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INEQUALITIES FROM LORENTZ–FINSLER NORMS

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Abstract. We show that Lorentz-Finsler geometry offers a powerful tool in obtaining inequalities. With this aim, we first point out that a series of famous inequalities such as: the (weighted) arithmetic-geometric mean inequality, Aczél's, Popoviciu's and Bellman's inequalities, are all particular cases of a reverse Cauchy-Schwarz, respectively, of a reverse triangle inequality holding in Lorentz-Finsler geometry. Then, we use the same method to prove some completely new inequalities, including two refinements of Aczél's inequality.

1. Introduction

The Cauchy-Schwarz inequality on the Euclidean space $\mathbb{R}^n$:

$$\left(\sum_{i=1}^{n} v_i^2\right) \cdot \left(\sum_{i=1}^{n} w_i^2\right) \geqslant \left(\sum_{i=1}^{n} v_i w_i\right)^2, \tag{1}$$

$\forall v = (v_1, \ldots, v_n),\ w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, is a basic result, with applications in almost all the branches of mathematics.

In 1956, Aczél [2] introduced the following inequality

$$\left(v_0^2 - v_1^2 - \ldots - v_n^2\right)\left(w_0^2 - w_1^2 - \ldots - w_n^2\right) \leqslant \left(v_0 w_0 - v_1 w_1 - \ldots - v_n w_n\right)^2, \tag{2}$$

(holding for all $v = (v_0, v_1, \ldots, v_n)$, $w = (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1}$ such that $v_0^2 - v_1^2 - \ldots - v_n^2 > 0$, $w_0^2 - w_1^2 - \ldots - w_n^2 > 0$), in relation to the theory of functional equations in one variable. The Aczél inequality (2), together with its generalization to Lorentzian manifolds, known by the name of *reverse Cauchy-Schwarz inequality,* proved to be crucial to relativity theory and to theories of physical fields.

Indeed, from a geometric standpoint, the two inequalities above are known to be two sides of the same coin; while the usual Cauchy-Schwarz inequality (1) is extended to positive definite inner product spaces – and further on, to Riemannian manifolds – leading to the triangle inequality $\|v+w\| \leqslant \|v\| + \|w\|$, (2) is naturally extended to spaces with a Lorentzian inner product (and more generally, to Lorentzian manifolds), leading to a reverse triangle inequality (see, e.g., [6, 24]).

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In this article we discuss a further generalization of the above picture which has not been exploited so far. The Cauchy-Schwarz inequality and its Lorentzian-reverse version can be extended to Finsler, [4], respectively, to Lorentz-Finsler spaces, [1, 13, 20, 21]. Roughly speaking, while a Riemannian manifold is a space equipped with a smoothly varying family of inner products, a Finsler manifold is a space equipped with a family of *norms*[1] that do not necessarily arise from a scalar product. Similarly, a Lorentz-Finsler manifold is equipped with a smoothly varying family of so-called *Lorentz-Finsler norms* of vectors, that do not necessarily arise as the square root of any quadratic expression – but are just positively 1-homogeneous in the considered vectors.

Usually, the Finslerian Cauchy-Schwarz inequality (also called the *fundamental inequality,* [4]) is proven under the assumption that the Finsler norm F of vectors has the property that the Hessian $Hess(F^2)$ is positive definite; in the particular case of Riemannian spaces, this will turn into the condition that the metric tensor g is positive definite. Similarly, its Lorentzian-reverse counterpart, [1, 13, 20, 21] is proven under the assumption that $Hess(F^2)$ has Lorentzian signature for all vectors in a strictly convex set. Under these assumptions, the obtained inequalities are *strict*, i.e., equality only holds when the vectors v and w are collinear.

As a preliminary step, we show that the above conditions can be relaxed. Namely, the respective inequalities still hold – just, non-strictly – if we allow $Hess(F^2)$ to be degenerate along some directions; also, for practical matters, we replace the strict convexity assumption on the set of interest with the more relaxed one that F is defined on a convex conic domain $\mathscr{T}$. An extension to the case when F is not smooth is also presented. Moreover, we prove two refinements of the reverse triangle inequality holding in general Lorentz-Finsler spaces in Section 3.4.

While the above generalizations are not spectacular for themselves, they allow us much more freedom in choosing the range of examples and applications. Indeed, we show in Section 4 that some of the most famous inequalities on $\mathbb{R}^n$ are nothing but reverse Cauchy-Schwarz inequalities for conveniently chosen (possibly, degenerate) Lorentz-Finsler norms:

1. The usual arithmetic-geometric mean inequality: $\frac{1}{n}\sum\limits_{i=1}^{n} v_i \geqslant (\prod\limits_{i=1}^{n} v_i)^{1/n}, \quad \forall v_i \geqslant 0,$ $i=\overline{1,n}$.

2. The weighted arithmetic-geometric mean inequality:
$$\frac{1}{a}\sum_{i=1}^{n} a_i v_i \geqslant [(v_1)^{a_1}\dots(v_n)^{a_n}]^{1/a}, \tag{3}$$
for all $a, a_i, v_i \in \mathbb{R}_+^*$, such that $\sum\limits_{i=1}^{n} a_i = a$.

3. Popoviciu's inequality, [30]:
$$\left(v_0^p - v_1^p - \dots - v_n^p\right)^{1/p}\left(w_0^q - w_1^q - \dots - w_n^q\right)^{1/q} \leqslant v_0w_0 - v_1w_1 - \dots - v_nw_n, \tag{4}$$

[1]Actually, the notion of Finsler norm is slightly more general than the usual one, as it is only required to be positively homogeneous instead of absolutely homogeneous.

holding for all $v_i, w_i > 0$, such that $v_0^p - v_1^p - \ldots - v_n^p > 0$ and $w_0^q - w_1^q - \ldots - w_n^q > 0$; the powers $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. In particular, for $p = q = \frac{1}{2}$, Popoviciu's inequality yields Aczél's inequality.

4. Another result, due to Bellman [5, 22, 23]:

$$\left(v_0^p - v_1^p - \ldots - v_n^p\right)^{1/p} + \left(w_0^p - w_1^p - \ldots - w_n^p\right)^{1/p}$$
$$\leqslant [(v_0 + w_0)^p - (v_1 + w_1)^p - \ldots - (v_n + w_n)^p]^{1/p}, \tag{5}$$

(with v_i, w_i as above and $p > 1$) is just the reverse triangle inequality corresponding to (4).

Further, in Sections 4.4 and 4.5, we use two particular classes of Lorentz-Finsler norms (the so-called *bimetric* [25, 26, 31] and (α, β)-metrics (with focus on a particular case, *Kropina* norms [16]), in order to prove some new inequalities.

In Section 5, we prove the following class of inequalities on $\mathbb{R}^{n+1}$:

$$[v_0 w_0 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})]^2 - [v_0^2 - \|\vec{v}\|^2][w_0^2 - \|\vec{w}\|^2] \geqslant 0 \tag{6}$$

(for all $\vec{v} = (v_1, \ldots, v_n), \vec{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $v_0, w_0 > 0$ such that $v_0^2 - \|\vec{v}\|^2 \geqslant 0$, $w_0^2 - \|\vec{w}\|^2 \geqslant 0$), where $\|\vec{v}\| = \bar{F}(\vec{v})$ is an arbitrary Finsler norm on $\mathbb{R}^n$ and $\bar{g}_{\vec{v}} = \frac{1}{2} Hess_{\vec{v}}(\bar{F}^2)$ is the corresponding Finsler metric tensor – thus generalizing the usual Aczél inequality. Using the positive definite version of the Finslerian Cauchy-Schwarz inequality, we then find two refinements thereof:

$$[v_0 w_0 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})]^2 - [v_0^2 - \|\vec{v}\|^2][w_0^2 - \|\vec{w}\|^2] \geqslant \frac{\left(w^0\right)^2 - \|\vec{w}\|^2}{\|\vec{w}\|^2} \left(\|\vec{v}\|^2 \|\vec{w}\|^2 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})\right); \tag{7}$$

$$[v_0 w_0 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})]^2 - [v_0^2 - \|\vec{v}\|^2][w_0^2 - \|\vec{w}\|^2] \geqslant \left[w^0 \frac{\bar{g}_{\vec{v}}(\vec{v}, \vec{w})}{\|\vec{w}\|} - v^0 \|\vec{w}\|\right]^2. \tag{8}$$

2. Reverse inequalities for Lorentzian bilinear forms

Before we study the extended Finslerian case, we briefly recall the classical inequalities for bilinear forms.

Throughout the paper, we denote by V a real $(n+1)$-dimensional space. We will use Einstein's summation convention, if not otherwise explicitly stated: whenever in an expression an index i appears both as a superscript and as a subscript, we will automatically understand summation over all possible values of i, i.e., instead of $\sum_{i=0}^{n} a_i b^i$, we will write simply, $a_i b^i$. This is why, we will typically number components of vectors with superscripts from 0 to n, rather than with subscripts; this way, the expression of a vector $v \in V$ in the basis $\{e_i\}_{i=\overline{0,n}}$ will be written as

$$v = v^i e_i.$$

Unless elsewhere specified, by "smooth", we will mean $\mathscr{C}^\infty$ (though usually, differentiability of some finite order is sufficient). We will denote by $i, j, k\dots$ indices running from 0 to n and by Greek letters $\alpha, \beta, \gamma, \dots,$ indices running from 1 to n.

