligned} \tag{5.11}$$

Integrating (5.11) over M^n and applying for the divergence theorem, we obtain that

$$\begin{aligned}0 \geqslant{}& \tfrac{n(n^2-1)}{2}\int_{M^n}\left[\chi^2 - (L_1)^2\right]\,dV_G \\ &- \left[\tfrac{4n^2+n-4}{8} - \tfrac{3n(n+2)}{4(n+4)}\right]\int_{M^n}\sum(\tilde{B}_{ij})^2\,dV_G,\end{aligned} \tag{5.12}$$

which is exactly equivalent to the inequality (5.1).

Finally, if the equality in (5.12) holds, then (5.10) becomes an equality identically and (M^n,G) is an Einstein manifold. It follows from the theorem of Kozlowski-Simon [18] (cf. Theorem 4.8 in [20]) that $x(M^n)$ is an ellipsoid. Conversely, the ellipsoids are affine hyperspheres with vanishing Pick invariant and having constant sectional curvature. Therefore, it holds that $\|\tilde{B}\|_G = 0$ and $\chi = L_1$ and the equality in (5.12) holds.

Case (2) If $n=2$, then according to (3.8), together with the use of (2.3), (2.7), (3.5) and (4.1), we can derive that

$$\begin{aligned}\Delta J &= \sum(A_{ijk,l})^2 + 6\chi J - 2\sum A_{ijk}B_{ij,k} \\ &= \sum(A_{ijk,l})^2 + 6\chi^2 - 6\chi L_1 - 2\sum(\tilde{B}_{ij})^2 - 2\sum(B_{ij}A_{ijk})_{,k} \\ &\geqslant 3\chi^2 - 3(L_1)^2 - \sum(\tilde{B}_{ij})^2 - 2\sum(B_{ij}A_{ijk})_{,k}\,,\end{aligned} \tag{5.13}$$

here, in the last step, we make use the fact $2\chi L_1 \leqslant \chi^2 + (L_1)^2$ with equality holding if and only if $\chi = L_1$, or equivalently $J=0$.

Then, integrating (5.13) over M^2, we get

$$0 \geqslant \int_{M^2}\left[3\chi^2 - 3(L_1)^2 - \sum(\tilde{B}_{ij})^2\right]dV_G, \tag{5.14}$$

which is again exactly equivalent to the inequality (5.1).

Moreover, if the equality in (5.14) holds, then we have $J=0$ identically. This implies by the classical theorem of Maschke-Pick (cf. Theorem 2.13 in [20]) that $x(M^2)$ is an ellipsoid. Conversely, it is well known that the ellipsoids satisfy that $\|\tilde{B}\|_G = 0$ and $\chi = L_1$, thus the equality in (5.14) holds.

In conclusion, we have completed the proof of Theorem 5.1. □

As a direct consequence of Theorem 5.1, we have the following pointwise rigidity theorem:

COROLLARY 5.1. *Let $x: M^n \to \mathbf{R}^{n+1}$ be a hyperovaloid with affine metric G and affine Weingarten operator B. If, for ε_n defined by (5.2), the equiaffine invariants χ, $\|\tilde{B}\|_G$ and $\|B\|_G$ satisfy the inequality*

$$\chi^2 + \varepsilon_n\|\tilde{B}\|_G^2 \geqslant \tfrac{1}{n}\|B\|_G^2 \tag{5.15}$$

identically on M^n, then $x(M^n)$ is an ellipsoid.

6. Affine hypersurfaces with semi-parallel cubic form

According to [14, 15] and the references therein, we have a complete classification of the locally strongly convex affine hypersurfaces with parallel cubic (Fubini-Pick) form. It follows easily from the classification that a hyperovaloid in $\mathbf{R}^{n+1}$ with parallel cubic form is exactly an ellipsoid. To relax the condition of *parallel* cubic form, in this section we will consider locally strongly convex affine hypersurfaces with *semi-parallel* cubic form. Here, the cubic form A is said to be semi-parallel if it satisfies $R\cdot A=0$, where R is the Riemannian curvature tensor corresponding to the affine metric G. Our first result of this section is the following theorem.

