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BEST CONSTANT OF THE CRITICAL HARDY–LERAY INEQUALITY FOR CURL–FREE FIELDS IN TWO DIMENSIONS

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Abstract. In this note, we prove that the best-possible constant of the critical Hardy-Leray inequality for curl-free fields is $1/4$, just the same value as the one for all smooth fields. This fact contrasts sharply with the recent result on the subcritical Hardy-Leray inequality for curl-free fields by the authors [6], and shows the criticality of the inequality.

1. Introduction

Let $[0,\infty)\times[0,2\pi)\ni(\rho,\varphi)\mapsto \mathbf{x}={}^t(x_1,x_2)={}^t(\rho\cos\varphi,\rho\sin\varphi)\in\mathbb{R}^2$ denote the polar coordinate system in $\mathbb{R}^2$ composed of the radius ρ and the angle φ. Along these coordinates, define the two vector fields

$$\mathbf{e}_\rho={}^t(\cos\varphi,\sin\varphi),\quad \mathbf{e}_\varphi={}^t(-\sin\varphi,\cos\varphi)$$

which form an orthonormal basis on $\mathbb{R}^2\setminus\{\mathbf{0}\}$ with respect to the standard scalar product $\mathbf{x}\cdot\mathbf{y}=\sum_{k=1,2}x_ky_k$. In terms of such a basis, let us expand every smooth vector field $\mathbf{u}={}^t(u_1,u_2)$ and the gradient operator $\nabla={}^t(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$ as

$$\mathbf{u}=\mathbf{e}_\rho u_\rho+\mathbf{e}_\varphi u_\varphi,\quad \nabla=\mathbf{e}_\rho\partial_\rho+\frac{1}{\rho}\mathbf{e}_\varphi\partial_\varphi,$$

where the scalar fields $u_\rho=\mathbf{e}_\rho\cdot\mathbf{u}$ and $u_\varphi=\mathbf{e}_\varphi\cdot\mathbf{u}$ are the radial-angular components of $\mathbf{u}$, and where $\partial_\rho=\mathbf{e}_\rho\cdot\nabla$ and $\partial_\varphi=\mathbf{e}_\varphi\cdot\nabla$ are the partial radial-angular derivatives. Now, let $B_1(0)$ denote the unit ball in $\mathbb{R}^2$ with center the origin, and let $C_c^\infty(B_1(0))^2$ denote the set of smooth vector fields with compact support on $B_1(0)$. Then the following *critical Hardy-Leray inequalities* hold for any $\mathbf{u}={}^t(u_1,u_2)\in C_c^\infty(B_1(0))^2$:

$$\frac14\int_{B_1(0)}\frac{|\mathbf{u}(\mathbf{x})|^2}{|\mathbf{x}|^2\left(\log\frac{1}{|\mathbf{x}|}\right)^2}dx\leqslant\int_{B_1(0)}\left|\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\nabla\mathbf{u}(\mathbf{x})\right|^2dx=\int_{B_1(0)}|\partial_\rho\mathbf{u}(\mathbf{x})|^2dx,$$
$$\frac14\int_{B_1(0)}\frac{|\mathbf{u}(\mathbf{x})|^2}{|\mathbf{x}|^2\left(\log\frac{1}{|\mathbf{x}|}\right)^2}dx\leqslant\int_{B_1(0)}|\nabla\mathbf{u}(\mathbf{x})|^2dx=\int_{B_1(0)}\left(|\partial_\rho\mathbf{u}(\mathbf{x})|^2+\frac{1}{\rho^2}|\partial_\varphi\mathbf{u}(\mathbf{x})|^2\right)dx,$$

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where $\nabla \mathbf{u}(\mathbf{x}) = \left(\frac{\partial u_i(\mathbf{x})}{\partial x_j} \right)_{1 \leqslant i,j \leqslant 2}$ denotes the Jacobi matrix of $\mathbf{u}$; see [9], also [8], [11]. Both the values $1/4$ on the left-hand sides are known to be the best, in the sense that

$$\frac{1}{4} = \inf_{\substack{\mathbf{u} \in C_c^\infty(B_1(0))^2 \\ \mathbf{u} \neq \mathbf{0}}} \frac{\int_{B_1(0)} \left| \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla \mathbf{u}(\mathbf{x}) \right|^2 dx}{\int_{B_1(0)} \frac{|\mathbf{u}(\mathbf{x})|^2}{|\mathbf{x}|^2 \left(\log \frac{1}{|\mathbf{x}|} \right)^2} dx} = \inf_{\substack{\mathbf{u} \in C_c^\infty(B_1(0))^2 \\ \mathbf{u} \neq \mathbf{0}}} \frac{\int_{B_1(0)} |\nabla \mathbf{u}(\mathbf{x})|^2 dx}{\int_{B_1(0)} \frac{|\mathbf{u}(\mathbf{x})|^2}{|\mathbf{x}|^2 \left(\log \frac{1}{|\mathbf{x}|} \right)^2} dx}$$

holds true.