A *Lorentzian scalar product,* [6, 19, 24], on the $(n+1)$-dimensional vector space V is a symmetric bilinear form $g : V \times V \to \mathbb{R}$ of index n. If V admits a Lorentzian scalar product, it is called an $(n+1)$-dimensional *Minkowski spacetime*. Choosing an arbitrary basis, we have:

$$g(v,v) = g_{ij} v^i v^j, \tag{9}$$

where (g_{ij}) is a matrix with constant entries. In particular, in a g-orthonormal basis of V, the bilinear form g has the expression

$$g(v,v) = \eta_{ij} v^i v^j = \left(v^0\right)^2 - \left(v^1\right)^2 - \ldots - (v^n)^2, \tag{10}$$

where $(\eta_{ij}) = diag(1, -1, -1, \dots, -1)$.

A nonzero vector $v \in V$ is called *timelike* if $g(v,v) > 0$ and *causal*, if $g(v,v) \geqslant 0$. The set of causal vectors consists of two connected components, corresponding to the choices $v^0 > 0$ and $v^0 < 0$ respectively in a given (arbitrary) g-orthonormal basis.

In the following, we will denote by C one of these two connected components. By conveniently choosing the basis, we can assume that, for all $v \in C$, we have $v^0 > 0$. The elements of C are called *future-directed causal vectors.*

Denote:

$$F(v) := \sqrt{g(v,v)}, \quad \forall v \in C. \tag{11}$$

The function $F : C \to \mathbb{R}^+$ defined by the above relation is sometimes called, by analogy with the Euclidean case, the Lorentzian (pseudo-)*norm* associated to the Lorentzian scalar-product g.

We will denote by:

$$\mathscr{T} = \{v \in C \mid F(v) > 0\}, \tag{12}$$

the subset of C consisting of timelike vectors. Elements of $\mathscr{T}$ are called *future-directed timelike vectors*. The set $\mathscr{T}$ is always convex.

On a Minkowski spacetime (V, g), the following inequalities hold (see, e.g., [24, Proposition 30])

- **Reverse Cauchy-Schwarz inequality**:

$$g(v,w) \geqslant F(v)F(w), \quad \forall v, w \in C; \tag{13}$$

- **Reverse triangle inequality:**

$$F(v+w) \geqslant F(v) + F(w), \quad \forall v, w \in C.$$

These inequalities are *strict*, in the sense that equality holds if and only if v and w are collinear.

Particular case (Aczél's inequality). For $V=\mathbb{R}^{n+1}$ equipped with a g-orthonormal basis, the reverse Cauchy-Schwarz inequality

$$(v^0w^0-v^1w^1-\dots.-v^nw^n)^2\geqslant[\left(v^0\right)^2-\left(v^1\right)^2\dots-(v^n)^2][\left(w^0\right)^2-\left(w^1\right)^2\dots-(w^n)^2], \tag{14}$$

$\forall v,w\in C$, becomes Aczél's inequality (2).

REMARK 1. (Positive definite bilinear forms): In the case when the metric g is positive definite, we have: $C=V$, $\mathscr{T}=V\setminus\{0\}$. The usual, non-reverse Cauchy-Schwarz inequality

$$g(v,w)\leqslant F(v)F(w) \tag{15}$$

and the usual triangle inequality:

$$F(v+w)\leqslant F(v)+F(w) \tag{16}$$

hold strictly on the entire space V, see for example [24, Proposition 18].

3. Finsler and Lorentz-Finsler functions on a vector space

We saw in the previous section that the famous reverse and non-reverse Cauchy-Schwarz inequalities, thus in particular Aczél's inequality, are closely connected to the geometric concepts of (pseudo)-Riemannian geometry. Here we show that further famous inequalities are also related to a geometric concept, namely to the concept of *(pseudo)-Finsler geometry*.

3.1. Finsler structures on a vector space

Let V be a real $(n+1)$-dimensional space as above.

A *Finsler* norm on V is "almost" a norm in the usual sense; the difference consists in the fact that it is only *positively* homogeneous, instead of absolutely homogeneous. The precise definition is given below.

DEFINITION 1. ([4]): A Finsler norm on the vector space V is a function $F:V\to[0,\infty)$ with the following properties:

1. F is smooth on $\mathscr{T}=V\setminus\{0\}$ and continuous at $v=0$;
2. F is positively homogeneous of degree 1, i.e., $F(\lambda v)=\lambda F(v)$, $\forall\lambda>0$;
3. For every $v\in\mathscr{T}$, the fundamental tensor $g_v:V\times V\to\mathbb{R}$,

$$g_v(u,w):=\frac{1}{2}\frac{\partial^2F^2}{\partial t\partial s}(v+tu+ws)\,|_{t=s=0} \tag{17}$$

   is positive definite.

In the Finsler geometry literature, a function F with the above properties is called a *Minkowski norm*. Yet, in order to avoid confusions with the *Minkowski metric* η as defined above, we will avoid this terminology here and call F instead, a *Finsler* norm.

Particular case. Euclidean spaces are recovered for $F(v) = \sqrt{a_{ij}v^i v^j}$, where (a_{ij}) is a constant matrix (i.e., $g_v = a$ does not depend on v). In this case, the Finsler norm F is a Euclidean one, since it arises from a scalar product.

Generally, a Finsler norm does not generally arise from a scalar product. Nevertheless, there exists a similar notion to a scalar product – namely, the *fundamental* (or *metric*) *tensor* g_v – but, in general g_v has a nontrivial dependence on the vector v. More precisely, the fundamental tensor of the Finsler space (V,F) is the mapping $g : V\setminus\{0\} \to T_2^0(V)$, $v \mapsto g_v$, which associates to each vector v the symmetric and positive definite bilinear form g_v defined above. With respect to an arbitrary basis $\{e_i\}_{i=\overline{0,n}}$ of V, the fundamental tensor g_v has the matrix:

$$g_{ij}(v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial v^j}(v). \tag{18}$$

that is:

$$g_v(u,w) = g_{ij}(v)u^i w^j. \tag{19}$$

Hence, for each $v \in V$, g_v is a scalar product (with “reference vector” v) on V. Moreover, due to the homogeneity of F, there holds a similar formula to the one in Euclidean geometry:

$$F(v) = \sqrt{g_v(v,v)}. \tag{20}$$

In the following, we denote the derivatives of F with subscripts: $F_i := \dfrac{\partial F}{\partial v^i}$, $F_{ij} = \dfrac{\partial^2 F}{\partial v^i \partial v^j}$ etc.

At any $v \in V\setminus\{0\}$, the Hessian of F:

$$F_{ij}(v) = \frac{1}{F}[g_{ij}(v) - F_i(v)F_j(v)], \tag{21}$$

is positive semidefinite, with radical spanned by v. This fact serves to prove, (see [4], p. 8–9):

1. *the fundamental (or Cauchy-Schwarz) inequality:*

$$dF_v(w) \leqslant F(w), \quad \forall v, w \in V\setminus\{0\}; \tag{22}$$

2. *the triangle inequality:*

$$F(v+w) \leqslant F(v) + F(w), \quad \forall v, w \in V. \tag{23}$$

The above inequalities are strict, i.e., equality only holds when v and w are collinear.

With respect to a given basis, the fundamental inequality takes the form:

$$F_i(v)w^i \leqslant F(w). \tag{24}$$

The name of Cauchy-Schwarz inequality for (24) is justified by the following. Noticing that

$$dF_v(w) = F_i(v)w^i = \frac{g_{ij}(v)v^j w^i}{F(v)} = \frac{g_v(v,w)}{F(v)}, \tag{25}$$

this inequality can be equivalently written as:

$$g_v(v,w) \leqslant F(v)F(w), \tag{26}$$

i.e., the fundamental inequality (22) is just a generalization of the usual Cauchy-Schwarz inequality (15).

3.2. Lorentz-Finsler structures

An important feature of Lorentz-Finsler functions is that, typically, they can only be defined on a conic subset of V.

In the following, by a *conic domain* of V, we will mean an open connected subset $\mathscr{Q}$ of $V \setminus \{0\}$ with the *conic property*:

$$\forall v \in \mathscr{Q}, \forall \lambda > 0: \ \lambda v \in \mathscr{Q}.$$

The definition below is slightly more general than the one by Javaloyes and Sanchez, [13, p. 21]:

DEFINITION 2. Let $\mathscr{T} \subset V \setminus \{0\}$ be a conic domain. We call a Lorentz-Finsler norm on $\mathscr{T}$ a smooth function $F : \mathscr{T} \to (0,\infty)$ such that:

1. F is positively homogeneous of degree 1: $F(\lambda v) = \lambda F(v)$, $\forall \lambda > 0$, $\forall v \in \mathscr{T}$.
2. For every $v \in \mathscr{T}$, the fundamental tensor $g_v : V \times V \to \mathbb{R}$,

$$g_v(u,w) := \frac{1}{2}\frac{\partial^2 F^2}{\partial t \partial s}(v + tu + ws)\,|_{t=s=0}$$

has Lorentzian signature $(+,-,-,\dots,-)$.

A Lorentz-Finsler norm can always be continuously extended as 0 at $v = 0$.

NOTES.

1. The difference between the above introduced notion and the one of *Lorentz-Minkowski norm* presented in [13] is that we will *not* require F to be extended as 0 on $\partial\mathscr{T}$; while this requirement is important to applications in physical theories, in our case, it would just uselessly limit the range of allowed examples (see, e.g., Section 4.1). Actually, as we will see in the next section, we will even allow g_v to be degenerate at some vectors $v \in \mathscr{T}$.

