Mathematical Inequalities & Applications
Volume 24, Number 2 (2021), 351–372 doi:10.7153/mia-2021-24-25

MEANS PRODUCED BY DISTANCES

VOLKER DIELS-GRABSCH*, MOWAFFAQ HAJJA AND PANAGIOTIS T. KRASOPOULOS

(*Communicated by H. Martini*)

Abstract. We describe a methodology that can be used to construct new distances which produce many famous means. Its main application is to construct a distance for the logarithmic mean, settling an old open problem. We also use it to construct alternative distances for already known means, such as the arithmetic and all quasi-arithmetic means. Moreover, we show how to construct distances for almost all means that can be obtained from Cauchy's Mean Value Theorem, and apply this to construct distances for all Stolarsky means. Finally, we show how to construct a distance for a mean $\mathscr{M}_q(a,b) = q^{-1}(\mathscr{M}(q(a),q(b)))$, where $\mathscr{M}$ is another mean for which a distance is already known, and q is a monotone bijection to a subinterval.

1. Introduction

The Pythagoreans knew three means, namely, the arithmetic, geometric, and harmonic mean. These are the means defined, respectively, by

$$\mathscr{A}(a,b) = \frac{a+b}{2},\ \mathscr{G}(a,b) = \sqrt{ab},\ \mathscr{H}(a,b) = \frac{2ab}{a+b}.$$

Few centuries later, Nichomachus, in his book *Introduction to Arithmetic* [16], added seven more means to the list; see [9, Chapter III, (g), pp. 110–125]. Nowadays, there is a huge number of means, or rather families of means, and the literature on them is voluminous. There are even complete books on them, such as the encyclopedic book [3].

Let us denote by $\mathbb{P} = (0,\infty)$ the set of positive real numbers, and by $\mathbb{J} \subseteq \mathbb{R}$ a general interval. The ten Greek means are means of two positive numbers. That is, they are functions $\mathscr{M} : \mathbb{P}\times\mathbb{P} \to \mathbb{P}$, where $\mathscr{M}$ has the internality property

$$\mathscr{M}(a,b) \in [a,b] \text{ for all closed subintervals } [a,b] \subseteq \mathbb{P}.$$

In contrast, the means that are studied in the current literature can involve any number of variables, not necessarily two. Moreover, they can have as their domain any interval $\mathbb{J}$, not necessarily $\mathbb{P}$.

Mathematics subject classification (2010): 26A24, 26E60, 51K05.
Keywords and phrases: Distances, logarithmic mean, means, mean value theorem, quasi-arithmetic means, Stolarsky means.
* Corresponding author.

It is observed in [7] that the arithmetic mean $\mathscr{A}$ is produced in a natural way by the distance d on $\mathbb{P}$ defined by

$$d : \mathbb{P} \times \mathbb{P} \to \mathbb{R}, \;\; d(a,b) = (a-b)^2$$

in the sense that for every $a,b \in \mathbb{P}$, $\mathscr{A}(a,b)$ is the unique point x whose distances from a and b have a minimal sum. In other words, $\mathscr{A}(a,b)$ is the unique point x at which the function $f : \mathbb{P} \to \mathbb{R}$ defined by

$$f(x) = d(x,a) + d(x,b) = (x-a)^2 + (x-b)^2$$

attains its minimum. This follows immediately from

$$f'(x) = 4\left(x - \frac{a+b}{2}\right), \;\; f''(x) = 4 > 0.$$

One expresses this observation by saying that $\mathscr{A}$ *is the distance mean produced by* d. Similarly, $\mathscr{H}$ and $\mathscr{G}$ are the distance means produced by the distances d_1 and d_2 defined by

$$d_1(a,b) = \left(\frac{1}{a} - \frac{1}{b}\right)^2, \;\; d_2(a,b) = (\ln a - \ln b)^2.$$

In fact, $\mathscr{H}$ is similar to $\mathscr{A}$ in the sense that

$$\mathscr{H} = \phi^{-1}\mathscr{A}\phi,$$

or more accurately

$$\mathscr{H}(a,b) = \phi^{-1}\left(\mathscr{A}\left(\phi(a), \phi(b)\right)\right),$$

where $\phi : \mathbb{P} \to \mathbb{P}$ is the monotone bijection given by

$$\phi(x) = \frac{1}{x}.$$

Similarly, if one thinks of $\mathscr{A}$ as defined on all of $\mathbb{R}$ (instead of $\mathbb{P}$), then $\mathscr{G}$ is similar to $\mathscr{A}$ in the sense that

$$\mathscr{G} = \psi^{-1}\mathscr{A}\psi,$$

where $\psi : \mathbb{P} \to \mathbb{R}$ is the monotone bijection given by

$$\psi(x) = \ln x.$$

These observations are what led to the distances d_1 and d_2.

The facts that $\mathscr{H}$ and $\mathscr{G}$ are similar to the arithmetic mean $\mathscr{A}$ are expressed in the existing literature by saying that $\mathscr{H}$ and $\mathscr{G}$ are quasi-arithmetic means. Thus if a mean $\mathscr{M}$ is quasi-arithmetic, then it is produced by a distance. However, if $\mathscr{M}$ is not quasi-arithmetic, then it is not clear how to find a distance that produces it, nor it is clear whether such a distance does indeed exist, i.e., whether $\mathscr{M}$ is a distance mean. In particular, the question whether the logarithmic mean $\mathscr{L}$ defined by

$$\mathscr{L}(a,b) = \frac{a-b}{\ln a - \ln b},$$

is a distance mean remained open until the results of this paper were established. This problem was raised by the second-named author, MH, in 1995, and resisted the attempts of the very many people whom MH challenged the problem with. It was later included as Open Problem 9 in [7]. Let us also stress that $\mathscr{L}(a,b)$ is not a quasi-arithmetic mean. To see this, let us consider the following functional equation (see [7, Section 9, p. 6]):

$$\mathscr{M}(\mathscr{M}(\mathscr{M}(a,b),b),\mathscr{M}(\mathscr{M}(a,b),a)) = \mathscr{M}(a,b), \tag{1}$$

for $a,b>0$. It is noted in [7] that any quasi-arithmetic mean of two variables $\mathscr{M}(a,b)$ must satisfy (1). So, one can verify by experimenting with random numbers $a,b>0$ that $\mathscr{L}(a,b)$ does not satisfy (1).

It seems that the only way to establish that a given mean is a distance mean is to explicitly find a distance that produces it. This is what makes such a problem very difficult, and what makes the problem of proving that a given mean is not a distance mean intractable. One of the open, and probably most challenging, problems left by this paper is to find a characterization of distance means via which one can prove that a given mean is not a distance mean, or can prove that it is a distance mean without having to find the producing distance.

In this paper we answer the aforementioned open problem pertaining to the mean $\mathscr{L}$. We give an explicit simple distance that produces $\mathscr{L}$, and we describe a general method by which we can handle the analogous problem for other means. We also answer other questions that were raised in [7].

