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JOINT NUMERICAL RADIUS OF SPHERICAL ALUTHGE TRANSFORMS OF TUPLES OF HILBERT SPACE OPERATORS

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Abstract. Let $\mathbf{T} = (T_1,\ldots,T_d)$ be a d-tuple of operators on a complex Hilbert space $\mathscr{H}$. The spherical Aluthge transform of $\mathbf{T}$ is the d-tuple given by $\widehat{\mathbf{T}} := (\sqrt{P}V_1\sqrt{P},\ldots,\sqrt{P}V_d\sqrt{P})$ where $P := \sqrt{T_1^*T_1+\ldots+T_d^*T_d}$ and $(V_1,\ldots,V_d)$ is a joint partial isometry such that $T_k = V_kP$ for all $1 \leqslant k \leqslant d$. In this paper, we prove several inequalities involving the joint numerical radius and the joint operator norm of $\widehat{\mathbf{T}}$. Moreover, a characterization of the joint spectral radius of an operator tuple $\mathbf{T}$ via n-th iterated of spherical Aluthge transform is established.

1. Introduction and Preliminaries

Throughout this paper, $\mathscr{H}$ will be a complex Hilbert space, with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. $\mathscr{B}(\mathscr{H})$ stands for the Banach algebra of all bounded linear operators on $\mathscr{H}$ and I denotes the identity operator on $\mathscr{H}$. In all that follows, by an operator we mean a bounded linear operator. The range and the null space of an operator T are denoted by $\mathscr{R}(T)$ and $\mathscr{N}(T)$, respectively. Also, T^* will be denoted to be the adjoint of T. An operator T is called positive if $\langle Tx,x\rangle \geqslant 0$ for all $x \in \mathscr{H}$, and we then write $T \geqslant 0$. Further, the square root of every positive operator T is denoted by $T^{\frac{1}{2}}$. If $T \in \mathscr{B}(\mathscr{H})$, then the absolute value of T is denoted by $|T|$ and given by $|T| = (T^*T)^{\frac{1}{2}}$.

For $T \in \mathscr{B}(\mathscr{H})$, the spectral radius of T is defined by

$$r(T) = \sup\{|\lambda|\,;\ \lambda \in \sigma(T)\},$$

where $\sigma(T)$ denotes the spectrum of T. Moreover, the numerical radius and operator norm of T are denoted by $\omega(T)$ and $\|T\|$ respectively and they are given by

$$\omega(T) = \sup\{|\langle Tx,x\rangle|\,;\ x \in \mathscr{H}, \|x\| = 1\}$$

and

$$\|T\| = \sup\{\|Tx\|\,;\ x \in \mathscr{H}, \|x\| = 1\}.$$

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It is well-known that for $T \in \mathscr{B}(\mathscr{H})$ we have

$$\frac{\|T\|}{2} \leqslant \max\left\{r(T), \frac{\|T\|}{2}\right\} \leqslant \omega(T) \leqslant \|T\|. \tag{1}$$

It has been shown in [36] that if $T \in \mathscr{B}(\mathscr{H})$, then

$$\omega(T) = \sup_{\theta \in \mathbb{R}} \left\|\Re(e^{i\theta}T)\right\|, \tag{2}$$

where $\Re(X) := \frac{X+X^*}{2}$ for a given operator X. For more results, we refer the reader to the book by Gustafson and Rao [20].

An operator $U \in \mathscr{B}(\mathscr{H})$ is said to be a partial isometry if $\|Ux\| = \|x\|$ for every $x \in \mathscr{N}(U)^{\perp}$. Let $T = U|T|$ be the polar decomposition of $T \in \mathscr{B}(\mathscr{H})$ with U is a partial isometry. The Aluthge transform of T was first defined in [1] by $\widetilde{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transformation has attracted considerable attention over the last two decades (see, for example, [2, 9, 16, 23, 24, 27, 37]). The following properties of $\widetilde{T}$ are well-known (see [23]):

(i) $\|\widetilde{T}\| \leqslant \|T\|$,

(ii) $r(\widetilde{T}) = r(T)$,

(iii) $\omega(\widetilde{T}) \leqslant \omega(T)$.

Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^d$ be a d-tuple of operators. The joint numerical range of $\mathbf{T}$ is introduced by A.T. Dash [15] as:

$$JtW(\mathbf{T}) = \{(\langle T_1x,x\rangle, \dots, \langle T_dx,x\rangle)\,;\, x \in \mathscr{H},\ \|x\| = 1\}.$$

If $d = 1$, we get the definition of the classical numerical range of an operator T, denoted by $W(T)$, which is firstly introduced by Toeplitz in [33]. It is well-known that $W(T)$ is convex (see [28, 19]). Unlike the classical numerical range, $JtW(\mathbf{T})$ may be non convex for $d \geqslant 2$. For a survey of results concerning the convexity of $JtW(\mathbf{T})$, the reader may see [15, 29] and their references. The joint numerical radius of an operator tuple $\mathbf{T} = (T_1, \dots, T_d)$ is defined in [12] as

$$\begin{aligned}\omega(\mathbf{T}) &= \sup\{\|\lambda\|_2\,;\,\lambda = (\lambda_1, \dots, \lambda_d) \in JtW(\mathbf{T})\} \\ &= \sup\left\{\left(\sum_{k=1}^{d} |\langle T_kx,x\rangle|^2\right)^{\frac{1}{2}}\,;\, x \in \mathscr{H},\ \|x\| = 1\right\}.\end{aligned}$$

It was shown in [4] that for an operator tuple $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^d$, we have

$$\omega(\mathbf{T}) = \sup_{(\lambda_1,\dots,\lambda_d) \in \overline{\mathbb{B}}_d} \omega(\lambda_1T_1 + \dots + \lambda_dT_d), \tag{3}$$

where $\mathbb{B}_d$ denotes the open unit ball in $\mathbb{C}^d$ with respect to the euclidean norm, and $\overline{\mathbb{B}}_d$ is its closure i.e.

$$\overline{\mathbb{B}}_d := \left\{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d;\ \|\lambda\|_2^2 := \sum_{k=1}^{d} |\lambda_k|^2 \leqslant 1 \right\}.$$

Given a d-tuple $\mathbf{T} = (T_1, \ldots, T_d)$ of operators on $\mathscr{H}$, the joint norm of $\mathbf{T}$ is defined as

$$\|\mathbf{T}\| := \sup \left\{ \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{\frac{1}{2}};\ x \in \mathscr{H},\ \|x\| = 1 \right\}.$$

Notice that $\|\cdot\|$ and $\omega(\cdot)$ are equivalent norms on $\mathscr{B}(\mathscr{H})^d$. More precisely, for every $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$ we have

$$\frac{1}{2\sqrt{d}} \|\mathbf{T}\| \leqslant \omega(\mathbf{T}) \leqslant \|\mathbf{T}\|. \tag{4}$$

Moreover, the inequalities in (4) are sharp (see [5, 31]).

Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$ be a d-tuple of operators, and consider $S = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix}$ as an operator from $\mathscr{H}$ into $\mathbb{H} := \oplus_{i=1}^{d} \mathscr{H}$, that is,

$$S = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} : \mathscr{H} \to \mathbb{H},\ x \mapsto {}^t(T_1 x, \ldots, T_d x). \tag{5}$$

Then, we have $S^*S = (T_1^*, \ldots, T_d^*) \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = \sum_{k=1}^{d} T_k^* T_k$. Since S is an operator from $\mathscr{H}$ into $\mathbb{H}$, then S has a classical polar decomposition $S = VP$, that is,

$$\begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} P = \begin{pmatrix} V_1 P \\ \vdots \\ V_d P \end{pmatrix},$$

where $V = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}$ is a partial isometry from $\mathscr{H}$ to $\mathbb{H}$ and P is the positive operator on $\mathscr{H}$ given by

$$P = (S^*S)^{\frac{1}{2}} = \sqrt{T_1^* T_1 + \ldots + T_d^* T_d}.$$

So $R := V^*V = (V_1^*, \ldots, V_d^*)\begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} = \sum_{k=1}^{d} V_k^* V_k$ is the orthogonal projection onto the initial space of V which is

$$\left(\bigcap_{i=1}^{d} \mathcal{N}(T_i)\right)^{\perp} = \mathcal{N}(S)^{\perp} = \mathcal{N}(P)^{\perp} = \left(\bigcap_{i=1}^{d} \mathcal{N}(V_i)\right)^{\perp}. \tag{6}$$