2. Equipping a differentiable manifold M with a smooth family of Lorentz Finsler norms $p \mapsto F(p)$ which define a Lorentz Finsler structure $F(p)$ on each tangent space $T_pM,\ p \in M$, and demanding that $F|_{\partial\mathscr{T}} = 0$, makes the pair $(M, L = F^2)$ a Finsler spacetime, [11, 13]. Finsler spacetimes gain attention in the application to gravitational physics [12, 29], as well as in the mathematical community as generalizations of Lorentzian manifolds [7].

3. If, in the above definition, one replaces the condition of Lorentzian signature with positive definiteness, one obtains the notion of (positive definite) *conic Finsler metric,* [13]. Thus, a usual Finsler metric is a conic Finsler metric with $\mathscr{T} = V \setminus \{0\}$.

For Lorentz-Finsler norms F, the matrix $g_{ij}(v)$ is defined by the same formula (18) (but this time, it has Lorentzian signature) and the relation $F(v) = \sqrt{g_v(v,v)}$ still holds. The Hessian F_{ij} is negative semidefinite with radical spanned by v, i.e.,

$$F_{ij}(v)w^i w^j \leqslant 0, \tag{27}$$

for all $v \in \mathscr{T}$, and $w \in V$, where equality implies that w is collinear to v. Conversely, if $(F_{ij}(v))$ is negative semidefinite with 1-dimensional radical, then g_v has $(+,-,-,\dots,-)$ signature, (see[2] [13], Proposition 4.8 and, respectively, Lemma 4.7).

EXAMPLES OF LORENTZ-FINSLER NORMS. Here we just briefly list some examples $F : \mathscr{T} \to \mathbb{R}$ (defined on conic subsets $\mathscr{T} \subset \mathbb{R}^{n+1}$), to be examined in the following sections.

1. The $(n+1)$-dimensional *Minkowski metric*: $F(v) = \sqrt{\eta_{ij}v^i v^j}$.

2. (α,β)*-spacetime metrics:* $F(v) = \varphi(s)\sqrt{\eta_{ij}v^i v^j}$, where $s = \dfrac{b_i v^i}{\sqrt{\eta_{ij}v^i v^j}}$ and $\varphi = \varphi(s)$ is a smooth function on its domain of definition.

3. *The p-pseudo-norm*: $F(v) = \left[(v^0)^p - (v^1)^p - \dots - (v^n)^p\right]^{\frac{1}{p}}$.

4. The $(n+1)$-dimensional Berwald-Moór metric $F(v) = (v^0 v^1 \dots v^n)^{\frac{1}{n+1}}$.

5. Bimetric spaces: $F(v) = [(\eta_{ij}v^i v^j)(h_{kl}v^k v^l)]^{\frac{1}{4}}$, where $h_{kl}v^k v^l$ has Lorentzian signature.

The latter three examples belong to a wider class of Lorentz-Finsler functions F, called *m-th root metrics*, expressed as the m-th root of some polynomial of degree $m > 2$ in v^i.

[2] The proof of (27) inside the open conic set $\mathscr{T}$ in the cited paper does not require F to be extendable as 0 on ∂T, hence the result holds with no modification in our case.

3.3. The degenerate/non-smooth case

In previous works on the topic, such as [1, 13, 20, 21], the Finslerian generalizations of the reverse Cauchy-Schwarz inequality and of the reverse triangle inequality were proven under the hypothesis that the set

$$B(1) = F^{-1}([1,\infty))$$

is strictly convex (a sufficient condition thereof is that F vanishes on $\partial\mathscr{T}$ – which, as we mentioned above, is not assumed here). These inequalities are strict, i.e., equality happens if and only if v and w are collinear. Also, in [13], it is proven that, if $B(1)$ is just (non-strictly) convex, then the inequalities hold non-strictly.

In this section, we will present a reformulation – and a slight extension – of the above results (by relaxing either the nondegeneracy condition on g_v or the smoothness, even the continuity, assumption on F). Also, we will only require as a hypothesis the convexity of the domain $\mathscr{T}$.

3.3.1. The non-smooth case

Let us drop, for the moment, any smoothness (or even continuity) assumption on F. We obtain the following result.

PROPOSITION 1. *For a positively 1-homogeneous function $F : \mathscr{T} \to (0,\infty)$, $v \mapsto F(v)$ defined on a convex conic domain $\mathscr{T} \subset V \setminus \{0\}$, the following statements are equivalent:*

(i) F obeys the reverse triangle inequality (32);

(ii) F is a concave function;

(iii) the set $B(1) = F^{-1}([1,\infty))$ is convex;

Moreover, the reverse triangle inequality of F is strict if and only if the convexity of $B(1)$ is strict.

Proof. *(i) $\to$ (ii):* Assume that F obeys the reverse triangle inequality and pick two arbitrary vectors $u, v \in \mathscr{T}$. Then, for any $\alpha \in [0,1]$, the convex combination $(1-\alpha)u + \alpha v$ lies in $\mathscr{T}$ (as $\mathscr{T}$ is assumed to be convex), hence, it makes sense to speak about $F((1-\alpha)u + \alpha v)$. Using *(i)* and the homogeneity of F, we find:

$$F((1-\alpha)u + \alpha v) \geqslant F((1-\alpha)u) + F(\alpha v) = (1-\alpha)F(u) + \alpha F(v), \tag{28}$$

i.e., F is concave.

(ii) $\to$(i): If F is concave, then, for any $u, v \in \mathscr{T}$: $F\left(\frac{u+v}{2}\right) \geqslant \frac{1}{2}F(u) + \frac{1}{2}F(v)$. Using the homogeneity of F, this yields the reverse triangle inequality (32).

(i)$\to$(iii): Assuming that the reverse triangle inequality holds, pick two arbitrary vectors $v, w \in B(1)$ (i.e., $F(v), F(w) \geqslant 1$) and an arbitrary $\alpha \in [0,1]$. Then,

$$F((1-\alpha)v + \alpha w) \geqslant F((1-\alpha)v) + F(\alpha w) = (1-\alpha)F(v) + \alpha F(w) \geqslant 1,$$

which means that $(1-\alpha)v+\alpha w \in B(1)$. Consequently, $B(1)$ is convex. Also, if the triangle inequality is strict, then, for all non-collinear v,w, the first inequality above is strict, which eventually leads to $F((1-\alpha)v+\alpha w) > 1$, i.e., the convexity of $B(1)$ is strict.

(iii)→(i): The idea of the proof is similar to the standard one in the positive semi-definite case (see e.g., [14]). Assume $B(1)$ is convex and pick two arbitrary vectors $v,w \in \mathscr{T}$. By the 1-homogeneity of F, it follows that the vectors $v' := \dfrac{v}{F(v)}$, $w' = \dfrac{w}{F(w)}$ obey $F(v') = F(w') = 1$, i.e., $v',w' \in \partial B(1)$. Set $\alpha := \dfrac{F(w)}{F(v)+F(w)} \in (0,1)$ and build the convex combination:

$$u := (1-\alpha)v' + \alpha w' = \frac{v+w}{F(v)+F(w)} \in \mathscr{T}.$$

By the homogeneity assumption on F, we have $u \in B(1)$, i.e., $F(u) \geqslant 1$. But, using again the homogeneity of F, this is: $F(v+w) \geqslant F(v)+F(w)$, q.e.d. If $B(1)$ is strictly convex, then u belongs to the interior of $B(1)$, i.e., $F(u) > 1$. This leads to the strictness of the reverse triangle inequality. $\square$

The above result extends a result in [1], by removing the restrictions on the continuity of F or on the boundary $\partial\mathscr{T}$.

Similarly, for a 1-homogeneous function $F : \mathscr{T} \to \mathbb{R}$, there holds (see also [14]) the equivalence:

$$\text{(non-reverse) triangle inequality} \Leftrightarrow F\text{-convex} \Leftrightarrow F^{-1}([0,1])\text{-convex}.$$

REMARK 2. For a general positively 1-homogeneous function $F : \mathscr{T} \to \mathbb{R}$ (where $\mathscr{T} \in V \setminus \{0\}$ is conic), the convexity of $B(1) = F^{-1}([1,\infty))$ is a stronger requirement than convexity of its domain $\mathscr{T}$. Indeed, if $B(1)$ is convex, then, the homogeneity of F implies that all the sets $aB(1) = F^{-1}([a,\infty))$, $a > 0$ are convex; this immediately implies the convexity of $\mathscr{T} = F^{-1}((0,\infty)))$. Yet, the converse statement is generally not true; as we have seen above, it actually depends on the concavity of the function F.

3.3.2. Degenerate Lorentz-Finsler norms

Now, let us assume again that the function F smooth. We will first prove a lemma, which extends (27) to degenerate Finsler structures.

LEMMA 1. *Consider a smooth, 1-homogeneous function $F : \mathscr{T} \to \mathbb{R}$ defined on a conic domain $\mathscr{T} \subset V \setminus \{0\}$, an arbitrary $v \in \mathscr{T}$ and denote, with respect to an arbitrary basis:*

$$g_{ij}(v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial v^j}(v)$$

Then:

(i) *The matrix $(g_{ij}(v))$ has only one positive eigenvalue if and only if the Hessian $(F_{ij}(v))$ is negative semidefinite.*

(ii) The matrix $(g_{ij}(v))$ is positive semidefinite if and only if the Hessian $(F_{ij}(v))$ is positive semidefinite.