In this note, we study whether the best constant $1/4$ could change when we put curl-free conditions on admissible vector fields. More precisely, we show the following:

THEOREM 1. *Let us define two constant numbers $C_1 \geqslant C_0 (\geqslant 1/4)$ by the formulae*

$$C_0 = \inf_{\mathbf{u} \in \mathscr{A}} \frac{\int_{B_1(0)} |\partial_\rho \mathbf{u}|^2 dx}{\int_{B_1(0)} \frac{|\mathbf{u}|^2}{\rho^2 \left(\log \frac{1}{\rho} \right)^2} dx}, \tag{1}$$

$$C_1 = \inf_{\mathbf{u} \in \mathscr{A}} \frac{\int_{B_1(0)} \left(|\partial_\rho \mathbf{u}|^2 + \frac{1}{\rho^2} |\partial_\varphi \mathbf{u}|^2 \right) dx}{\int_{B_1(0)} \frac{|\mathbf{u}|^2}{\rho^2 \left(\log \frac{1}{\rho} \right)^2} dx}, \tag{2}$$

where

$$\mathscr{A} = \left\{ \mathbf{u} \in C_c^\infty(B_1(0))^2 \setminus \{\mathbf{0}\} \,\middle|\, \operatorname{curl} \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0 \right\}.$$

Then we have $C_0 = C_1 = 1/4$.

REMARK 1. We note that the vector field $\mathbf{u}^\perp = {}^t(-u_2, u_1)$ is divergence-free (solenoidal) if and only if $\mathbf{u}$ is curl-free, and that the identities

$$|\mathbf{u}| = |\mathbf{u}^\perp|, \quad |\partial_\rho \mathbf{u}| = |\partial_\rho \mathbf{u}^\perp| \quad \text{and} \quad |\partial_\varphi \mathbf{u}| = |\partial_\varphi \mathbf{u}^\perp|$$

always hold true. Thus the above theorem still holds even if we replace $\mathscr{A}$ by

$$\mathscr{B} = \left\{ \mathbf{u} \in C_c^\infty(B_1(0))^2 \setminus \{\mathbf{0}\} \,|\, \operatorname{div} \mathbf{u} = 0 \right\}.$$

In addition, noticing (from the Poincaré lemma) that every curl-free field $\mathbf{u}$ satisfies $\mathbf{u} = \nabla \phi$ for some $\phi \in C_c^\infty(B_1(0))$, we obtain the following corollary, which seems interesting in itself:

COROLLARY 1. *Define*

$$C_2 = \inf_{\substack{\phi \in C_c^\infty(B_1(0)) \\ \phi \neq 0}} \frac{\int_{B_1(0)} |\Delta \phi|^2 dx}{\int_{B_1(0)} \frac{|\nabla \phi|^2}{|\mathbf{x}|^2 \left(\log \frac{1}{|\mathbf{x}|} \right)^2} dx}.$$

Then we have $C_2 = \frac{1}{4}$.

The result of Theorem 1 is a striking contrast to the recent work by the authors [6], where we studied the (subcritical) Hardy-Leray inequality (with a radial power weight) for curl-free vector fields; we proved that the best constant is strictly larger than that of the same inequality for unconstrained vector fields, by explicitly computing it with the aid of the spectral decomposition of the Laplace-Beltrami operator on the sphere. If we put solenoidal (divergence-free) constraint on the admissible vector fields, similar phenomena occur for subcritical Hardy-Leray inequalities [1], [3], [4]. See also [5], [2], [7], [10] for related results.

2. Proofs

It suffices to check $C_1 = 1/4$, since this equation together with $C_1 \geqslant C_0 \geqslant 1/4$ directly proves $C_0 = 1/4$.