2. Preliminaries

An n-dimensional (or n-variable) mean on an interval $\mathbb{J}\subseteq\mathbb{R}$ is a function $\mathscr{M}$: $\mathbb{J}^n\to\mathbb{J}$ that has the so-called *internality* or *intermediacy* property

$$\min(a_1,\dots,a_n)\leqslant\mathscr{M}(a_1,\dots,a_n)\leqslant\max(a_1,\dots,a_n). \tag{2}$$

Internality is a very natural property and indicates the requirement that a mean is a real number somewhere between the minimum and the maximum value of the original tuple. Internality is the absolute property that a function must satisfy in order to be a mean, and many authors define a proper mean by using only (2) (see [7], [18]).

Another natural property of a mean function $\mathscr{M}$ is *symmetry*. So, if $\mathscr{M}$ is symmetric in the sense that

$$\mathscr{M}(a_1,\dots,a_n)=\mathscr{M}(\hat{a}_1,\dots,\hat{a}_n)$$

for all permutations $(\hat{a}_1,\dots,\hat{a}_n)$, then we may think of $\mathscr{M}$ as defined not only on ordered tuples, but also on data sets. These are sets in which repetition is allowed and of course like all sets the order is not important.

Besides symmetry, another property of a mean function $\mathscr{M}$ that is also considered in the literature is *1-homogeneity*:

$$\mathscr{M}(ta_1,\cdots,ta_n)=t\mathscr{M}(a_1,\cdots,a_n),$$

for any real t. It is interesting to note that for a mean function $\mathscr{M}$, homogeneity is always equivalent to 1-homogeneity. To see this, observe that from (2) and for $a_1 = \ldots = a_n = a$ we get $\mathscr{M}(a,\ldots,a) = a$ and likewise $\mathscr{M}(ta,\ldots,ta) = ta$. Thus $\mathscr{M}$ cannot be p-homogeneous for $p \neq 1$ and at the same time have the internality property.

Let us also note that all famous means that are considered in the literature satisfy these three basic properties, but it must be also stressed that there is no general accepted definition of a "proper" mean among the authors (see [5]).

For a two-variable mean $\mathscr{M} : \mathbb{J} \times \mathbb{J} \to \mathbb{J}$, the three properties specialize to:

$$\begin{aligned}
\mathscr{M}(a,b) &\in [a,b] && \forall [a,b] \subseteq \mathbb{J} && \text{internality,}\\
\mathscr{M}(a,b) &= \mathscr{M}(b,a) && \forall a,b \in \mathbb{J} && \text{symmetry,}\\
\mathscr{M}(ta,tb) &= t\mathscr{M}(a,b) && \forall [a,b] \subseteq \mathbb{J}, t \in \mathbb{R} \text{ with } [ta,tb] \subseteq \mathbb{J} && \text{homogeneity.}
\end{aligned}$$

Two-variable means have been also considered and discussed by many authors, see e.g. [6], [12], [13], [14], [17], [18]. In [7] a natural way of producing means by using distances is described. A *distance* d on a set $\mathbb{J}$ is defined to be a function $d : \mathbb{J} \times \mathbb{J} \to [0,\infty)$ such that

$$\begin{aligned}
d(a,b) &= d(b,a) && \forall a,b \in \mathbb{J},\\
d(a,b) &= 0 \iff a = b && \forall a,b \in \mathbb{J}.
\end{aligned}$$

Now, according to [7], a natural way to produce a two-variable mean $\mathscr{M}$ on $\mathbb{J}$ is to find an appropriate distance d on $\mathbb{J}$ such that for every closed subinterval $[a,b] \subseteq \mathbb{J}$, the function

$$f(x) = d(a,x) + d(x,b)$$

attains its minimum at a unique point x^* in the interval $[a,b]$. A distance having this property is called a *mean-producing distance*. If the point x^* at which f attains its minimum is denoted by

$$x^* = \mathscr{M}(a,b),$$

then $\mathscr{M}$ is a mean on $\mathbb{J}$ that will be referred to as the *distance mean produced by* d. To put it differently, the mean $\mathscr{M}$ on $\mathbb{J}$ is defined by the requirement that

$$d(a,x) + d(x,b) > d(a,\mathscr{M}(a,b)) + d(\mathscr{M}(a,b),b) \tag{3}$$

for all $x \in [a,b]$ with $x \neq \mathscr{M}(a,b)$, for all closed subintervals $[a,b] \subseteq \mathbb{J}$.

Let us note that in [7] the same idea of producing means is also described more generally for means of several variables. As an example let us consider the interesting case of quasi-arithmetic means of several variables. A quasi-arithmetic mean $\mathscr{M}$ of a data set of n real numbers $(a_1,\ldots,a_n)$ is defined as

$$\mathscr{M}(a_1,\ldots,a_n) = \phi^{-1}\left(\frac{\phi(a_1) + \ldots + \phi(a_n)}{n}\right), \tag{4}$$

where ϕ is a monotone bijection and ϕ^{-1} is its inverse function. We may assume that ϕ is defined on an appropriate set $\mathbb{J} \subseteq \mathbb{R}$ in which any data set of interest is also defined.

Quasi-arithmetic means generalize naturally the concept of the arithmetic mean and many famous means belong to this category (e.g. geometric mean, harmonic mean, see also Section 1).

It is interesting to note that A. Kolmogorov ([11]), by assuming four regularity properties that a mean function $\mathscr{M}$ should have, showed that $\mathscr{M}$ is necessarily a quasi-arithmetic mean. Note that among the regularity properties there are internality and symmetry, but not 1-homogeneity.

Moreover, G. H. Hardy, J. E. Littlewood and G. Pólya showed in [8, p. 68] that if a quasi-arithmetic mean does have 1-homogeneity, it must be the geometric mean or one of the power means. Note that the geometric mean can be considered as a power mean for the limit case $p \to 0$.

Our interest on quasi-arithmetic means stems from the fact that all quasi-arithmetic means are also distance means. The usual distance that produces (4) is given by

$$d(x,y) = (\phi(x) - \phi(y))^2, \tag{5}$$

and the function f we want to minimize is

$$f(x) = \sum_{i=1}^{n} d(a_i,x) = \sum_{i=1}^{n} (\phi(a_i) - \phi(x))^2. \tag{6}$$

It can be easily seen that (4) is the unique minimum of (6).

Now, returning to two-variable means, it is easy to see that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = (x-y)^2$ is a mean producing distance and that it produces the arithmetic mean on $\mathbb{R}$ (see also Section 1). In other words, the arithmetic mean is the distance mean produced by d. Similar statements can be made about geometric, harmonic, and actually about all power means on $\mathbb{P}$. This follows immediately from the considerations on quasi-arithmetic means that we saw.

However, these considerations do not settle the question whether the logarithmic mean $\mathscr{L}$ defined on $\mathbb{P}$ is a distance mean or not. As mentioned in the Introduction, this problem, which appears on page 6 of [7], as Open Problem No. 9, was to defy a great many attempts, and settling it, together with answering or partially answering some other questions in [7], is the main raison d'être of this paper.