THEOREM 6.1. *Let $x: M^n \to \mathbf{R}^{n+1}$ $(n \geqslant 3)$ be a locally strongly convex affine hypersurface with semi-parallel cubic form. Then the equiaffine invariants χ, $\|\tilde{B}\|_G$ and $\|B\|_G$ satisfy the inequality*

$$\chi^2+\tfrac{1}{2(n+1)}\|\tilde{B}\|_G^2 \leqslant \tfrac{1}{n}\|B\|_G^2. \tag{6.1}$$

Moreover, the equality in (6.1) holds identically on M^n if and only if $x(M^n)$ is locally a hyperquadric.

Proof. Following the proof of Lemma 3.1, we have

$$\tfrac{n(n-1)}{2}\Delta J=\sum(A_{ijk,l})^2+\sum A_{ijk}A_{ijl,kl}-\sum A_{ijk}B_{ij,k}. \tag{6.2}$$

From the assumption $R\cdot A=0$ and the fact

$$(R(e_l,e_m)A)(e_i,e_j,e_k)=A_{ijk,ml}-A_{ijk,lm}$$

we have $A_{ijl,kl}=A_{ijl,lk}$. It follows from (6.2), (2.3) and (2.6) that

$$\tfrac{1}{2}n(n-1)\Delta J=\sum(A_{ijk,l})^2-\tfrac{n+2}{2}\sum A_{ijk}B_{ij,k}. \tag{6.3}$$

Then, comparing (6.3) with (3.1), we obtain that

$$\sum(R_{ijkl})^2+\sum(R_{ij})^2-\tfrac{n+2}{2}\sum B_{ij}R_{ij}=\tfrac{n}{2}\kappa L_1. \tag{6.4}$$

Next, on the one hand, we apply for (4.10), (5.7) and (5.10) to obtain that

$$\begin{aligned}
&\sum(R_{ijkl})^2+\sum(R_{ij})^2-\tfrac{n+2}{2}\sum B_{ij}R_{ij}\\
&=\sum(W_{ijkl})^2+\tfrac{n+2}{n-2}\sum(R_{ij})^2-\tfrac{2\kappa^2}{(n-1)(n-2)}-\tfrac{n+2}{2}\sum B_{ij}R_{ij}\\
&\geqslant \tfrac{(3n-2)(n+2)}{4(n-1)(n-2)}\sum(R_{ij})^2-\tfrac{2\kappa^2}{(n-1)(n-2)}-\tfrac{(n-1)(n+2)}{4}\sum(B_{ij})^2\\
&\geqslant \tfrac{n(3n+2)(n-1)}{4}\chi^2-\tfrac{(n-1)(n+2)}{4}\sum(B_{ij})^2,
\end{aligned} \tag{6.5}$$

and the last equality holds if and only if $W=0$, $R_{ij}=(n-1)B_{ij}$ for all indices, and that (M^n,G) is Einstein.

On the other hand, by using (5.8), we obtain that

$$\kappa L_1 \leqslant \tfrac{1}{2}n(n-1)\left[\chi^2+(L_1)^2\right], \tag{6.6}$$

and the equality holds if and only if $\chi = L_1$ and thus $J=0$.

The combination of (6.4), (6.5) and (6.6) then gives

$$\chi^2+\tfrac{1}{2(n+1)}\sum(\tilde{B}_{ij})^2 \leqslant \tfrac{1}{n}\sum(B_{ij})^2. \tag{6.7}$$

The obvious equivalence between (6.1) and (6.7) verifies the assertion (6.1).

Moreover, if the equality in (6.7) holds identically, then the inequalities (6.5) and (6.6) all become equalities, so we have $J=0$ on M^n and, by the well-known Maschke-Pick-Berwald theorem (cf. Theorem 2.13 in [20]), $x(M^n)$ is locally a hyperquadric. Conversely, the hyperquadrics are affine hyperspheres with vanishing Pick invariant and having constant sectional curvature. Therefore, it holds that $\|\tilde{B}\|_G=0$, $\|B\|_G^2=n(L_1)^2$ and $\chi=L_1$. Hence the equality in (6.7) holds.