First of all, the curl of any vector field $\mathbf{u} = \mathbf{e}_\rho u_\rho + \mathbf{e}_\varphi u_\varphi$ can be expressed as

$$\operatorname{curl}\mathbf{u} = \nabla \times \mathbf{u} = \partial_\rho u_\varphi + \frac{1}{\rho} u_\varphi - \frac{1}{\rho}\partial_\varphi u_\rho$$

in terms of the polar coordinates, which one can directly verify by the elementary vector calculus. Hence the condition that $\mathbf{u}$ is curl-free is equivalent to the equation

$$u_\varphi + \rho\partial_\rho u_\varphi = \partial_\varphi u_\rho. \tag{3}$$

In order to evaluate C_1 in (2), let us start with the inequality

$$C_1 \int_{B_1(0)} \frac{|\mathbf{u}|^2}{\rho^2\left(\log\frac{1}{\rho}\right)^2} dx \leqslant \int_{B_1(0)} \left(|\partial_\rho \mathbf{u}|^2 + \frac{1}{\rho^2}|\partial_\varphi \mathbf{u}|^2\right) dx.$$

Change the radius ρ into a (alternative) radial coordinate t by the Emden transformation

$$t = \log(1/\rho)$$

together with its differential rule and the measure transformation:

$$\partial_t = -\rho\partial_\rho, \quad dx = \rho\, d\rho\, d\varphi = -\rho^2 dt\, d\varphi.$$

Then the curl-free condition (3) and the above integral inequality are changed into

$$\begin{aligned} &u_\varphi - \partial_t u_\varphi = \partial_\varphi u_\rho, \\ &C_1 \int_0^{2\pi}\int_0^\infty \frac{|\mathbf{u}|^2}{t^2} dt\, d\varphi \leqslant \int_0^{2\pi}\int_0^\infty \left(|\partial_t \mathbf{u}|^2 + |\partial_\varphi \mathbf{u}|^2\right) dt\, d\varphi. \end{aligned}$$

Next, we introduce a new vector field $\mathbf{v}(= v_\rho \mathbf{e}_\rho + v_\varphi \mathbf{e}_\varphi)$ by the formula

$$\mathbf{u} = \sqrt{t}\,\mathbf{v} \quad \left(\text{or equivalently } \mathbf{u}(\mathbf{x}) = \sqrt{\log(1/|\mathbf{x}|)}\,\mathbf{v}(\mathbf{x})\right).$$

Then the curl-free equation and the integral inequality above can be re-written as

$$\left(1-\frac{1}{2t}\right)v_\varphi-\partial_t v_\varphi=\partial_\varphi v_\rho,$$

$$\left(C_1-\frac{1}{4}\right)\int_0^{2\pi}\int_0^\infty\frac{|\mathbf{v}|^2}{t}dtd\varphi\leqslant\int_0^{2\pi}\int_0^\infty\left(|\partial_t\mathbf{v}|^2+|\partial_\varphi\mathbf{v}|^2\right)tdtd\varphi.$$

To further proceed, let us rechange the radial coordinate by the transformation formula $s=\log t$ (together with the differential rules $\partial_t=e^{-s}\partial_s$ and $ds=dt/t$). Then the above equation and inequality are again re-written as

$$\left(e^s-\frac{1}{2}\right)v_\varphi-\partial_s v_\varphi=e^s\partial_\varphi v_\rho, \tag{4}$$

$$\left(C_1-\frac{1}{4}\right)\int_0^{2\pi}\int_{-\infty}^\infty|\mathbf{v}|^2dsd\varphi\leqslant\int_0^{2\pi}\int_{-\infty}^\infty\left(|\partial_s\mathbf{v}|^2+e^{2s}|\partial_\varphi\mathbf{v}|^2\right)dsd\varphi. \tag{5}$$

Now let us choose a test vector field of the form

$$\mathbf{v}(s,\varphi)=\mathbf{e}_\rho f(s) \tag{6}$$

or equivalently $v_\rho=f(s)$ and $v_\varphi\equiv 0$, where $f\in C_c^\infty(\mathbb{R})\setminus\{0\}$ is a function depending only on s. Then it is clear that the $\mathbf{v}$ in (6) satisfies the curl-free condition (4). By testing (5) by $\mathbf{v}=\mathbf{v}(s,\varphi)$ in (6), we have

$$\begin{aligned}0\leqslant\left(C_1-\frac{1}{4}\right)&=\inf_{\mathbf{v}\in\mathscr{A}}\frac{\int_0^{2\pi}\int_{-\infty}^\infty\left(|\partial_s\mathbf{v}|^2+e^{2s}|\partial_\varphi\mathbf{v}|^2\right)dsd\varphi}{\int_0^{2\pi}\int_{-\infty}^\infty|\mathbf{v}|^2dsd\varphi}\\&\leqslant\inf_{f\in C_c^\infty(\mathbb{R})\setminus\{0\}}\frac{\int_{-\infty}^\infty((f'(s))^2+e^{2s}(f(s))^2)ds}{\int_{-\infty}^\infty(f(s))^2ds}.\end{aligned} \tag{7}$$