For $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$, the spherical Aluthge transform of $\mathbf{T}$ is defined as

$$\widehat{\mathbf{T}} = (\widehat{T_1}, \ldots, \widehat{T_d}) := \left(\sqrt{P}V_1\sqrt{P}, \ldots, \sqrt{P}V_d\sqrt{P}\right) \text{ (cf. [10], [11], [25]).}$$

This transformation has been recently investigated by C. Benhida et al. in [6]. It should be mention here that $\widehat{T_i} = \sqrt{P}V_i\sqrt{P}$ is not the Aluthge transform of T_i (for $i \in \{1, \ldots, d\}$). Further, the spherical Duggal transform of $\mathbf{T}$ is defined, as in [26], by

$$\mathbf{T}^D = (T_1^D, \ldots, T_d^D) := (PV_1, \ldots, PV_d).$$

Notice that for $i \in \{1, \ldots, d\}$, the operator $T_i^D = PV_i$ is not the Duggal transform of T_i which is first referred to in [17]. When the operators T_k are pairwise commuting, we say that $\mathbf{T}$ is a commuting d-tuple.

Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators.There are several different notions of a spectrum. For a good description, the reader is referred to [14] and the references therein. There is a well-known notion of a joint spectrum of a commuting d-tuple $\mathbf{T}$ called the Taylor joint spectrum denoted by $\sigma_T(\mathbf{T})$ (see [34]). It is shown in [6] that $\sigma_T(\widehat{\mathbf{T}}) = \sigma_T(\mathbf{T})$ for commuting $\mathbf{T} \in \mathcal{B}(\mathcal{H})^d$. The joint spectral radius of $\mathbf{T}$ is defined to be the number

$$r(\mathbf{T}) = \sup\{\|\lambda\|_2 ; \lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_T(\mathbf{T})\}.$$

It should be mention here that Chō and Żelazko proved in [13] that this definition of $r(\mathbf{T})$ is independent of the choice of the joint spectrum of $\mathbf{T}$. Furthermore, an analogue of the Gelfand-Beurling spectral radius formula for single operators has been established by Müller and Soltysiak in [30] for commuting tuples. Let $\mathbf{T} = (T_1, \ldots, T_m) \in \mathcal{B}(\mathcal{H})^m$ and $\mathbf{S} = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n$. Then the product $\mathbf{TS}$ is defined by

$$\mathbf{TS} = (T_1S_1, \ldots, T_1S_n, T_2S_1, \ldots, T_2S_n, \ldots, T_mS_1, \ldots, T_mS_n) \in \mathcal{B}(\mathcal{H})^{mn}.$$

Especially, $\mathbf{T}^2 = \mathbf{TT}$ and $\mathbf{T}^{n+1} = \mathbf{TT}^n$. It was shown in [30] (cf. [7]) that if $\mathbf{T}$ is commuting, then the joint spectral radius of $\mathbf{T}$ is given by

$$r(\mathbf{T}) = \lim_{n\to\infty} \|\mathbf{T}^n\|^{\frac{1}{n}}. \tag{7}$$

In this paper, we shall show several inequalities for spherical Aluthge transform which are known in the single operator case in Sections 2 and 3. Then, in Section 4 we shall show a characterization of joint spectral radius via n-th iterated of spherical Aluthge transform. It is an extension of the formula $\lim_{n\to\infty} \|\widetilde{T}_n\| = r(T)$, which is proved by the second author in [37], where $\widetilde{T}_n$ means the n-th iterated of Aluthge transform of a single operator (see [37]).

2. Basic inequalities

In this section, we present basic inequalities for spherical Aluthge transform.

THEOREM 1. *Let* $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^d$. *Then,*

$$\|\widehat{\mathbf{T}}\| \leqslant \|\mathbf{T}\|.$$

In order to prove our first result, we need the following lemmas.