Proof.

(i) $\rightarrow$: Assume $g_{ij}(v)$ has only one positive eigenvalue. Using $g_{ij}(v) = F(v)F_{ij}(v) + F_i(v)F_j(v)$, we find for any $u \in V$:

$$F(v)F_{ij}(v)u^i u^j = g_{ij}(v)u^i u^j - (F_i(v)u^i)^2. \tag{29}$$

As the signature of g_v does not depend on the choice of the basis $\{e_i\}_{i=\overline{0,n}}$, we can freely choose this basis. For instance, we can choose an orthogonal basis for g_v, with $e_0 = v$. Since $g_v(e_0,e_0) = g_{ij}(v)v^i v^j = F^2(v) > 0$, it follows from the hypothesis that all the other diagonal entries $g_{ii}(v)$ are nonpositive. Setting $u = e_\alpha$ for $\alpha \neq 0$, the orthogonality condition is written, taking into account (25), as $F_i(v)u^i = 0$; therefore,

$$F_{ij}(v)u^i u^j = \frac{1}{F(v)} g_{ij}(v)u^i u^j, \tag{30}$$

which entails $F_{ij}(v)u^i u^j \leqslant 0$. But, on the other hand, we have: $F_{ij}(v)e_0^i e_0^j = F_{ij}(v)v^i v^j = 0$; that is, evaluating the bilinear form $F_{ij}(v)$ on any basis vector e_i, we get nonpositive values, i.e., $F_{ij}(v)$ is negative semidefinite for any $v \in \mathscr{T}$.

$\leftarrow$: Conversely, assume $F_{ij}(v)$ is negative semidefinite. Using the same g_v-orthogonal basis as above, we find from (30) that, for $u = e_\alpha$, $\alpha = 1,\ldots,n$, there holds $0 \geqslant g_{ij}(v)u^i u^j$, i.e., $g_{ij}(v)$ has n nonpositive eigenvalues. But, on the other hand, $g_v(e_0,e_0) = F^2(v) > 0$, i.e., the eigenvector $e_0 = v$ corresponds to a (unique) positive eigenvalue for g.

(ii) is proven similarly, taking into account that, this time,

$$F(v)F_{ij}(v)u^i u^j = g_{ij}(v)u^i u^j \geqslant 0. \quad \square$$

From the above Lemma and Proposition 1, we immediately find:

THEOREM 2. *For a smooth, positively 1-homogeneous function $F : \mathscr{T} \to (0,\infty)$ defined on a convex conic domain $\mathscr{T} \subset V \setminus \{0\}$, the following statements are equivalent:*

(i) F-concave $\Leftrightarrow$ g_v has exactly one positive eigenvalue $\Leftrightarrow$ F obeys the non-strict reverse triangle inequality $\Leftrightarrow$ the set $B(1) = F^{-1}([1,\infty))$ is convex;

(ii) F-convex $\Leftrightarrow$ g_v is positive semidefinite $\Leftrightarrow$ F obeys the non-strict triangle inequality $\Leftrightarrow$ the set $F^{-1}([0,1])$ is convex (where, in the latter, we have defined $F(0) := 0$).

Finally, we can state (albeit in a somewhat redundant way):

THEOREM 3. (The degenerate-Lorentzian case): *Let $\mathscr{T} \subset V \setminus \{0\}$ be a convex conic domain and $F : \mathscr{T} \rightarrow (0,\infty)$ a smooth, positively 1-homogeneous function. If the Hessian g_v of F^2 has only one positive eigenvalue for all $v \in \mathscr{T}$, then, for any $v,w \in \mathscr{T}$, there hold:*

(i) the fundamental (or reverse Cauchy-Schwarz) inequality:

$$dF_v(w) \geqslant F(w). \tag{31}$$

(ii) The reverse triangle inequality:

$$F(v+w) \geqslant F(v)+F(w); \tag{32}$$

If g_v is everywhere nondegenerate (i.e., Lorentzian), then both the above inequalities are strict.

Proof.

(i) The technique follows roughly the same steps as in the positive definite case (see, e.g., [4], p. 8–9). Consider two arbitrary vectors $u,v \in \mathscr{T}$. Since $\mathscr{T}$ is convex, it follows that $\frac{u+v}{2} \in \mathscr{T}$; but as it is also conic, we find $u+v \in \mathscr{T}$, which means that it makes sense to speak about $F(u+v)$. Now, perform a Taylor expansion around v, with the remainder in Lagrange form:

$$F(u+v) = F(v)+F_i(v)u^i + \frac{1}{2}F_{ij}(v+\varepsilon u)u^i u^j. \tag{33}$$

From the above Lemma, we obtain that F_{ij} is negative semidefinite, that is, $F_{ij}(v+\varepsilon u)u^i u^j \leqslant 0$ and therefore,

$$F(u+v) \leqslant F(v)+F_i(v)u^i. \tag{34}$$

Then, denoting $w := u+v$, the above becomes $F(w) \leqslant F(v)+F_i(v)(w^i - v^i)$, which, using the 1-homogeneity of F, leads to: $F(w) \leqslant F_i(v)w^i$, which is the coordinate form of *(i)*.

(ii) The reverse triangle inequality follows immediately from Theorem 2.

Strictness: Assume g_v is nondegenerate. Then, the equality $F_{ij}(v+\varepsilon u)u^i u^j = 0$ can only happen when v and u are collinear; this leads to the strictness (34) and consequently, of (31). Further, set $\xi := v+w$ (which belongs to $\mathscr{T}$, since $\mathscr{T}$ is conic and convex) and calculate:

$$F(\xi) = F_i(\xi)\xi^i = F_i(\xi)(v^i + w^i) = F_i(\xi)v^i + F_i(\xi)w^i.$$

Applying the reverse Cauchy-Schwarz inequality twice in the right hand side, we get: $F(v+w) = F(\xi) > F(v)+F(w)$, i.e., the reverse triangle inequality is strict. $\square$

Using (25), the fundamental inequality can be equivalently written as:

$$F_i(v)w^i \geqslant F(w), \tag{35}$$

or as:

$$g_v(v,w) \geqslant F(v)F(w). \tag{36}$$

EXAMPLE. To convince ourselves that the reverse Cauchy-Schwarz inequality becomes non-strict if F is degenerate, consider

$$F : \mathscr{T} \to \mathbb{R},\ \ F(v) = \sqrt{(v^0)^2 - (v^1)^2 - \ldots - (v^k)^2};$$

here, $n > 3$, $k \leqslant n-2$ and the cone $\mathscr{T} \subset \mathbb{R}^{n+1}$ is the Cartesian product $\mathscr{T} = \mathscr{T}_k \times \mathbb{R}^{n-k}$, where $\mathscr{T}_k = \{u \in \mathbb{R}^{k+1} \mid (u^0)^2 - (u^1)^2 - \ldots - (u^k)^2 > 0,\ u^0 > 0\}$. Since $\mathscr{T}_k$ is a convex cone in $\mathbb{R}^{k+1}$, it follows that $\mathscr{T}$ is also convex. The corresponding metric tensor is $g_v = diag(1,-1,\ldots,-1,0,\ldots,0)$, where the number of -1 entries is k. Picking $v = (1,0,0,\ldots 0,1,0)$ and $w = (1,0,0,\ldots,0,1)$, we get: $g_v(v,w) = 1$, $F(v) = 1$ and $F(w) = 1$, which means that $g_v(v,w) = F(v)F(w)$, while, obviously, v and w are not collinear.

In the maximally degenerate case, when g_v has everywhere signature $(+,0,0,\ldots,0)$, $(F_{ij}(v))$ is the zero matrix, hence, (31) and (32) become equalities for *all* $v,w \in \mathscr{T}$.

REMARK 3. Generally, if F is not smooth, it makes no sense to ask about the (reverse) fundamental inequality, as it involves dF_v. Yet, if F is smooth at some *fixed* $v \in \mathscr{T}$ and $F^{-1}([1,\infty))$ is a convex set, then, it was proven in [1] (Proposition *(ii)),* that we can still state that $dF_v(w) \geqslant F(w)$, for all $w \in \mathscr{T}$. The conditions that $\partial\mathscr{T}$ must be smooth and $F_{|\partial\mathscr{T}} = 0$ (assumed in the cited paper) play no role in the proof, so the result works with no modification under our milder assumptions.

Similarly, it holds:

PROPOSITION 4. (The positive semidefinite case): *Let $\mathscr{T} \subset V \setminus \{0\}$ be a convex conic domain and $F : \mathscr{T} \to (0,\infty)$, a smooth positively 1-homogeneous function such that the Hessian g_v of F^2 is positive semidefinite for all $v \in \mathscr{T}$. Then, the Cauchy-Schwarz inequality (22) and the triangle inequality (23) still hold, but they are generally non-strict.*

The proof is identical to the one of Proposition 3, with the only difference that, in (33), the matrix $F_{ij}(v+\varepsilon u)$ is positive semidefinite, which leads to the opposite inequality: $F(u+v) \geqslant F(v) + F_i(v)u^i$, hence, to the usual (non-reverse) Cauchy-Schwarz and triangle inequalities.

3.4. Refinements of the Finslerian reverse triangle inequality

Here are two refinements of the reverse triangle inequality, holding for (possibly degenerate, or non-smooth) Lorentz-Finsler functions.