3. Main result and the logarithmic mean

From now on we restrict our attention to two-variable means $\mathscr{M}(a,b)$ over any interval $\mathbb{J} \subseteq \mathbb{R}$. We thoroughly present our method in three steps, applying it to the logarithmic mean over $\mathbb{J} := \mathbb{P}$ as an example.

The logarithmic mean appears in physics in several contexts, e.g. for calculating the rate of heat flow along two coaxial cylinders (see [1]). There are many articles which deal with inequalities involving the logarithmic mean and other famous means, see e.g. [1], [2], [4], [10] and [15]. In [17], the logarithmic mean was generalized to a whole family of means, which are now known as *Stolarsky means*. We will examine those in the next section.

For convenience, from now on we use the symbols $\uparrow$ and $\downarrow$ to denote "strictly increasing" and "strictly decreasing", respectively. For functions with multiple arguments, the relevant argument is appended to the arrow. For example, "$\uparrow y$" means "strictly increasing with respect to y". Moreover, we always assume $a \neq b$ unless otherwise stated.

Step 1. Firstly, we want to find a function $e : \mathbb{J} \times \mathbb{J} \to \mathbb{R}$ with the property that for every closed subinterval $[a,b] \subseteq \mathbb{J}$, the restriction of

$$h(x) = e(a,x) + e(x,b)$$

to $[a,b]$ attains its minimum at a unique

$$x^* = \mathscr{M}(a,b)$$

in $[a,b]$. We do not require that $e(x,y)$ is a distance. It may attain negative values and it may even be non-symmetric. Actually, it may be defined only for $x \leqslant y$.

For the logarithmic mean over $\mathbb{P}$, it seems quite natural to introduce, as a possible candidate for $e : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$, the function

$$e(x,y) = x \ln y - y \ln x. \tag{7}$$

To justify this, let us rewrite the definition

$$x^* = \frac{a-b}{\ln a - \ln b}$$

of the logarithmic mean as

$$\frac{a}{x^*} - \ln a - \frac{b}{x^*} + \ln b = 0,$$

which is valid since $x^* > 0$, because all $x > 0$. Now we are looking for a function $f(x)$ such that its derivative is given by

$$f'(x) = \frac{a}{x} - \ln a - \frac{b}{x} + \ln b.$$

By integrating with respect to x, and ignoring the possible constant, we get

$$f(x) = (a \ln x - x \ln a) + (x \ln b - b \ln x),$$

which is nothing but $e(a,x) + e(x,b)$, where e is as in (7). This should explain why we chose $e(x,y)$ as a good candidate. Now, working straightforward, we see that $e(x,y)$ does indeed work.

Thus let

$$h(x) = e(a,x) + e(x,b) = a \ln x - x \ln a + x \ln b - b \ln x.$$

The first derivative of h is given by

$$h'(x) = \frac{a}{x} - \ln a + \ln b - \frac{b}{x}.$$

So the critical points of h are the solutions of $h'(x) = 0$, i.e., the single value

$$x^* = \frac{a-b}{\ln a - \ln b}.$$

This critical point of h is the logarithmic mean that we investigate. Furthermore, the second derivative of h is

$$h'' = \frac{b-a}{x^2} > 0$$

in $[a,b]$, and the function h is strictly convex in $[a,b]$. Thus x^* is the unique minimum of h in $[a,b]$.

Step 2. Secondly, we want to write this function $e(x,y)$ as a symmetric difference of another function $g(x,y)$ which is $\downarrow x$ or $\uparrow y$. So we want

$$e(x,y) = g(x,y) - g(y,x).$$

Let us note here that, equivalently, we could have written

$$e(x,y) = g(y,x) - g(x,y),$$

with g being $\downarrow y$ or $\uparrow x$. To see the equivalence note that

$$e(x,y) = g(x,y) - g(y,x) = [-g(y,x)] - [-g(x,y)].$$

In our example we have $g(x,y) = x \ln y$, which is $\uparrow y$ and so we are fine.

Step 3. Thirdly, suppose that the assumptions in steps 1 and 2 hold. Then there is a way to construct a distance d on $\mathbb{J}$ that produces the mean $\mathscr{M}$, and thus $\mathscr{M}$ is a distance mean produced by d. In this final step, we will prove this fact. So let

$$e(x,y) = g(x,y) - g(y,x),$$

and we have the next two cases:

(A) If g is $\uparrow y$, we use as a distance

$$d(x,y) = |g(x,y) - g(x,x)| + |g(y,x) - g(y,y)|.$$

Let us first see that d is indeed a distance. Obviously, $d(x,y) \geqslant 0$ and the strict monotonicity of g in its second argument guarantees that

$$d(x,y) = 0 \Longleftrightarrow \Big(g(x,y) = g(x,x) \text{ and } g(y,x) = g(y,y)\Big) \Longleftrightarrow x = y.$$

Next, for each $x \in [a,b]$ we have

$$\begin{aligned} &d(a,x)+d(x,b)\\ &=g(a,x)-g(a,a)+g(x,x)-g(x,a)+g(x,b)-g(x,x)+g(b,b)-g(b,x)\\ &=g(a,x)-g(x,a)+g(x,b)-g(b,x)+[g(b,b)-g(a,a)]\\ &=e(a,x)+e(x,b)+[g(b,b)-g(a,a)]\\ &=h(x)+[g(b,b)-g(a,a)], \end{aligned}$$

and it is clear that $d(a,x)+d(b,x)$ has the same unique minimum as $h(x)$, at $x^* = \mathscr{M}(a,b)$ in $[a,b]$, because the last term $g(b,b)-g(a,a)$ is just a constant that is independent of x. The proof for this case is complete.

(B) If g is $\downarrow x$, we use as a distance

$$d(x,y) = |g(x,y)-g(y,y)| + |g(y,x)-g(x,x)|,$$

and the proof is analogous to case (A).

By applying the above result to the logarithmic mean function we easily get

$$\begin{aligned} d(x,y) &= |x\ln y - x\ln x| + |y\ln y - y\ln x|\\ &= x|\ln y - \ln x| + y|\ln y - \ln x|\\ &= (x+y)|\ln y - \ln x|, \end{aligned}$$

and we have constructed a distance for the logarithmic mean. Hence, we have shown that the logarithmic mean is a distance mean. This answers affirmatively the open question posed in [7].