LEMMA 1. *Let* $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^d$. *Then*

$$\|\mathbf{T}\| = \left\| \sum_{k=1}^{d} T_k^* T_k \right\|^{\frac{1}{2}}.$$

Proof. Since $\sum_{k=1}^{d} T_k^* T_k \geqslant 0$, then it follows that

$$\|\mathbf{T}\|^2 = \sup_{\|x\|=1} \sum_{k=1}^{d} \|T_k x\|^2 = \sup_{\|x\|=1} \langle \sum_{k=1}^{d} T_k^* T_k x, x \rangle = \left\| \sum_{k=1}^{d} T_k^* T_k \right\|. \quad \square$$

LEMMA 2. *Let* $A, X_k \in \mathscr{B}(\mathscr{H})$ *for* $k = 1, 2, \dots, d$. *Then*

$$\left\| \sum_{k=1}^{d} X_k^* A X_k \right\| \leqslant \left\| \sum_{k=1}^{d} X_k^* X_k \right\| \|A\|.$$

Proof. It can be seen that

$$\begin{aligned}
\left\| \sum_{k=1}^{d} X_k^* A X_k \right\| &= \left\| \begin{pmatrix} X_1^* & \cdots & X_d^* \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ X_d & 0 & \cdots & 0 \end{pmatrix} \right\| \\
&\leqslant \left\| \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \right\| \left\| \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ X_d & 0 & \cdots & 0 \end{pmatrix} \right\|^2 \\
&= \|A\| \left\| \begin{pmatrix} X_1^* & \cdots & X_d^* \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ X_d & 0 & \cdots & 0 \end{pmatrix} \right\| \\
&= \|A\| \left\| \sum_{k=1}^{d} X_k^* X_k \right\|.
\end{aligned}$$

This proves the desired inequality. □

Proof of Theorem 1. First of all, we notice that, in view of Lemma 1, we have

$$\|\mathbf{T}\|^2 = \left\| \sum_{k=1}^{d} T_k^* T_k \right\| = \|P\|^2.$$

Further, by using Lemma 2, we see that

$$\begin{aligned} \|\widehat{\mathbf{T}}\|^2 &= \left\| \sum_{k=1}^{d} \widehat{T_k}^* \widehat{T_k} \right\| \\ &= \left\| \sum_{k=1}^{d} P^{\frac{1}{2}} V_k^* P V_k P^{\frac{1}{2}} \right\| \\ &\leqslant \|P\| \left\| \sum_{k=1}^{d} P^{\frac{1}{2}} V_k^* V_k P^{\frac{1}{2}} \right\| = \|P\| \cdot \|P\| = \|\mathbf{T}\|^2, \end{aligned}$$

where the third equation follows from the fact that $\sum_{k=1}^{d} V_k^* V_k$ is a projection onto $\overline{\mathscr{R}(P)}$.

Next, we shall show inequalities of joint numerical radius for spherical Aluthge transform. This discussion will be divided into two parts. We treat non-commuting tuples of operators in the first part.

THEOREM 2. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$. *Then,*

$$\omega(\widehat{\mathbf{T}}) \leqslant \frac{1}{2}\omega(\mathbf{T}) + \frac{1}{2}\omega\big(\mathbf{T}^D\big). \tag{8}$$

To prove the result, we will use the following theorems.

THEOREM A. ([21, 32]) Let $T \in \mathscr{B}(\mathscr{H})$. Then

$$\overline{W(T)} = \bigcap_{\mu \in \mathbb{C}} \{\lambda \in \mathbb{C}\,;\, |\lambda - \mu| \leqslant \|T - \mu I\|\}.$$

THEOREM B. ([8], [18, Theorem 3.12.1]) Let A be a self-adjoint invertible operator and $X \in \mathscr{B}(\mathscr{H})$. Then

$$2\|X\| \leqslant \|AXA^{-1} + A^{-1}XA\|.$$

Proof of Theorem 2. In view of (3), we have

$$\omega(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega(\lambda_1 T_1 + \ldots + \lambda_d T_d) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega(U_\lambda P), \tag{9}$$

$$\omega(\widehat{\mathbf{T}}) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}) \ \text{ and } \ \omega\big(\mathbf{T}^D\big) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega(P U_\lambda), \tag{10}$$

where $U_\lambda = \lambda_1 V_1 + \ldots + \lambda_d V_d$. We shall prove

$$\overline{W(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})} \subseteq \overline{W\left(\frac{U_\lambda P + PU_\lambda}{2}\right)},$$

where $\overline{W(X)}$ means the closure of numerical range of $X \in \mathscr{B}(\mathscr{H})$. By taking into consideration Theorem A, it suffices to prove the following norm inequality.

$$\|P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}} - \mu I\| \leqslant \left\| \frac{U_\lambda P + PU_\lambda}{2} - \mu I \right\| \tag{11}$$

for all $\mu \in \mathbb{C}$.