THEOREM 5. *If a positively 1-homogeneous function* $F:\mathscr{T}\to(0,\infty)$, *defined on a convex conic subset* $\mathscr{T}\subset V\setminus\{0\}$, *obeys the reverse triangle inequality, then, for all* $v,w\in\mathscr{T}$ *and for any* $0<a\leqslant b$, *we have:*

$$a[F(v+w)-F(v)-F(w)]\leqslant F(av+bw)-aF(v)-bF(w)\leqslant b[F(v+w)-F(v)-F(w)]. \tag{37}$$

If the reverse triangle inequality of F *is strict, then the above inequalities are also strict.*

Proof. The first inequality is equivalent (after canceling out the $-aF(v)$ terms and grouping the $F(w)$ ones into the left hand side) to:

$$aF(v+w)+(b-a)F(w)\leqslant F(av+bw).$$

But, since F is positively homogeneous and $b-a\geqslant 0$, we get: $aF(v+w)=F(av+aw)$ and $(b-a)F(w)=F(bw-aw)$. Then, the reverse triangle inequality yields:

$$aF(v+w)+(b-a)F(w)=F(av+aw)+F(bw-aw)\leqslant F(av+bw)$$

as required.

The second inequality is proven in a completely similar way to be equivalent to: $F(av+bw)+F(bv-av)\leqslant F(bv+bw)$, which, again, holds by virtue of the reverse triangle inequality. □

PROPOSITION 6. *If a continuous, positively 1-homogeneous function* $F:\mathscr{T}\to\mathbb{R}$ *defined on a convex conic subset* $\mathscr{T}\subset V\setminus\{0\}$, *obeys the reverse triangle inequality, then:*

$$F(v)+F(w)\leqslant 2\int_0^1 F(tv+(1-t)w)dt\leqslant F(v+w), \tag{38}$$

for all $v,w\in\mathscr{T}\subset V\setminus\{0\}$.

Proof. We use the same idea as in [18]. Using the reverse triangle inequality and homogeneity, we find:

$$F(tv+(1-t)w))\geqslant F(tv)+F((1-t)w))=tF(v)+(1-t)F(w),$$

for every $v,w\in\mathscr{T}$, $t\in[0,1]$. Integrating with respect to t, from 0 to 1, we obtain:

$$\frac{F(v)+F(w)}{2}\leqslant\int_0^1 F(tv+(1-t)w)dt,$$

i.e., the first inequality (38). Similarly, using the reverse triangle inequality, we have $F(v+w)=F(tv+(1-t)w+(1-t)v+tw)\geqslant F(tv+(1-t)w)+F((1-t)v+tw)$. Integrating from 0 to 1, we deduce: $F(v+w)\geqslant\int_0^1 F(tv+(1-t)w)dt+\int_0^1 F((1-t)v+tw)dt=2\int_0^1 F(tv+(1-t)w)dt$, which is just the second inequality (38). □

The two above results trivially hold when one of the vectors v,w is zero.

4. Lorentz-Finsler norms and their inequalities

The set of Lorentz-Finsler norms is rich in interesting examples whose reverse Cauchy-Schwarz or reverse triangle inequality yield immediately famous inequalities from the literature and open a pathway to reveal further interesting inequalities.

We already pointed out in Section 2 that, for the simplest example of Lorentz-Finsler structure on $\mathbb{R}^{n+1}$, the Minkowski metric $F(v) = \sqrt{\eta_{ij}v^i v^j}$, for which $\mathscr{T} = \left\{v \in V \mid \eta_{ij}v^i v^j > 0, v^0 > 0\right\}$, its reverse Cauchy-Schwarz inequality led directly to Aczél's inequality (2).

In the following, we explore some nontrivial Finslerian cases.

4.1. Popoviciu's inequality

PROPOSITION 7. *Let $\mathscr{T} \subset \mathbb{R}^{n+1}$ be the conic domain:*

$$\mathscr{T} := \{v \in \mathbb{R}^{n+1} \mid v^0, v^1, \ldots, v^n > 0, \left(v^0\right)^p - \left(v^1\right)^p - \ldots - (v^n)^p > 0\}.$$

Moreover let $F : \mathscr{T} \to \mathbb{R}^+$ be the Lorentz-Finsler structure defined by

$$F(v) = H(v)^{\frac{1}{p}}, \quad H(v) = \left(v^0\right)^p - \left(v^1\right)^p - \ldots - (v^n)^p, \tag{39}$$

where $p > 1$. Then:

(i) the fundamental inequality $F_i(v)w^i \geqslant F(w)$ is Popoviciu's inequality:

$$\eta_{ij}a^i b^j \geqslant \left[(a^0)^q - \left(a^1\right)^q - \ldots - (a^n)^q\right]^{\frac{1}{q}} \left[\left(b^0\right)^p - \left(b^1\right)^p - \ldots - (b^n)^p\right]^{\frac{1}{p}}, \forall a, b \in \mathscr{T}, \tag{40}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$;

(ii) the reverse triangle inequality of F is Bellman's inequality:

$$\begin{aligned} &\left(v_0^p - v_1^p - \ldots - v_n^p\right)^{1/p} + \left(w_0^p - w_1^p - \ldots - w_n^p\right)^{1/p} \\ &\leqslant [(v_0 + w_0)^p - (v_1 + w_1)^p - \ldots - (v_n + w_n)^p]^{1/p}. \end{aligned} \tag{41}$$

Proof.

(i) To see that the fundamental inequality holds, we realise that the Hessian of H is

$$H_{ij}(v) = p(p-1)diag\left((v^0)^{p-2}, -(v^1)^{p-2}, \ldots, -(v^n)^{p-2}\right).$$

and thus has Lorentzian signature on $\mathscr{T}$. Moreover, $\mathscr{T}$ is convex (but not strictly convex), as it can be identified with the epigraph of the convex function $\tilde{H}(v^1, \ldots, v^n) = (v^1)^p + \ldots + (v^n)^p$, defined for all $v^\alpha > 0$.

By Proposition 14 (see Appendix), we obtain that $g_{ij}(v)$ is Lorentzian for all $v \in \mathscr{T}$ and hence, the fundamental inequality inequality (35) holds.

By a straightforward calculation, we get:

$$F_i(v) = F(v)^{1-p}\eta_{ij}(v^j)^{p-1};$$

therefore, the fundamental inequality $F_i(v)w^i \geqslant F(w)$ becomes:

$$\eta_{ij}(v^j)^{p-1}w^i \geqslant F(v)^{p-1}F(w) = H(v)^{\frac{p-1}{p}}H(w)^{\frac{1}{p}}.$$

Evaluating the above equation with the following notation:

$$q := \frac{p}{p-1}, \quad a^i := \left(v^i\right)^{p-1}, \quad b^j := w^j,$$

(in particular, $\dfrac{1}{p}+\dfrac{1}{q}=1$), yields Popoviciu's inequality.

(ii) is obvious. □

REMARK 4. Similarly, Hölder's inequality

$$\delta_{ij}a^ib^j \leqslant \left[\left(a^0\right)^q+\ldots+(a^n)^q\right]^{\frac{1}{q}}\left[\left(b^0\right)^p+\ldots+(b^n)^p\right]^{\frac{1}{p}}, \quad \forall a^i, b^i > 0, i = \overline{0,n},$$

where δ_{ij} is the Kronecker symbol, can be treated as fundamental inequality of the Finsler norm $F(v) = [\left(v^0\right)^p+\ldots+(v^n)^p]^{\frac{1}{p}}$ – which is positive definite for $v^i > 0$, $i = \overline{0,n}$ and Minkowski's inequality

$$\left[\left(a^0+b^0\right)^p+\ldots+(a^n+b^n)^p\right]^{\frac{1}{p}} \leqslant \left[\left(a^0\right)^p+\ldots+(a^n)^p\right]^{\frac{1}{p}}+\left[\left(b^0\right)^p+\ldots+(b^n)^p\right]^{\frac{1}{p}},$$

$\forall a^i, b^i > 0, i = \overline{0,n}, p > 1$ is just the corresponding triangle inequality.

4.2. The arithmetic-geometric mean inequality

PROPOSITION 8. *Let $\mathscr{T} \subset \mathbb{R}^{n+1}$ be the convex conic domain*

$$\mathscr{T} := \left\{v \in \mathbb{R}^{n+1} \mid v^0, v^1, \ldots, v^n > 0\right\} \subset \mathbb{R}^{n+1}.$$

Moreover let $F : \mathscr{T} \to \mathbb{R}^+$ be the Berwald-Moór Finsler structure defined by

$$F(v) = (v^0v^1\ldots v^n)^{\frac{1}{n+1}}.$$

Then, the fundamental inequality $F_i(v)w^i \geqslant F(w)$ is the aritmetic-geometric mean inequality:

$$\frac{a_0+\ldots.+a_n}{n+1} \geqslant (a_0a_1\ldots a_n)^{\frac{1}{n+1}}, \quad \forall a_i \in \mathbb{R}^*_+. \tag{42}$$

Proof. The n-dimensional Berwald-Moór metric is known, [3], to be of Lorentzian signature. Yet, for the sake of completeness, we sketch a proof of this fact below. To this aim, we will use Proposition 14.