The next theorem, which is the main result of the article, summarizes our investigation:

THEOREM 1. *Let $\mathbb{J} \subseteq \mathbb{R}$ be an interval and $\mathscr{M} : \mathbb{J}\times\mathbb{J} \to \mathbb{J}$ be a mean over $\mathbb{J}$. Let $e : \mathbb{J}\times\mathbb{J} \to \mathbb{R}$ be a function such that for any closed subinterval $[a,b] \subseteq \mathbb{J}$ with $a < b$ the function $h : [a,b] \to \mathbb{R}$ defined by $h(x) = e(a,x)+e(x,b)$ attains its minimum at a unique $x^* \in [a,b]$ with $x^* = \mathscr{M}(a,b)$. Let $e(x,y)$ be expressible as*

$$e(x,y) = g(x,y) - g(y,x),$$

where $g : \mathbb{J}\times\mathbb{J} \to \mathbb{R}$ is a function for which $g(x,y)$ is $\downarrow x$ or $\uparrow y$. (Equivalently, we may assume that $e(x,y) = g(y,x)-g(x,y)$ and $g(x,y)$ is $\uparrow x$ or $\downarrow y$.)

Then $\mathscr{M}$ is produced by the distance $d : \mathbb{J}\times\mathbb{J} \to [0,\infty)$ defined by

1. *$d(x,y) = |g(x,y)-g(x,x)| + |g(y,x)-g(y,y)|$, if $g(x,y)$ is $\uparrow y$, or*
2. *$d(x,y) = |g(x,y)-g(y,y)| + |g(y,x)-g(x,x)|$, if $g(x,y)$ is $\downarrow x$.*

One might claim that the above assumptions are quite strict, but as we will see in the next sections they are not. Theorem 1 will help us to find appropriate new distances for famous means, where among them are the Stolarsky means which appeared in [17]. This Section ends up by applying Theorem 1 to find a distance that produces the geometric mean.

Let
$$g(x,y) = \frac{y}{x}$$
for $x,y > 0$. Then g is $\uparrow y$ and we have that
$$e(x,y) = g(x,y) - g(y,x) = \frac{y}{x} - \frac{x}{y},$$
and for $0 < a < b$,
$$h(x) = e(a,x) + e(x,b) = \frac{x}{a} - \frac{a}{x} + \frac{b}{x} - \frac{x}{b}.$$
Thus,
$$h'(x) = \frac{1}{a} + \frac{a}{x^2} - \frac{b}{x^2} - \frac{1}{b},$$
and for $h'(x^*) = 0$ we get $x^* = \sqrt{ab}$, i.e. the geometric mean. The second derivative is
$$h''(x) = \frac{2(b-a)}{x^2} > 0.$$

So, the geometric mean $\sqrt{ab}$ is the unique minimum of h in $[a,b]$. From Theorem 1 we get that the distance defined by
$$d(x,y) = |g(x,y) - g(x,x)| + |g(y,y) - g(y,x)| = \left|\frac{x}{y} - \frac{y}{x}\right| = \left(\frac{1}{x} + \frac{1}{y}\right)|y - x|$$
produces the geometric mean.

Since $\sqrt{ab}$ is a quasi-arithmetic mean, it is also a distance mean. The known distance that produces it is given by equation (5), namely, $d(x,y) = (\ln x - \ln y)^2$. Note how our distance is very different from the known distance. We will further investigate this phenomenon in Section 5.

4. Distances of MVT-constructed means

In this section we will investigate the family of means which can be constructed using Cauchy's Mean Value Theorem (MVT). We will provide a partial answer to the Open Problem 12 in [7]. Under certain conditions, Theorem 1 can be applied to generate distances which produce those means. Our approach here is to repeat the construction of those means, and then to reuse parts of their construction to generate respective distance functions.

We start with two functions $s,u : \mathbb{J} \to \mathbb{R}$ which are continuously differentiable. We further assume that for all $x \in \mathbb{J}$ we have $u'(x) \neq 0$. Finally, we assume that the continuous function r defined by

$$r(x) = \frac{s'(x)}{u'(x)}$$

is a bijection $r : \mathbb{J} \to \mathbb{B}$ to some arbitrary subset $\mathbb{B} \subseteq \mathbb{R}$. For such functions s,u let us apply Cauchy's MVT to a closed subinterval $[a,b] \subseteq \mathbb{J}$ with $a < b$. Since u' is continuous and nonzero over the whole interval $\mathbb{J}$, it follows that u' is either positive everywhere, or negative everywhere. Hence, u is either $\uparrow$ or $\downarrow$, thus $u(a) \neq u(b)$. We then get that there is a $c \in (a,b)$ such that

$$r(c) = \frac{s'(c)}{u'(c)} = \frac{s(a)-s(b)}{u(a)-u(b)}. \tag{8}$$

Note that in particular, Cauchy's MVT guarantees that the right-hand side is always in the image of r, that is,

$$\frac{s(a)-s(b)}{u(a)-u(b)} \in \mathbb{B}.$$

This is why we don't need to make any assumptions on $\mathbb{B}$, apart from being a subset of $\mathbb{R}$.

Applying the inverse of r, which is $r^{-1} : \mathbb{B} \to \mathbb{J}$, it follows that

$$c = r^{-1}\left(\frac{s(a)-s(b)}{u(a)-u(b)}\right).$$

So c is unique and we even have a closed form for it. Finally, we want to define our mean function $\mathscr{M} : \mathbb{J} \times \mathbb{J} \to \mathbb{J}$ to be exactly that c. However, so far we defined $\mathscr{M}(a,b)$ only for $a < b$ and ensured the internality property for that case. For $b < a$ we observe that the above form is symmetric in a and b, so we just use it also for $b < a$ and notice that $\mathscr{M}(a,b) = \mathscr{M}(b,a)$. For $a = b$ we define $\mathscr{M}(a,a) = a$, as this is the only possible choice that preserves the internality property.

In summary, for each pair of functions (s,u) we obtain a function $\mathscr{M} : \mathbb{J} \times \mathbb{J} \to \mathbb{J}$ with

$$\mathscr{M}(a,b) = \begin{cases} r^{-1}\left(\frac{s(a)-s(b)}{u(a)-u(b)}\right) & a \neq b \\ a & a = b \end{cases} \tag{9}$$

that satisfies both the internality and the symmetry property and so is indeed a mean function.

The idea of using a MVT to describe a family of means is not new. In [7], Cauchy's MVT was also used as a method of generating new means. In [17] the MVT was used to create new means and specifically a class that generalizes the logarithmic mean, and in [6] and [14] the MVT for integrals was used to generate another family of means.

Let us next try to apply Theorem 1 to this family of means. We want to apply it to

$$g(x,y) = s(x)u(y),$$

because as we will see its corresponding h function has its critical point exactly at $\mathscr{M}(a,b)$. But before we can do that, we need to ensure that $g(x,y)$ is either $\downarrow x$ or $\uparrow y$ in $\mathbb{J}$. This requires some preparation.

First, note that if we add a constant value to $s(x)$, equation (8) still holds, for the same value of c. The constant vanishes in the difference $s(a)-s(b)$ as well as in the derivative $s'(c)$. Moreover, reversing the sign of $s(x)$ won't change the resulting mean, since this also reverses the sign of $s(a)-s(b)$ and at the same time reverses the sign of $s'(c)$, so (8) still holds for the same c. Those changes can also be applied to $u(x)$ without changing the resulting mean function $\mathscr{M}(a,b)$ in any way.