Subsequently, let us choose a sequence $\{f_n\}_{n\in\mathbb{N}}\subset C_c^\infty(\mathbb{R})\setminus\{0\}$ of functions by the formula

$$f_n(s)=f\left(\frac{s}{n}+n\right)\qquad\forall n\in\mathbb{N},$$

and test by f_n the last-hand side of (7). Then we confirm that

$$\begin{aligned}&\frac{\int_{-\infty}^\infty\left((f_n'(s))^2+e^{2s}(f_n(s))^2\right)ds}{\int_{-\infty}^\infty(f_n(s))^2ds}\\&=\frac{\int_{-\infty}^\infty\left(n^{-2}(f'(s))^2+e^{2(ns-n^2)}(f(s))^2\right)ds}{\int_{-\infty}^\infty(f(s))^2ds}\\&\leqslant\frac{1}{n^2}\frac{\int_{-\infty}^\infty(f'(s))^2ds}{\int_{-\infty}^\infty(f(s))^2ds}+e^{2(nR_f-n^2)}\qquad\forall n\in\mathbb{N},\end{aligned}$$

where $R_f:=\sup_{s\in\mathbb{R},f(s)\neq 0}|s|$ is a finite positive number independent of n. Passing to the limit $n\to\infty$, we then see that

$$\frac{\int_{-\infty}^\infty\left((f_n'(s))^2+e^{2s}(f_n(s))^2\right)ds}{\int_{-\infty}^\infty(f_n(s))^2ds}=O(n^{-2})+O\left(\exp\left(2(nR_f-n^2)\right)\right)\to 0,$$

and hence that

$$\inf_{f\in C_c^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_{-\infty}^{\infty}\left((f'(s))^2+e^{2s}(f(s))^2\right)ds}{\int_{-\infty}^{\infty}(f(s))^2 ds}=0.$$

Therefore, we obtain $C_1 = 1/4$ from (7). □

Proof of Corollary 1. Notice that $B_1(0)$ is simply connected, and we have the equivalence relation

$$\mathbf{u}\in\mathscr{A} \iff \text{there exists } \phi\in C_c^\infty(B_1(0))\setminus\{0\} \text{ such that } \mathbf{u}=\nabla\phi$$

by the Poincaré lemma. Hence, applying this fact to Theorem 1, we have

$$\frac{1}{4}=\inf_{\mathbf{u}\in\mathscr{A}} \frac{\int_{B_1(0)}|\nabla\mathbf{u}(\mathbf{x})|^2\,dx}{\int_{B_1(0)}\frac{|\mathbf{u}(\mathbf{x})|^2}{|\mathbf{x}|^2\left(\log\frac{1}{|\mathbf{x}|}\right)^2}dx}=\inf_{\phi\in C_c^\infty(B_1(0))\setminus\{0\}} \frac{\int_{B_1(0)}\left|D^2\phi(\mathbf{x})\right|^2dx}{\int_{B_1(0)}\frac{|\nabla\phi(\mathbf{x})|^2}{|\mathbf{x}|^2\left(\log\frac{1}{|\mathbf{x}|}\right)^2}dx}, \tag{8}$$

where $D^2\phi = (\frac{\partial^2\phi}{\partial x_i x_j})_{1\leqslant i,j\leqslant 2}$ is the Hessian matrix of ϕ. On the other hand, by using the elementary identity

$$|D^2\phi|^2=\sum_{i,j=1}^{2}\left(\frac{\partial^2\phi}{\partial x_i x_j}\right)^2=\operatorname{div}\left(\frac{1}{2}\nabla|\nabla\phi|^2-(\Delta\phi)\nabla\phi\right)+(\Delta\phi)^2,$$

an integration by parts yields that $\int_{B_1(0)}|D^2\phi|^2dx=\int_{B_1(0)}|\Delta\phi|^2dx$. Combining this result with the numerator on the last-hand side of (8), we get $C_2 = 1/4$. □

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