The Hessian of the $(n+1)$-th power $H(v) := v^0 v^1 \dots v^{n+1}$ of F is:

$$H_{ij}(v) = \begin{cases} 0, & \text{if } i = j \\ \dfrac{H(v)}{v^i v^j}, & \text{if } i \neq j. \end{cases} \tag{43}$$

On $\mathscr{T}$, the matrix $(H_{ij}(v))$ has Lorentzian signature. To see this, fix an arbitrary $v \in \mathscr{T}$ and introduce the vectors $e_0 := v$ and $\{e_\alpha\}$, $\alpha = \overline{1,n}$ as follows:

$$e^i_\alpha = A^i_\alpha v^i, \ \ i = \overline{0,n}$$

(where no summation is understood over i), such that:

$$\sum_{i=0}^{n} A^i_\alpha = A^0_\alpha + \sum_{\beta=1}^{n} A^\beta_\alpha = 0, \qquad \det(A^\beta_\alpha)_{\alpha,\beta=\overline{1,n}} \neq 0.$$

The vectors $\{e_0, e_\alpha\}$ are linearly independent, as the matrix with the columns e_0, e_α has the determinant $H(v)\det(A^\beta_\alpha) \neq 0$. Moreover, e_α span the $(H_{ij}(v))$-orthogonal complement of $e_0 = v$, since, using (43), we find:

$$H_{ij}(v) v^i e^j_\alpha = 0, \quad \forall \alpha = 1, \dots, n.$$

Then, on one hand, we have:

$$H_{ij}(v) v^i v^j = n(n-1)H(v) > 0$$

and, on the other hand, $H_{ij}(v)$ is negative definite on $Span\{e_\alpha\}$, since:

$$H_{ij}(v) e^i_\alpha e^j_\alpha = H(v) \sum_{i \neq j} \frac{v^i}{v^i} \frac{v^j}{v^j} A^i_\alpha A^j_\alpha = H(v) \left(\left(\sum_{i=0}^{n} A^i_\alpha \right)^2 - \sum_{i=0}^{n} (A^i_\alpha)^2 \right) = 0 - H(v) \sum_{i=0}^{n} (A^i_\alpha)^2.$$

Consequently, $H_{ij}(v)$ has Lorentzian signature. Then, by Proposition 14, also $g_{ij}(v)$ has Lorentzian signature on $\mathscr{T}$ and the fundamental inequality $F_i(v) w^i \geqslant F(w)$ holds $\forall v, w \in \mathscr{T}$. We easily find:

$$F_i(v) = \frac{1}{n+1} H^{\frac{1}{n+1}-1} \frac{H(v)}{v^i} = \frac{F(v)}{n+1} \frac{1}{v^i}$$

and thus

$$\frac{F(v)}{n+1} \sum_{i=0}^{n} \frac{w^i}{v^i} \geqslant F(w)$$

or equivalently

$$\frac{1}{n+1} \sum_{i=0}^{n} \frac{w^i}{v^i} \geqslant \frac{F(w)}{F(v)} = \left(\frac{w^0}{v^0} \frac{w^1}{v^1} \dots \frac{w^n}{v^n} \right)^{\frac{1}{n+1}}.$$

Setting $a_i := \dfrac{w^i}{v^i}$, $i = \overline{0,n}$, a_i take all possible values in $\mathbb{R}^*_+$ and the fundamental inequality becomes the aritmetic-geometric mean inequality (42). $\square$

4.3. Weighted arithmetic-geometric mean inequality

PROPOSITION 9. *Let $\mathscr{T} \subset \mathbb{R}^{n+1}$ be the convex conic domain*

$$\mathscr{T} := \left\{ v \in \mathbb{R}^{n+1} \mid v^0, v^1, \ldots, v^n > 0 \right\} \subset \mathbb{R}^{n+1}.$$

Then, the function $F : \mathscr{T} \to \mathbb{R}^+$, defined by

$$F(v) = \left(v^0\right)^{a_0} \left(v^1\right)^{a_1} \ldots (v^n)^{a_n}, \ \sum_{i=0}^{n} a_i = 1 \quad a_i \geqslant 0.$$

is a Lorentz-Finsler norm whose fundamental inequality is the weighted arithmetic-geometric mean inequality

$$\sum_{i=0}^{n} a_i v^i \geqslant (v^0)^{a_0} (v^1)^{a_1} \ldots (v^n)^{a_n}, \qquad \forall v^i \in \mathbb{R}_+^*.$$

Proof. Fix $u \in \mathscr{T}$. The components of the fundamental tensor are (no sum convention is employed in the following expression)

$$g_{ij}(u) = \frac{1}{2} \frac{\partial^2 F^2}{\partial u^i \partial u^j} = F(u)^2 \left(\frac{2a_i a_j}{u^i u^j} - \frac{a_i \delta_{ij}}{(u^i)^2} \right).$$

Its signature can be determined similarly to the previous case. Introducing the vectors $e_0 = u$ and $e_\alpha, \alpha = \overline{1,n}$ with components

$$e_\alpha^i = B_\alpha^i u^i$$

such that

$$\sum_{i=0}^{n} a_i B_\alpha^i = a_0 B_\alpha^0 + \sum_{\beta=1}^{n} a_\beta B_\alpha^\beta = 0, \tag{44}$$

and $\det(B_\alpha^\beta) \neq 0$, the vectors $\{e_0, e_\alpha\}$ are linearly independent and

$$g_{ij}(u) e_0^i e_\alpha^j = 0.$$

Thus, $\{e_\alpha\}_{\alpha=\overline{1,n}}$ span the orthogonal complement of $e_0 = u$. Moreover, $g_{ij}(u)$ is negative definite on $Span\{e_\alpha\}$, since, by (44), we have:

$$g_{ij}(u) e_\alpha^i e_\alpha^j = F(u)^2 \left(2 \left(\sum_{i=0}^{n} a_i B_\alpha^i \right)^2 - \sum_{i=0}^{n} a_i (B_\alpha^i)^2 \right) = -F(u)^2 \sum_{i=0}^{n} a_i (B_\alpha^i)^2 < 0;$$

taking into account that $g_{ij}(u) e_0^i e_0^j = F(u)^2 > 0$, we obtain that $g_{ij}(u)$ is Lorentzian. Calculating

$$F_i(u) = a_i \frac{F(u)}{u^i},$$

the fundamental inequality (35) becomes

$$F(u)\sum_{i=0}^{n}\frac{a_i w^i}{u^i} \geqslant F(w)\,.$$

Introducing $v_i = \frac{w^i}{u^i} \in \mathbb{R}^*_+$, it can be rewritten as: $\sum_{i=0}^{n} a_i v^i \geqslant (v^0)^{a_0}(v^1)^{a_1}\dots(v^n)^{a_n}$. The generalization (3) then follows immediately. $\square$

4.4. Bimetric structures on $\mathbb{R}^{n+1}$

Finsler functions of the type

$$F(v) = [(g_{ij}v^iv^j)(h_{lk}v^kv^l)]^{\frac{1}{4}}\,,$$

where g_{ij} and h_{kl} are Lorentzian metrics of same signature type $(+,-,-,\dots,-)$, are relevant in physics in four spacetime dimensions, when one describes the propagation of light in birefringent crystals, see for example [25, 28, 31].

Yet, here we will discuss the general $(n+1)$-dimensional case**.** We can always choose a basis the tangent spaces of the manifold such that one of the bilinear forms h or g assumes its normal form, i.e. it is locally diagonal with entries $g_{ij} = \eta_{ij}$.

PROPOSITION 10. *Consider $\mathbb{R}^{n+1}$ equipped with the Minkowski metric η and another bilinear form h of Lorentzian signature. Let $\mathscr{T} \subset \mathbb{R}^{n+1}$ be the convex conic domain given by the intersection of the future pointing timelike vectors of η and h:*

$$\mathscr{T} := \left\{v \in \mathbb{R}^{n+1} | \eta_{ij}v^iv^j > 0,\ h_{ij}v^iv^j > 0, v^0 > 0\right\} \subset \mathbb{R}^{n+1}$$

and $F : \mathscr{T} \to \mathbb{R}^+$, the bimetric Finsler structure defined by

$$F(v) = [(\eta_{ij}v^iv^j)^{\frac{1}{4}}(h_{lk}v^kv^l)]^{\frac{1}{4}}\,.$$

Then, the fundamental inequality $F_i(v)w^i \geqslant F(w)$ is

$$\frac{1}{2}\left(\frac{\eta_{ij}w^iv^j}{\eta_{kl}v^kv^l} + \frac{h_{ij}w^iv^j}{h_{kl}v^kv^l}\right) \geqslant \frac{F(w)}{F(v)}, \qquad \forall v,w \in \mathscr{T}\,. \tag{45}$$

Proof. The Finsler structure is built from a fourth order polynomial

$$H(v) := (\eta_{ij}v^iv^j)(h_{lk}v^kv^l)$$

whose Hessian is given by

$$H_{ij} = 2\eta_{ij}(h_{lk}v^kv^l) + 4(\eta_{ik}h_{jl} + \eta_{jk}h_{il})v^kv^l + 2h_{ij}(\eta_{kl}v^kv^l);$$

it was proven, see [26, 27, 33], that H_{ij} has Lorentzian signature on $\mathscr{T}$. Consequently, again using Proposition 14, g_{ij} is of Lorentzian signature on $\mathscr{T}$ and the fundamental inequality $F_i(v)w^i \geqslant F(w)$ holds. We easily calculate:

$$F_i(v) = \frac{1}{2}\frac{1}{H(v)^{\frac{3}{4}}}[\eta_{ij}v^j\ (h_{lk}v^kv^l) + (\eta_{kl}v^kv^l)\ h_{ij}v^j],$$

and thus the inequality

$$\frac{1}{2}\frac{1}{H(v)^{\frac{3}{4}}}\left(\eta_{ij}w^iv^j\,(h_{lk}v^kv^l)+(\eta_{kl}v^kv^l)\,h_{ij}w^iv^j\right)\geqslant H(w)^{\frac{1}{4}}$$

holds. Both sides of the inequality can be multiplied with the positive factor $H(v)^{-\frac{1}{4}}$ to obtain

$$\frac{1}{2}\left(\frac{\eta_{ij}w^iv^j}{\eta_{kl}v^kv^l}+\frac{h_{ij}w^iv^j}{h_{kl}v^kv^l}\right)\geqslant\left(\frac{H(w)}{H(v)}\right)^{\frac{1}{4}}. \quad \square$$