Second, let us remember that u is either $\uparrow$ or $\downarrow$. Without loss of generality we may assume that u is $\uparrow$. Otherwise, we just reverse the sign of u and arrive at the same mean function $\mathscr{M}(a,b)$.

Reconsidering $g(x,y)=s(x)u(y)$, we are pleasantly surprised that the property of g being $\uparrow y$ is almost given. We just need to ensure that $s(x)>0$ for all $x\in\mathbb{J}$. Note that it is sufficient for s to have any upper or lower bound. If s has a lower bound, i.e. if we know that $s(x)>s_B$ for some perhaps negative $s_B\in\mathbb{R}$, we just use instead $s(x)-s_B$, moving s to the positive range. If s has an upper bound, we first reverse its sign and then move it up as needed. Either way, we arrive at a new $s(x)>0$ without altering the mean function $\mathscr{M}(a,b)$.

Let us now analyze the h function, where

$$\begin{aligned} h(x) &= e(a,x)+e(x,b) \\ &= g(a,x)-g(x,a)+g(x,b)-g(b,x) \\ &= s(a)u(x)-s(x)u(a)+s(x)u(b)-s(b)u(x) \\ &= s(x)\Big(u(b)-u(a)\Big)-u(x)\Big(s(b)-s(a)\Big). \end{aligned}$$

The first derivative of h is

$$h'(x)=s'(x)\Big(u(b)-u(a)\Big)-u'(x)\Big(s(b)-s(a)\Big).$$

The critical points are given by $h'(x^*)=0$, which is equivalent to (8), which has the unique solution $x^*=\mathscr{M}(a,b)$. So $h:[a,b]\to\mathbb{R}$ is continuously differentiable and has exactly one critical point. If we demand that $\mathscr{M}(a,b)$ is a local minimum we can finally apply Theorem 1 and arrive at the distance function

$$\begin{aligned} d(x,y) &= |g(x,y)-g(x,x)|+|g(y,x)-g(y,y)| \\ &= |s(x)u(y)-s(x)u(x)|+|s(y)u(x)-s(y)u(y)|. \end{aligned}$$

Using $s(x)>0$, we simplify this to

$$\begin{aligned} d(x,y) &= s(x)|u(y)-u(x)|+s(y)|u(x)-u(y)| \\ &= \Big(s(x)+s(y)\Big)|u(x)-u(y)|. \end{aligned}$$

There's one final trick to mention here. Sometimes we find s and u for which all assumptions hold, but finally notice that h has a local maximum rather than minimum at $\mathcal{M}(a,b)$. In this case we can try again with s and u switched. Now we need to re-check that s satisfies all requirements we had formerly on u and vice versa. If they do, we will arrive at the same mean function $\mathcal{M}(a,b)$ but the sign of $h(x)$ is reversed, converting the maximum to a minimum. This trick will be used when we analyze Stolarsky means. However, before we get to that, let us summarize everything in a theorem:

THEOREM 2. *Let $\mathbb{J} \subseteq \mathbb{R}$ be an interval and $s,u : \mathbb{J} \to \mathbb{R}$ be continuously differentiable. Let $s_B \in \mathbb{R}$ be such that*

$$\textit{either } s(x) > s_B \textit{ for all } x \in \mathbb{J}, \textit{ or } s(x) < s_B \textit{ for all } x \in \mathbb{J}.$$

For all $x \in \mathbb{J}$, let $u'(x) \neq 0$. Let the function r, defined by

$$r(x) = \frac{s'(x)}{u'(x)},$$

be a bijection $r : \mathbb{J} \to \mathbb{B}$ to a subset $\mathbb{B} \subseteq \mathbb{R}$. Let $s_2, u_2 : \mathbb{J} \to \mathbb{R}$ be given by

$$s_2(x) = \begin{cases} s(x) - s_B & \text{if } s(z) > s_B \text{ for all } z \in \mathbb{J} \\ -s(x) + s_B & \text{if } s(z) < s_B \text{ for all } z \in \mathbb{J} \end{cases} \quad \text{and}$$

$$u_2(x) = \begin{cases} u(x) & \text{if } u \text{ is } \uparrow \\ -u(x) & \text{if } u \text{ is } \downarrow \end{cases}.$$

For every $[a,b] \subseteq \mathbb{J}$ with $a < b$, let the function $h : [a,b] \to \mathbb{R}$, defined by

$$h(x) = s_2(x)\Big(u_2(b) - u_2(a)\Big) - u_2(x)\Big(s_2(b) - s_2(a)\Big),$$

have a local minimum at its one critical point.

Then the function $\mathcal{M} : \mathbb{J} \times \mathbb{J} \to \mathbb{J}$ with

$$\mathcal{M}(a,b) = \begin{cases} r^{-1}\left(\frac{s(a)-s(b)}{u(a)-u(b)}\right) & a \neq b \\ a & a = b \end{cases}$$

is a mean that is produced by the distance $d : \mathbb{J} \times \mathbb{J} \to [0,\infty)$ defined by

$$d(x,y) = \Big(s_2(x) + s_2(y)\Big)|u_2(x) - u_2(y)|,$$

which can be obtained from Theorem 1 by

$$g(x,y) = s_2(x)u_2(y).$$

In [17] the following family of means, which generalize the logarithmic mean, was presented (called Stolarsky means):

$$S_p : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$$

$$S_p(a,b) = \begin{cases} \left(\frac{a^p - b^p}{p(a-b)}\right)^{\frac{1}{p-1}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases},$$

where the parameter p is a real number distinct from 0 and 1. We stress that Stolarsky means are not generally quasi-arithmetic means. This can be seen by numerical experimentation using the functional equation (1). However, for specific values of p we get some already known quasi-arithmetic means, e.g. for $p=2$ we get the arithmetic mean, for $p=-1$ we get the geometric mean and for $p=1/2$ we get the corresponding power mean.

Let us now use Theorem 2 to prove that all Stolarsky means are distance means, where we must remember that we now work in $\mathbb{J}=\mathbb{P}$:

Case 1. For $p>1$ or $p<0$ we choose

$$s(x)=x^p \text{ and } u(x)=x.$$

Both are continuously differentiable and

$$u'(x)=1\neq 0.$$

Since $s(x)>0$ and u is $\uparrow$, we have

$$s_2=s \text{ and } u_2=u.$$

Also,

$$r(x)=\frac{s'(x)}{u'(x)}=px^{p-1}$$

and is thus a bijection with

$$r^{-1}(x)=\left(\frac{x}{p}\right)^{\frac{1}{p-1}}.$$

Expanding

$$\mathscr{M}(a,b)=r^{-1}\left(\frac{s(a)-s(b)}{u(a)-u(b)}\right)=\left(\frac{a^p-b^p}{p(a-b)}\right)^{\frac{1}{p-1}}=S_p(a,b),$$

we verify to have arrived indeed at the intended mean. Moreover,

$$h(x)=x^p(b-a)-x(b^p-a^p) \quad \text{and}$$
$$h''(x)=p(p-1)x^{p-2}(b-a)>0,$$

so h is convex and its critical point is a minimum. The application of Theorem 2 gives the distance

$$d(x,y)=(x^p+y^p)|x-y|$$

that produces $S_p(a,b)$.