This completes the proof of Theorem 6.1. □

In case $n=2$, a result better than Theorem 6.1 can be proved.

THEOREM 6.2. *A locally strongly convex affine surface* $x: M^2\to\mathbf{R}^3$ *is of semi-parallel cubic form if and only if either* $x(M^2)$ *is locally a quadric or* (M^2,G) *is flat.*

Proof. First, we assume that $x: M^2\to\mathbf{R}^3$ is a locally strongly convex affine surface with semi-parallel cubic form. Then, following the proof of Theorem 6.1, we get

$$\sum(R_{ijkl})^2+\sum(R_{ij})^2-2\sum B_{ij}R_{ij}-\kappa L_1=0. \tag{6.8}$$

On the other hand, using (3.8) we obtain

$$\sum(R_{ijkl})^2=4\chi^2,\ \ \sum(R_{ij})^2=2\chi^2,\ \ \sum B_{ij}R_{ij}=2\chi L_1. \tag{6.9}$$

Hence, (6.8) becomes equivalently

$$\chi J=0. \tag{6.10}$$

This implies that, due to that our concern is in local, we have either $J=0$ on M^2 and, according to the classical theorem of Maschke-Pick (cf. Theorem 2.13 in [20]), $x(M^2)$ is locally a quadric, or, $J\neq 0$ and $\chi=0$ on M^2 so that (M^2,G) is flat.

Conversely, $R\cdot A=0$ is trivially true if $x(M^2)$ is a quadric or (M^2,G) is flat.

This completes the proof of Theorem 6.2. □

REMARK 6.1. The class of locally strongly convex affine surfaces in $\mathbf{R}^3$ with flat affine metric should be complicated. Its classification is a very hard problem and far from completed. For several of the related partial results, we refer to the earlier articles of Magid-Ryan [26] and Vrancken [38], and recent interesting progress [3].

Combining with the theorem of Schneider on the solution of Blaschke's conjecture (cf. Theorem 4.8 in [20]), Theorem 6.2 immediately implies the following

COROLLARY 6.1. *An ovaloid $x : M^2 \to \mathbf{R}^3$ with semi-parallel cubic form must be an ellipsoid.*

Finally, noting that if the affine metric is flat, then the cubic form is trivially semi-parallel. Now, besides the quadrics, we are going to state some explicit examples of affine surfaces with flat affine metric, including the ones that are of either parallel, or semi-parallel but not of parallel cubic form.

REMARK 6.2. We know that, besides the quadrics, there exists another class of locally strongly convex affine surfaces which are flat and parallel, i.e., the surfaces with equation $xyz = c$ for constant $c \neq 0$. Furthermore, there are also a lot of affine surfaces with flat affine metric. As typical examples, we have the following four affine rotation surfaces described in [13]:

$$x_1(u,v) = \big(\cos(3u)^{\frac{1}{3}}\cos(\sqrt{3}v), \cos(3u)^{\frac{1}{3}}\sin(\sqrt{3}v), \int_0^u \cos(3t)^{-\frac{2}{3}}dt\big),$$

$$x_2(u,v) = \big(\cosh(3u)^{\frac{1}{3}}\cos(\sqrt{3}v), \cosh(3u)^{\frac{1}{3}}\sin(\sqrt{3}v), \int_0^u \cosh(3t)^{-\frac{2}{3}}dt\big),$$

$$x_3(u,v) = \big(\cos(3u)^{\frac{1}{3}}\cosh(\sqrt{3}v), \cos(3u)^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u \cos(3t)^{-\frac{2}{3}}dt\big),$$

$$x_4(u,v) = \big(\cosh(3u)^{\frac{1}{3}}\cosh(\sqrt{3}v), \cosh(3u)^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u \cosh(3t)^{-\frac{2}{3}}dt\big).$$

Here, the last surface x_4 is locally strongly convex with complete and flat affine metric, which is semi-parallel but not parallel.