As an explicit example, take $\mathbb{R}^2$ with $v=(v^0,v^1)$, $w=(w^0,w^1)$ and set $h_{kl}v^kv^l:=2(v^0)^2-(v^1)^2$. As $\eta_{ij}v^iv^j=(v^0)^2-(v^1)^2$, we find for all v^0,v^1,w^0,w^1 such that $(v^0)^2>(v^1)^2$ and $(w^0)^2>(w^1)^2$:

$$\frac{1}{16}\left(\frac{v^0w^0-v^1w^1}{(v^0)^2-(v^1)^2}+\frac{2v^0w^0-v^1w^1}{2(v^0)^2-(v^1)^2}\right)^4\geqslant\frac{((w^0)^2-(w^1)^2)(2(w^0)^2-(w^1)^2)}{((v^0)^2-(v^1)^2)(2(v^0)^2-(v^1)^2)}. \tag{46}$$

4.5. (α,β)-metrics. Kropina metrics

An intensively studied class of Finsler metrics are the so-called (α,β)-metrics, defined in terms of two building blocks: $\alpha(v)=\sqrt{\eta(v,v)}$ and $\beta(v)=b_iv^i$, where b_i are components of an element $b\in V^*$. A general (α,β)-Finsler function can be expressed as

$$F(\alpha,\beta)=\alpha\phi(s), \tag{47}$$

where ϕ is a function of the 0-homogeneous variable $s=\frac{\beta}{\alpha}$.

Formally, the fundamental inequality (31) of this class of Finsler metrics takes the form

$$\frac{\eta(v,w)}{\alpha(v)}\left[\phi(s(v))-s(v)\phi'(s(v))\right]+\phi'(s(v))\beta(w)\geqslant\alpha(w)\phi(s(w)). \tag{48}$$

For the positive definite case, the conditions which need to be satisfied such that (47) gives a well defined Finsler space are already known, [4, Lemma 1.1.2] (see also [14, Remark 4.26]). For the indefinite case, the precise conditions under which (47) defines a Lorentzian-Finsler structure will be studied in a future work.

We now consider two examples of (α,β)-metrics. The Randers metric which leads to somewhat trivial results and afterwards a first non-trivial example is represented by Kropina metrics.

4.5.1. Randers metrics

The classical Randers metric, [32], is defined by the function $\phi(s)=1+s$. They are known to be Lorentzian if β obeys: $\eta^{ij}b_ib_j\in(0,1)$, see [12]. The fundamental inequality (48) becomes

$$\frac{\eta(v,w)}{\alpha(v)}+\beta(w)\geqslant\alpha(w)\left(\frac{\beta(w)}{\alpha(w)}+1\right),$$

which, after canceling the β terms, is nothing but the classical reverse Cauchy-Schwarz inequality for the undeformed Lorentzian scalar product η.

The same argument can be applied when the Lorentzian scalar product η is replaced by a positive definite scalar product h. Then the classical positive definite fundamental inequality (22) yields the non-reverse Cauchy-Schwarz inequality for the scalar product h.

A more interesting case is when one considers the deformed Randers metric $\tilde{F} = -\tilde{\alpha}+\beta = \tilde{\alpha}(-1+\tilde{s})$, where $\tilde{\alpha} = \sqrt{h(\dot{x},\dot{x})}$ is constructed from a positive definite scalar product h, as in [10]. The deformed Randers length defines a Lorentzian Finsler structure when $1-h^{ij}b_ib_j > 0$. Replacing α with $\tilde{\alpha}$ and η with h in (48) one immediately deduces the classical non-reverse Cauchy-Schwarz inequality for the scalar product h. Thus, in this case, the reverse fundamental inequality of a Lorentzian Finsler metric implies the non-reverse Cauchy-Schwarz inequality for a positive definite scalar product.

4.5.2. Kropina metrics

Another famous example of (α,β)-metrics, are Kropina metrics, which are applied for example in the context of Zermelo navigation, [15]. We will use a Kropina-type deformation of the Minkowski metric η in order to find out an inequality regarding η. They are defined by choosing $\phi(s) = s^{-1}$ in (47).

PROPOSITION 11. *Let $\mathscr{T} \subset \mathbb{R}^{n+1}$ be the convex conic domain*

$$\mathscr{T} := \left\{ v \in \mathbb{R}^{n+1} \mid \eta_{ij}v^iv^j > 0, v^0 > 0 \right\} \subset \mathbb{R}^{n+1}$$

and the smooth, 1-homogeneous function $F : \mathscr{T} \to \mathbb{R}^+$ be defined by

$$F(v) = \frac{\eta_{ij}v^iv^j}{v^0} = \frac{1}{v^0}[(v^0)^2 - (v^1)^2 - \ldots. - (v^1)^2].$$

Then, F obeys the strict fundamental inequality (31), which becomes:

$$2\eta(v,w) \geqslant \frac{w^0}{v^0}\eta(v,v) + \frac{v^0}{w^0}\eta(w,w), \qquad \forall v,w \in \mathscr{T}. \tag{49}$$

Proof. Let us rewrite F as:

$$F(v) = v^0 - \frac{\vec{v}\cdot\vec{v}}{v^0},$$

where $v = (v^0,\vec{v})$, $\vec{v} = (v^1,\ldots,v^n)$ and $\vec{v}\cdot\vec{v} = \delta_{\alpha\beta}v^\alpha v^\beta$ denotes the standard Euclidean product on $\mathbb{R}^n$. We easily get the derivatives of F as:

$$F_0(v) = 1 + \frac{\vec{v}\cdot\vec{v}}{(v^0)^2}, \quad F_\alpha(v) = -2\frac{\delta_{\alpha\beta}v^\beta}{v^0}, \quad \alpha = 1,\ldots,n; \tag{50}$$

$$F_{00}(v) = -2\frac{\vec{v}\cdot\vec{v}}{(v^0)^3}, \quad F_{\alpha 0} = 2\frac{\delta_{\alpha\beta}v^\beta}{(v^0)^2}, \quad F_{\alpha\beta} = -2\frac{\delta_{\alpha\beta}}{v^0}. \tag{51}$$

Now, we can prove that the matrix $g_{ij}(v)$ has Lorentzian signature for all $v \in \mathscr{T}$.

First, we notice that $g_v(v,v) = g_{ij}(v)v^i v^j = F(v)^2 > 0$ on $\mathscr{T}$.

Second, we show that g_v is negative semidefinite on the g_v-orthogonal complement of v. This complement is defined by $g_{ij}(v)v^j w^i = 0$, which is equivalent to: $F_i(v)w^i = 0$. Taking into account the identity $g_{ij}(v) = FF_{ij}(v) + F_i(v)F_j(v)$, for g_v-orthogonal vectors w, we can write:

$$g_{ij}(v)w^i w^j = F(v)(F_{ij}(v)w^i w^j). \tag{52}$$

Taking into account (51), we get:

$$\begin{aligned} F_{ij}(v)w^i w^j &= \frac{-2}{(v^0)^3}\left[(\vec{v}\cdot\vec{v})\left(w^0\right)^2 - 2\left(\vec{w}\cdot\vec{v}\right)v^0 w^0 + \left(\vec{w}\cdot\vec{w}\right)(v^0)^2\right] \\ &= \frac{-2}{(v^0)^3}(w^0\vec{v} - v^0\vec{w})\cdot(w^0\vec{v} - v^0\vec{w}) \leqslant 0. \end{aligned}$$

Since $F(v) > 0$ on $\mathscr{T}$, we obtain from (52) that $g_{ij}(v)w^i w^j \leqslant 0$ on the g_v-orthogonal complement of v, where $F_{ij}(v)w^i w^j = 0$ implies $\vec{v} = \dfrac{v^0}{w^0}\vec{w}$; further, this leads to $v = \dfrac{v^0}{w^0}w$, i.e., the vectors $v, w \in \mathscr{T} \subset \mathbb{R}^{n+1}$ are collinear. Consequently, g_v is Lorentzian and the resulting fundamental inequality is strict.

The fundamental inequality can quickly be calculated either from (48) with $\phi = s^{-1}$, or with help of

$$F_i(v) = 2\frac{\eta_{ij}v^j}{v^0} - \frac{\eta_{jk}v^j v^k}{(v^0)^2}\delta_i^0$$

which eventually leads to the desired inequality (49). □

5. Aczél inequality: generalizations and refinements

In the following, we will use a particular class of Lorentzian Finsler metrics to obtain a generalization and some refinements of the Aczél inequality (2).