Case 2. For $0 < p < 1$, the previous approach would lead to $h''(x) < 0$, so we use the aforementioned trick and choose instead

$$s(x) = x \text{ and } u(x) = x^p.$$

We verify that

$$u'(x) = px^{p-1} \neq 0,\ s(x) > 0,\ \text{ and } u \text{ is } \uparrow,$$

so again we have

$$s_2 = s \text{ and } u_2 = u.$$

Note that

$$r(x) = \frac{s'(x)}{u'(x)} = \frac{1}{px^{p-1}}$$

changed to its reciprocal, but is still a bijection with

$$r^{-1}(x) = \left(\frac{1}{px}\right)^{\frac{1}{p-1}}.$$

Expanding

$$\mathscr{M}(a,b) = r^{-1}\left(\frac{s(a)-s(b)}{u(a)-u(b)}\right) = \left(\frac{a^p-b^p}{p(a-b)}\right)^{\frac{1}{p-1}} = S_p(a,b),$$

we see that we again arrived at the intended mean. Moreover,

$$h(x) = x(b^p - a^p) - x^p(b-a),$$

so we indeed managed to reverse its sign. Now

$$h''(x) = -p(p-1)x^{p-2}(b-a) > 0,$$

so h is convex and its critical point is a minimum. The application of Theorem 2 gives the distance

$$d(x,y) = (x+y)|x^p - y^p|$$

that produces $S_p(a,b)$.

Case 3. For $p \to 0$ Stolarsky noted that its mean approaches the logarithmic mean, which we have already proved to be a distance mean with

$$d(x,y) = (x+y)|\ln y - \ln x|.$$

Alternatively, we could have constructed the same distance by applying Theorem 2 to

$$s(x) = x \text{ and } u(x) = \ln(x).$$

Case 4. For $p \to 1$ the Stolarsky mean becomes the identric mean:

$$S_1(a,b) = \begin{cases} \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/a-b} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}.$$

Observe that the identric mean is not a quasi-arithmetic mean (we can see this by experimenting again with the functional equation (1)). We can rewrite S_1 as

$$S_1(a,b) = \begin{cases} \exp\left(\frac{a(\ln a-1)-b(\ln b-1)}{a-b}\right) & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}$$

and this form of S_1 resembles (9). So we choose

$$s(x) = x(\ln x - 1) \text{ and } u(x) = x.$$

Both are continuously differentiable and

$$u'(x) = 1 \neq 0.$$

Since u is $\uparrow$, we have

$$u_2 = u.$$

But $s(x)$ can attain negative values, so we need to shift it. This requires some analysis on s. Since

$$s'(x) = \ln x,$$

it has exactly one critical point at $x^* = 1$. Moreover,

$$s''(x) = \frac{1}{x} > 0,$$

so s is convex and attains its minimum at x^*, where $s(x^*) = s(1) = -1$. However, we need our lower bound s_B to be strictly less than $s(x)$, so we choose $s_B = -2$. Hence

$$s_2(x) = s(x) - s_B = x(\ln x - 1) + 2 > 0.$$

Now we continue with

$$r(x) = \frac{s'(x)}{u'(x)} = \ln x,$$

which is a bijection with $r^{-1}(x) = e^x$. Expanding

$$\mathscr{M}(a,b) = r^{-1}\left(\frac{s(a)-s(b)}{u(a)-u(b)}\right) = \exp\left(\frac{a(\ln a - 1) - b(\ln b - 1)}{a-b}\right) = S_1(a,b),$$

we verify that we arrived indeed at the identric mean. Moreover,

$$\begin{aligned} h(x) &= \Big(x(\ln x - 1) + 2\Big)(b-a) - x\Big(b(\ln b - 1) - a(\ln a - 1)\Big), \\ h'(x) &= (\ln x)(b-a) - \Big(b(\ln b - 1) - a(\ln a - 1)\Big), \\ h''(x) &= \frac{b-a}{x} > 0, \end{aligned}$$

so h is convex and its critical point is a minimum. Theorem 2 now provides the distance

$$d(x,y) = \Big(x(\ln x - 1) + y(\ln y - 1) + 4\Big)|x-y|$$

that produces $S_1(a,b)$.

In all cases we have found a family of distances that produce the Stolarsky means.

5. Alternative distances and quasi-$\mathscr{M}$ means

In this section we will use Theorem 1 to find distances for the arithmetic mean and many other quasi-arithmetic means. While we have seen that those are distance means (see Sections 1 and 2), we stress that our method does not just reproduce those results, but generates different means from those already known. Moreover, we will introduce quasi-arithmetic means through the more general concept of *quasi-$\mathscr{M}$ means*, whose utility we will demonstrate by quickly generalizing our distance for the logarithmic mean to a whole class of *quasi-logarithmic means*.

First, let us use Theorem 1 to create a distance for the arithmetic mean. We start with $\mathbb{J} = \mathbb{R}$ and

$$g(x,y) = x^2 y,$$

so g is $\uparrow y$. Then

$$e(x,y) = g(x,y) - g(y,x) = x^2 y - y^2 x,$$

and in $[a,b]$ we want to minimize the function

$$h(x) = e(a,x) + e(x,b) = (b-a)\left(\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2\right).$$

This convex parabola attains its minimum at the unique

$$x^* = \frac{a+b}{2} = \mathscr{M}(a,b),$$

which is the arithmetic mean. From Theorem 1 we can construct the distance through

$$d(x,y) = |g(x,y) - g(x,x)| + |g(y,x) - g(y,y)|,$$

which yields

$$d(x,y) = (x^2 + y^2)|y - x|.$$

Note that this new distance is quite different from the usual distance for the arithmetic mean, i.e. $d(x,y) = (x-y)^2$, see also equation (1).

However, this new distance does equal the distance that we constructed in Section 4 for Stolarsky mean S_p with $p = 2$. This was to be expected, since S_2 is indeed the arithmetic mean, and moreover we constructed the distance from the same function g. However, we considered Stolarsky means only over $\mathbb{J} = \mathbb{P}$, while we now know that in the case of the arithmetic mean this distance works for all of $\mathbb{R}$.

We have seen that a quasi-arithmetic mean $\mathscr{M}$ in two variables can be expressed as

$$\mathscr{M}(a,b) = q^{-1}\left(\frac{q(a)+q(b)}{2}\right),$$

where q is a monotone bijection and q^{-1} is its inverse function. Quasi-arithmetic means were discussed in Sections 1 and 2, where it was stressed that they are distance means. The definition of quasi-arithmetic means as well as their distances generalize to any number of variables in the natural way (see equations (4), (5) and (6)), but here we are interested in a different direction of generalization.