Similarly, for higher dimensional cases, we refer to [3], Vrancken-Li-Simon [40] and Vrancken [39], where several typical examples of locally strongly convex affine hypersurfaces in $\mathbf{R}^{n+1}$ $(n \geqslant 3)$ with flat affine metric but non-parallel cubic form are presented.

Acknowledgements. The authors are much indebted to the referee for his/her valuable comments and suggestions. Following those helpful comments, we have illustrated our results in the last section by adding Remark 6.2.

REFERENCES

[1] B. ANDREWS, *Contraction of convex hypersurfaces by their affine normal*, J. Differ. Geom. **43** (1996), 207–230.

[2] L. J. ALIAS, A. G. COLARES, *A further characterization of ellipsoids*, Results Math. **48** (2005), 1–8.

[3] M. ANTIĆ, H. LI, L. VRANCKEN, X. WANG, *Affine hypersurfaces with constant sectional curvature*, Pacific J. Math. **310** (2021), 275–302.

[4] W. BLASCHKE, *Über affine Geometrie I: Isoperimetrische Eigenschaften von Ellipse und Ellipsoid*, Ber. Verh. Sächs. Akad. Leipzig, Math.-Phys. Kl. **68** (1916), 217–239.

[5] W. BLASCHKE, *Vorlesungen über Differentialgeometrie, II*, Springer, Berlin, 1923.

[6] E. CALABI, *An extension of E. Hopf's maximum principle with an application to Riemannian Geometry*, Duke Math. J. **25** (1958), 45–56.

[7] X. CHENG, Z. HU, *On the isolation phenomena of locally conformally flat manifolds with constant scalar curvature – submanifolds versions*, J. Math. Anal. Appl. **464** (2018), 1147–1157.

[8] X. CHENG, Z. HU, A.-M. LI, H. LI, *On the isolation phenomena of Einstein manifolds – submanifolds versions*, Proc. Am. Math. Soc., **146** (2018), 1731–1740.

[9] X. CHENG, Z. HU, L. VRANCKEN, *Every centroaffine Tchebychev hyperovaloid is ellipsoid*, arXiv:1911.05222v1.

[10] X. CHENG, Z. HU, Z. YAO, *A rigidity theorem for centroaffine Chebyshev hyperovaloids*, Colloq. Math. **157** (2019), 133–141.

[11] S. Y. CHENG, S. T. YAU, *Complete affine hyperspheres, Part I. The completeness of affine metrics*, Comm. Pure Appl. Math. **39** (1986), 17–30.

[12] A. DEICKE, *Über die Finsler-Räume mit* $A_i = 0$, Arch. Math. **4** (1953), 45–51.

[13] F. DILLEN, A. MARTINEZ, F. MILAN, F. G. SANTOS, L. VRANCKEN, *On the Pick invariant, the affine mean curvature and the Gauss curvature of affine surfaces*, Results Math. **20** (1991), 622–642.

[14] Z. HU, H. LI, L. VRANCKEN, *Locally strongly convex affine hypersurfaces with parallel cubic form*, J. Differ. Geom. **87** (2011), 239–307.

[15] Z. HU, H. LI, U. SIMON, L. VRANCKEN, *On locally strongly convex affine hypersurfaces with parallel cubic form. Part I*, Differ. Geom. Appl. **27** (2009), 188–205.

[16] C. C. HSIUNG, J. K. SHAHIN, *Affine differential geometry of closed hypersurfaces*, Proc. London Math. Soc. **17** (1967), 715–735.

[17] J. HU, G. XIONG, D. ZOU, *Affine isoperimetric inequalities for intersection mean ellipsoids*, Calc. Var. Partial Diff. Equat. **58** (2019), Art. 191, pp. 20.

[18] M. KOZLOWSKI, U. SIMON, *Hyperflächen mit äquiaffiner Einsteinmetrik*, Mathematica, Festschrift E. Mohr, TU Berlin, (1985), 179–190.