Consider, on $\mathbb{R}^{n+1}\setminus\{0\}$, a smooth, positive definite Finsler norm $\bar{F}$ and set:

$$F(v) := \sqrt{\left(v^0\right)^2 - \bar{F}^2(\vec{v})}, \tag{53}$$

where $v = \left(v^0, \vec{v}\right)$ belongs to the open conic subset of $\mathbb{R}^{n+1}\setminus\{0\}$:

$$\mathscr{T} = \left\{v = \left(v^0, \vec{v}\right) \in \mathbb{R}^{n+1} \mid v^0 > \bar{F}(\vec{v})\right\}. \tag{54}$$

Such functions F are used in the context of stationary Lorentz Finsler norms studied in [17, 9] as well as in the context of the Zermelo navigation [10].

As the function $\bar{F}$ is convex, its epigraph $\mathscr{T}$ is convex; it is also connected as it is the preimage of $(0,\infty)$ through the continuous function $v \mapsto \left(v^0\right) - \bar{F}(\vec{v})$. Hence, $\mathscr{T}$ is a convex conic domain.

The metric tensor $g = \frac{1}{2} Hess(F^2)$ is Lorentzian on $\mathscr{T}$, except for the half line $\left\{ \left(v^0, 0, 0, \ldots, 0\right) \mid v^0 > 0 \right\}$, where, due to the fact that $\bar{F}$ is only continuous at $0 \in \mathbb{R}^n$, g_v is not guaranteed to exist. Yet, the Aczél inequality can still be extended to such metrics, as follows.

PROPOSITION 12. (Finslerian Aczél inequality): *Let $\bar{F} : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary Finsler norm on $\mathbb{R}^n$ and set*

$$\|v\| = \bar{F}(\vec{v}), \qquad \bar{g}_{\vec{v}}(\vec{v}, \vec{w}) := \frac{1}{2} \frac{\partial^2 \bar{F}^2}{\partial v^\alpha \partial v^\beta}(\vec{v}) v^\alpha w^\beta. \tag{55}$$

Then, for all $v^0, w^0 > 0$ and for all $\vec{v}, \vec{w} \in \mathbb{R}^n \setminus \{0\}$ such that $\left(v^0\right)^2 - \|\vec{v}\|^2 > 0$, $\left(w^0\right)^2 - \|\vec{w}\|^2 > 0$, there holds:

$$\left[v^0 w^0 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})\right]^2 - [\left(v^0\right)^2 - \|\vec{v}\|^2][\left(w^0\right)^2 - \|\vec{w}\|^2] \geqslant 0, \tag{56}$$

where equality takes place if and only if the vectors $v = (v^0, \vec{v})$, $\left(w^0, \vec{w}\right) \in \mathbb{R}^{n+1}$ are collinear.

The proof is a direct one and relies on the following lemma:

LEMMA 2. *For all $v, w \in \mathbb{R}^{n+1}$, $v = (v^0, \vec{v})$, $w = (w^0, \vec{w})$ with $\vec{w} \neq 0$, there holds:*

$$\begin{aligned} &\left[v^0 w^0 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})\right]^2 - [\left(v^0\right)^2 - \|\vec{v}\|^2][\left(w^0\right)^2 - \|\vec{w}\|^2] = \\ &= \left[w^0 \frac{\bar{g}_{\vec{v}}(\vec{v}, \vec{w})}{\|\vec{w}\|} - v^0 \|\vec{w}\|\right]^2 + \frac{\left(w^0\right)^2 - \|\vec{w}\|^2}{\|\vec{w}\|^2} \left[\|\vec{v}\|^2 \|\vec{w}\|^2 - \bar{g}_{\vec{v}}(\vec{v}, \vec{w})\right]. \end{aligned} \tag{57}$$

Proof of the lemma. By direct computation, we find that both hand sides are actually equal to:

$$\left[\|\vec{v}\|^2 \left(w^0\right)^2 + \left(v^0\right)^2 \|\vec{w}\|^2 - 2 v^0 w^0 \bar{g}_{\vec{v}}(\vec{v}, \vec{w})\right] + \left[\bar{g}_{\vec{v}}(\vec{v}, \vec{w}) - \|\vec{v}\|^2 \|\vec{w}\|^2\right]. \quad \Box \tag{58}$$

Proof of Proposition 12. Since $w \in \mathscr{T}$, we have $\left(w^0\right)^2 - \|\vec{w}\|^2 > 0$. Also, using the (non-reverse) Cauchy-Schwarz inequality for $\bar{F}$, we find that the right hand side of (57) is nonnegative; hence, the left hand side must also be nonnegative, which is exactly (56). Moreover, equality to zero can only happen when both square brackets in the right hand side of (57) vanish. This means, on one hand, $\bar{g}_{\vec{v}}(\vec{v}, \vec{w}) - \|\vec{v}\|^2 \|\vec{w}\|^2 = 0$, which implies: $\vec{w} = \alpha \vec{v}$; the vanishing of the other bracket then leads to $w^0 = \alpha v^0$, i.e., the vectors $v, w \in \mathbb{R}^{n+1}$ are collinear. $\Box$

The above Lemma immediately yields two even more powerful results:

PROPOSITION 13. (Refinements of the Finslerian Aczél inequality). *Let g_v be defined by a Finslerian norm $\bar{F} = \|\cdot\|$ on $\mathbb{R}^n$ as in (55). Then, for any $v^0, w^0 \in \mathbb{R}$ and any $\vec{v}, \vec{w} \in \mathbb{R}^n \setminus \{0\}$ such that $\left(v^0\right)^2 - \|\vec{v}\|^2 > 0$, $\left(w^0\right)^2 - \|\vec{w}\|^2 > 0$, there hold the inequalities:*

(i) $\left[v^0 w^0 - \vec{g}_{\vec{v}}(\vec{v},\vec{w})\right]^2 - [\left(v^0\right)^2 - \|\vec{v}\|^2][\left(w^0\right)^2 - \|\vec{w}\|^2] \geqslant \dfrac{\left(w^0\right)^2 - \|\vec{w}\|^2}{\|\vec{w}\|^2}(\|\vec{v}\|^2 \|\vec{w}\|^2 - \vec{g}_{\vec{v}}(\vec{v},\vec{w}))$;

(ii) $\left[v^0 w^0 - \vec{g}_{\vec{v}}(\vec{v},\vec{w})\right]^2 - [\left(v^0\right)^2 - \|\vec{v}\|^2][\left(w^0\right)^2 - \|\vec{w}\|^2] \geqslant [w^0 \dfrac{\vec{g}_{\vec{v}}(\vec{v},\vec{w})}{\|\vec{w}\|} - v^0 \|\vec{w}\|]^2.$

A somewhat similar statement can be obtained by considering, on the entire space $V \simeq \mathbb{R}^{n+1}$, a positive definite Finsler norm $\hat{F}$ and a 1-form $\omega = \omega_i dx^i$. Then, the mapping $v \mapsto F(v) = \sqrt{\omega^2(v) - \hat{F}^2(v)}$ defines (see Theorem 4.1 in [13]), a smooth, nondegenerate Lorentz-Finsler norm on the convex conic domain $\mathscr{T} = \{v \in V \mid \omega(v) > \hat{F}(v)\} \subset V \setminus \{0\}$. The reverse Cauchy-Schwarz inequality then reads:

$$[\omega(v)\omega(w) - \hat{g}_v(v,w)]^2 \geqslant [\omega^2(v) - \hat{F}^2(v)][\omega^2(w) - \hat{F}^2(w)]. \tag{59}$$

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Appendix A The signature of m-th root metrics

Among the Lorentz-Finsler norms in Section 4, we encountered m-th root metrics, which are functions of the type:

$$F : \mathscr{T} \to \mathbb{R}, \quad F(v) = H(v)^{\frac{1}{m}}, \quad \text{with} \quad H(v) := a_{i_1 \dots i_m} v^{i_1} \dots v^{i_m},$$

where $a_{i_1 \dots i_m}$ are constants and $\mathscr{T}$ is a convex conic domain in $\mathbb{R}^{n+1}$ where $H(v) > 0$. Here, $m > 2$ is fixed.

To determine the signature of $g_{ij} = \frac{1}{2}\frac{\partial F^2}{\partial v^i \partial v^j}$ in some of our examples, we relate it to the signature of the Hessian $H_{ij} := \frac{\partial H}{\partial v^i \partial v^j}$ of H. By direct calculation, we get (see also [8], [26]

$$H_{ij} = mF^{m-2}[g_{ij} + (m-2)F_i F_j]. \tag{60}$$

The following result will greatly help us simplify calculations in concrete examples.

PROPOSITION 14. *If, for a vector $v \in \mathscr{T}$, the matrix $H_{ij}(v)$ has Lorentzian signature $(+,-,-,..,-)$, then $g_{ij}(v)$ has also Lorentzian signature.*

Proof. Assume that $H_{ij}(v)$ has index n; therefore, there exists an n-dimensional subspace of $\mathbb{R}^{n+1}$ where it is negative definite. Pick any vector $w \in \mathscr{T}$ with the property $H_{ij}(v)w^i w^j < 0$. From (60) we find: $g_{ij}(v)w^i w^j + (m-2)(l_i(v)w^j)^2 < 0$. Since $m > 2$, this implies $g_{ij}(v)w^i w^j < 0$, that is, (g_{ij}) is also negative definite on at least the same n-dimensional subspace.

But, $g_{ij}(v)v^i v^j = F^2(v) > 0$, which means that g_{ij} cannot be negative (semi-) definite on the whole V. Consequently, it must be Lorentzian. □

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