Let $\mathbb{J} \subseteq \mathbb{R}$ be an interval,

$$\mathscr{M} : \mathbb{J} \times \mathbb{J} \to \mathbb{J}$$

be an arbitrary mean function over $\mathbb{J}$ and

$$q : \mathbb{J}_q \to \hat{\mathbb{J}}$$

be a monotone bijection from a possibly different interval $\mathbb{J}_q$ to a subinterval $\hat{\mathbb{J}} \subseteq \mathbb{J}$. In analogy to quasi-arithmetic means, we want to define the *quasi-$\mathscr{M}$ mean* of $\mathscr{M}$ with respect to q to be the function $\mathscr{M}_q : \mathbb{J}_q \times \mathbb{J}_q \to \mathbb{J}_q$ with

$$\mathscr{M}_q(a,b) = q^{-1}\Big(\mathscr{M}(q(a),q(b))\Big).$$

To justify this definition, we will prove that $\mathscr{M}_q$ is well-defined, symmetric and has the internality property. Note that $\hat{\mathbb{J}} \subseteq \mathbb{J}$ being a subinterval, rather than an arbitrary subset, plays a vital role here. Since q is a monotone bijection between intervals, both q and q^{-1} are strictly monotone functions. Let

$$a,b \in \mathbb{J}_q,$$

so

$$q(a),q(b) \in \hat{\mathbb{J}}.$$

Let $(\bar{a},\bar{b})$ and $(q(\hat{a}),q(\hat{b}))$ be the increasingly ordered versions of those pairs, that is,

$$\bar{a} = \min(a,b),\ \bar{b} = \max(a,b) \text{ and}$$

$$(\hat{a},\hat{b}) = \begin{cases} (\bar{a},\bar{b}) & \text{if } q \text{ is } \uparrow \\ (\bar{b},\bar{a}) & \text{if } q \text{ is } \downarrow \end{cases}.$$

Since $\hat{\mathbb{J}}$ is an interval and $\mathscr{M}$ has the internality property, it follows that

$$\mathscr{M}(q(a),q(b)) \in [q(\hat{a}),q(\hat{b})] \subseteq \hat{\mathbb{J}}.$$

Thus we can apply q^{-1} to that term, showing that $\mathscr{M}_q(a,b)$ is well-defined. Since q^{-1} is strictly monotone, it follows that

$$\mathscr{M}_q(a,b) \in q^{-1}([q(\hat{a}),q(\hat{b})]) = [\bar{a},\bar{b}],$$

so $\mathscr{M}_q$ has the internality property. Finally, the symmetry of $\mathscr{M}_q$ follows directly from the symmetry of $\mathscr{M}$.

Note that if $\hat{\mathbb{J}} = \mathbb{J}$, the relation between $\mathscr{M}$ and $\mathscr{M}_q$ is interchangeable. For example, the geometric mean is a quasi-arithmetic mean with $q(x) = \ln x$, where $\mathbb{J}_q = \mathbb{P}$ and $\hat{\mathbb{J}} = \mathbb{J} = \mathbb{R}$, and conversely the arithmetic mean is a quasi-geometric mean with $q(x) = e^x$, where $\mathbb{J}_q = \mathbb{R}$ and $\hat{\mathbb{J}} = \mathbb{J} = \mathbb{P}$.

Let $d : \mathbb{J} \times \mathbb{J} \to [0,\infty)$ be a distance that produces $\mathscr{M}$. Using (3), this implies

$$d(a',x') + d(x',b') > d(a',\mathscr{M}(a',b')) + d(\mathscr{M}(a',b'),b') \tag{10}$$

for every closed subinterval $[a',b'] \subseteq \mathbb{J}$ and $x' \in [a',b']$ with $x' \neq \mathscr{M}(a',b')$.

We now want to use d to create a distance d_q for $\mathscr{M}_q$. So let

$$x \in [a,b] \subseteq \mathbb{J}_q, \text{ with } x \neq \mathscr{M}_q(a,b).$$

If q is $\uparrow$, then

$$q(a) \leqslant q(x) \leqslant q(b), \text{ and } q(x) \neq q(\mathscr{M}_q(a,b)).$$

Using the definition of $\mathscr{M}_q$, the latter is equivalent to

$$q(x) \neq \mathscr{M}(q(a),q(b)).$$

This means that we are allowed to apply (10) to

$$a' = q(a),\ x' = q(x),\ b' = q(b),$$

which leads to

$$d(q(a),q(x)) + d(q(x),q(b)) > d(q(a),\mathscr{M}(q(a),q(b))) + d(\mathscr{M}(q(a),q(b)),q(b)).$$

Applying the definition of $\mathscr{M}_q$, we get

$$d(q(a),q(x)) + d(q(x),q(b)) > d(q(a),q(\mathscr{M}_q(a,b))) + d(q(\mathscr{M}_q(a,b)),q(b)).$$

So we define $d_q : \mathbb{J}_q \times \mathbb{J}_q \to [0,\infty)$ to be

$$d_q(x,y) = d(q(x),q(y))$$

and arrive at

$$d_q(a,x) + d_q(x,b) > d_q(a,\mathscr{M}_q(a,b)) + d_q(\mathscr{M}_q(a,b),b).$$

Since d is a distance, it is clear that d_q is symmetric and that

$$d_q(x,y) = 0 \Leftrightarrow q(x) = q(y).$$

The latter is equivalent to $x = y$ because q is $\uparrow$. So d_q is really a distance, and according to (3), d_q produces $\mathscr{M}_q$.

If q is $\downarrow$, we use instead

$$a' = q(b),\ x' = q(x),\ b' = q(a)$$

and arrive at the same conclusion thanks to the symmetry of d and $\mathscr{M}$.

There is an interesting connection with Theorem 1 here: Assume that d was constructed from g via Theorem 1. There, $d(x,y)$ was constructed purely in terms of $g(x,y)$, without using x or y directly. So if we start instead with

$$g_q(x,y) = g(q(x),q(y)),$$

we arrive at the distance function

$$d_q(x,y) = d(q(x),q(y)).$$

The following theorem summarizes our results.