[19] A. LI, Q. HUANG, D. XI, *New sine ellipsoids and related volume inequalities*, Adv. Math. **353** (2019), 281–311.

[20] A.-M. LI, U. SIMON, G. S. ZHAO, Z. HU, *Global Affine Differential Geometry of Hypersurfaces*, 2nd ed., de Gruyter Expositions in Mathematics 11, Walter de Gruyter, Berlin/Boston, 2015.

[21] A.-M. LI, *Some theorems in affine differential geometry*, Acta Math. Sinica N. S. **5** (1989), 345–354.

[22] A.-M. LI, *A characterization of ellipsoids*, Results Math. **20** (1991), 657–659.

[23] E. LUTWAK, *Selected affine isoperimetric inequalities*, A survey article, In: P. M. Gruber and J. M. Wills (eds.), Handbook Convex Geometry, pp. 153–176, North Holland, Amsterdam, 1993.

[24] E. LUTWAK, D. YANG, G. ZHANG, *Orlicz projection bodies*, Adv. Math. **223** (2010), 220–242.

[25] E. LUTWAK, D. YANG, G. ZHANG, *Orlicz centroid bodies*, J. Differ. Geom. **84** (2010), 365–387.

[26] M. MAGID, P. RYAN, *Flat affine spheres in* $\mathbf{R}^3$, Geom. Dedicata **33** (1990), 277–288.

[27] K. NOMIZU, T. SASAKI, *Affine Differential Geometry*, Cambridge University Press, Cambridge, (1994).

[28] A. V. POGORELOV, *Complete affine minimal hypersurfaces*, Soviet Math. Dokl. **38** (1989), 217–219.

[29] L. SANTALÓ, *An affine invariant for convex bodies of* n*-dimensional space* (Spanish), Portugal. Math. **8** (1949), 155–161.

[30] R. SCHNEIDER, *Zur affinen Differentialgeometrie im Grossen, I*, Math. Z. **101** (1967), 375–406.

[31] R. SCHNEIDER, *Convex bodies: the Brunn–Minkowski theory*, Encyclopedia of Mathematics and Its Applications, **151**, Cambridge University Press, Cambridge, (2014).

[32] T. SASAKI, *On affine isoperimetric inequality for a strongly convex closed hypersurface in the unimodular affine space* A^{n+1}, Kumamoto J. Sci. (Math.) **16** (1984), 23–38.

[33] U. SIMON, *Minkowskische Integralformeln und ihre Anwendungen in der Differentialgeometrie im Grossen*, Math. Ann. **173** (1967), 307–321.

[34] U. SIMON, *The Pick invariant in equiaffine differential geometry*, Abh. Math. Sem. Univ. Hamburg **53** (1983), 225–228.

[35] U. SIMON, *Hypersurfaces in equiaffine differential geometry*, Geom. Dedicata **17** (1984), 157–168.

[36] W. SÜSS, *Zur relativen Differentialgeometrie V: Über Eihyperflächen im* $\mathbf{R}^{n+1}$, Tohoku Math. J. **31** (1929), 202–209.

[37] U. SIMON, A. SCHWENK-SCHELLSCHMIDT, H. VIESEL, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University Tokyo, 1991.

[38] L. VRANCKEN, *Affine surfaces with higher order parallel cubic form*, Tohoku Math. J. **43** (1991), 127–139.

[39] L. VRANCKEN, *Affine quasi umbilical hypersurfaces which are flat with respect to the affine metric*, Results Math. **20** (1991) 756–776.

[40] L. VRANCKEN, A.-M. LI, U. SIMON, *Affine spheres with constant sectional curvature*, Math. Z. **206** (1991), 651–658.

(Received April 11, 2020)

Zejun Hu
School of Mathematics and Statistics
Zhengzhou University
Zhengzhou 450001, People's Republic of China
e-mail: huzj@zzu.edu.cn

Cheng Xing
School of Mathematics and Statistics
Zhengzhou University
Zhengzhou 450001, People's Republic of China
e-mail: xingchengchn@yeah.net