THEOREM 3. *Let $\mathbb{J} \subseteq \mathbb{R}$ be an interval. Let $\mathscr{M} : \mathbb{J} \times \mathbb{J} \to \mathbb{J}$ be a mean that is produced by the distance $d : \mathbb{J} \times \mathbb{J} \to [0,\infty)$. Let $\mathbb{J}_q \subseteq \mathbb{R}$ be an interval, $\hat{\mathbb{J}} \subseteq \mathbb{J}$ be a subinterval and $q : \mathbb{J}_q \to \hat{\mathbb{J}}$ be a monotone bijection.*

Then the function $\mathscr{M}_q : \mathbb{J}_q \times \mathbb{J}_q \to \mathbb{J}_q$ given by

$$\mathscr{M}_q(a,b) = q^{-1}(\mathscr{M}(q(a),q(b)))$$

is a mean that is produced by the distance $d_q : \mathbb{J}_q \times \mathbb{J}_q \to [0,\infty)$ defined by

$$d_q(x,y) = d(q(x),q(y)).$$

If d was constructed from g via Theorem 1, then d_q can be constructed from

$$g_q(x,y) = g(q(x),q(y)).$$

If we apply Theorem 3 to our alternative distance for the arithmetic mean, we get the following new distances for all quasi-arithmetic means, which are different from the usual distances that produce them (see Section 2, equation (5)):

$$\begin{aligned} \mathscr{M}_q(a,b) &= q^{-1}\left(\frac{q(a)+q(b)}{2}\right), \\ d_q(x,y) &= \left(q(x)^2 + q(y)^2\right)|q(y)-q(x)|. \end{aligned}$$

For $q(x) = \ln x$, we get the following distance for the geometric mean, which is also different from the distance that was found at the end of Section 3:

$$\begin{aligned} \mathscr{M}(a,b) &= \sqrt{ab}, \\ d(x,y) &= \left((\ln x)^2 + (\ln y)^2\right)|\ln y - \ln x|. \end{aligned}$$

For $q(x) = x^p$, we get the following alternative distances for the power means:

$$\mathscr{M}(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}},$$
$$d(x,y) = \left(x^{2p} + y^{2p}\right)|y^p - x^p|.$$

Finally, let us apply Theorem 3 to the logarithmic mean. This leads to a new family of distances for *quasi-logarithmic means*:

$$\mathscr{M}_q(a,b) = q^{-1}\left(\frac{q(a) - q(b)}{\ln q(a) - \ln q(b)}\right),$$
$$d_q(x,y) = \Big(q(x) + q(y)\Big)|\ln q(y) - \ln q(x)|.$$

For $q(x) = e^x$ we get the following interesting distance-produced mean that is defined on whole $\mathbb{R}$:

$$\mathscr{M}(a,b) = \ln(e^a - e^b) - \ln(a - b),$$
$$d(x,y) = (e^x + e^y)|y - x|.$$

For $q(x) = x^p$ with $p \in \mathbb{R}$ and $p \neq 0$ we get the following family of distance-produced means that are defined on $\mathbb{P}$:

$$\mathscr{M}(a,b) = \left(\frac{a^p - b^p}{p(\ln a - \ln b)}\right)^{\frac{1}{p}},$$
$$d(x,y) = (x^p + y^p)|p(\ln y - \ln x)|.$$

6. Conclusions

We described a methodology that can be used to construct new distances which produce many famous means. Our method is summarized in Theorem 1 and it can be specialized in different directions. We showed in Theorem 2 how to construct distances for almost all means that can be obtained from Cauchy's Mean Value Theorem. Furthermore, we proved in Theorem 3 how to construct a distance for a mean $\mathscr{M}_q(a,b) = q^{-1}(\mathscr{M}(q(a),q(b)))$, where $\mathscr{M}$ is another mean for which a distance is already known, and q is a monotone bijection to a subinterval. Interested readers may find other possible specializations and applications of our method, and may also try to construct distances that produce other known or less known means.

We will close with a few open problems:

1. It may be worth investigating the following class of Cauchy's means, which are also introduced in [17] and examined further in [12]:

$$E(a,b;p,q) = \left(\frac{q(a^p - b^p)}{p(a^q - b^q)}\right)^{\frac{1}{p-q}},$$

for reals p, q where $p,q \neq 0$ and $p \neq q$, and for $a,b > 0$. Observe that for $p = 1$ or $q = 1$ these means give the family of Stolarsky means which we have already analyzed in Section 4.

2. The contraharmonic mean $\mathscr{M} : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$, which is given by

$$\mathscr{M}(a,b) = \frac{a^2+b^2}{a+b},$$

showed a surprising resistance against our method, despite being a very simple mean function. We failed to construct a distance for this mean using any of our theorems. It would be interesting to know if we missed an important specialization of Theorem 1, or if the contraharmonic mean is actually not a distance-produced mean at all.

3. We demonstrated that the same mean can have very different distance functions. For example, the arithmetic mean is produced by the distances

$$d(x,y) = (x^2+y^2)|y-x| \qquad \text{and} \qquad d(x,y) = (x-y)^2.$$

Do these new examples give rise to even more alternative distances? Does this get us closer to a classification of the family of distances that produce the same mean?

REFERENCES

[1] R. BHATIA, *The logarithmic mean*, Resonance, **13** (2008), 583–594.
[2] I. BRUCE, *The logarithmic mean*, Math. Gaz., **81** (1997), 89–92.
[3] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2003.
[4] B. C. CARLSON, *The logarithmic mean*, Amer. Math. Monthly, **79** (1972), 615–618.
[5] E. L. DODD, *The substitutive mean and certain subclasses of this general mean*, Ann. Math. Statist., **11** (1940), 163–176.
[6] B. EBANKS, *Looking for a few good means*, Amer. Math. Monthly, **119** (2012), 658–669.
[7] M. HAJJA, *Some elementary aspects of means*, Int. J. Math. Math. Sci. (2013) Article ID 689560.
[8] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[9] J. HENDERSON (editor), *Greek Mathematics I*, translated by I. Thomas, Harvard University Press, Cambridge, 2006.
[10] G. JAMESON AND P. R. MERCER, *The logarithmic mean revisited*, Amer. Math. Monthly, **126** (2019), 641–645.
[11] A. KOLMOGOROV, *Sur la notion de la moyenne*, Atti Accad. Naz. Lincei Rend., **12** (1930), 388–391.
[12] E. B. LEACH AND M. C. SHOLANDER, *Extended mean values*, Amer. Math. Monthly, **85** (1978), 84–90.
[13] D. H. LEHMER, *On the compounding of certain means*, J. Math. Anal. Appl., **36** (1971), 183–200.
[14] M. E. MAYS, *Functions which parametrize means*, Amer. Math. Monthly, **90** (1983), 677–683.
[15] P. R. MERCER, *Cauchy's mean value theorem meets the logarithmic mean*, Math. Gaz., **101** (2017), 108–115.
[16] M. L. D'OOGE (editor and translator), *Nichomachus of Gerasa: Introduction to Arithmetic*, N. Y., 1926.

[17] K. B. STOLARSKY, *Generalizations of the logarithmic mean*, Math. Mag., **48** (1975), 87–92.
[18] ZH.-H. YANG *Three families of two-parameter means constructed by trigonometric functions*, J. Inequal. Appl. (2013) Article 541.

(Received July 11, 2020)

Volker Diels-Grabsch
m-click.aero GmbH
Friedrichstr. 79, 10117 Berlin – Germany
https://njh.eu
e-mail: v@njh.eu

Mowaffaq Hajja
Department of Basic Sciences and Mathematics
Philadelphia University
P. O. Box 1, 13932 Amman, Jordan
e-mail: mowhajja1234@gmail.com

Panagiotis T. Krasopoulos
Department of Informatics, KEAO
Electronic National Social Security Fund
12 Patision St., 10677 Athens, Greece
e-mail: pan_kras@yahoo.gr