For $\varepsilon > 0$, let $P_\varepsilon := P + \varepsilon I > 0$. Then P_ε is positive invertible. Then by Theorem B, we have

$$\begin{aligned} 2\|P_\varepsilon^{\frac{1}{2}} U_\lambda P_\varepsilon^{\frac{1}{2}} - \mu I\| &\leqslant \|P_\varepsilon^{\frac{1}{2}} (P_\varepsilon^{\frac{1}{2}} U_\lambda P_\varepsilon^{\frac{1}{2}} - \mu I) P_\varepsilon^{-\frac{1}{2}} + P_\varepsilon^{-\frac{1}{2}} (P_\varepsilon^{\frac{1}{2}} U_\lambda P_\varepsilon^{\frac{1}{2}} - \mu I) P_\varepsilon^{\frac{1}{2}}\| \\ &= \|P_\varepsilon U_\lambda + U_\lambda P_\varepsilon - 2\mu I\|. \end{aligned}$$

By letting $\varepsilon \searrow 0$, we get (11), and hence

$$\overline{W(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})} \subseteq \overline{W\left(\frac{U_\lambda P + PU_\lambda}{2}\right)} \subseteq \frac{1}{2}\left\{\overline{W(PU_\lambda)} + \overline{W(U_\lambda P)}\right\}.$$

Therefore, we get

$$\omega(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}) \leqslant \frac{1}{2}\Big(\omega(PU_\lambda) + \omega(U_\lambda P)\Big),$$

which in turn implies, by taking the supremum over all $(\lambda_1, \ldots, \lambda_d) \in \overline{\mathbb{B}}_d$, that

$$\omega(\widehat{\mathbf{T}}) \leqslant \frac{1}{2}\omega(\mathbf{T}) + \frac{1}{2}\omega(\mathbf{T}^D).$$

Hence, the proof is complete.

In the second part of this discussion, we shall treat commuting tuples of operators.

THEOREM 3. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$ be a commuting tuple of operators. Then*

$$\omega(\widehat{\mathbf{T}}) \leqslant \omega(\mathbf{T}).$$

To prove this, we will introduce the following lemma.

LEMMA 3. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$, and let $T_j = V_j P$ with $P = (\sum_{j=1}^d T_j^* T_j)^{\frac{1}{2}}$. Then $\mathbf{T}$ is commuting if and only if*

$$V_j P V_k = V_k P V_j$$

holds for $j, k = 1, \ldots, d$.

Proof. Since $T_jT_k = T_kT_j$, we have $V_jPV_kP = V_kPV_jP$, that is, $V_jPV_k = V_kPV_j$ holds on $\overline{\mathscr{R}(P)}$. By (6), $\overline{\mathscr{R}(P)}^{\perp} = \mathscr{N}(P) = \bigcap_{k=1}^{d} \mathscr{N}(V_k) \subseteq \mathscr{N}(V_k)$ for $k = 1,\ldots,d$. Hence we have $V_jPV_k = V_kPV_j = 0$ on $\mathscr{N}(P)$. Therefore $V_jPV_k = V_kPV_j$ holds on $\mathscr{H} = \overline{\mathscr{R}(P)} \oplus \mathscr{N}(P)$. The converse implication is obvious. Thus the proof is completed. $\square$

Proof of Theorem 3. Since (8), we have only to prove the following inequality:

$$\omega\left(\mathbf{T}^D\right) \leqslant \omega(\mathbf{T}),$$

that is, we will prove that for every $(\lambda_1,\ldots,\lambda_d) \in \overline{\mathbb{B}}_d$, we have

$$\omega(PU_\lambda) \leqslant \omega(U_\lambda P), \tag{12}$$

where $U_\lambda = \sum_{j=1}^{d} \lambda_j V_j$. Let $x \in \mathscr{H}$ with $\|x\| = 1$ and $(\lambda_1,\ldots,\lambda_d) \in \overline{\mathbb{B}}_d$. Since $\sum_{k=1}^{d} V_k^* V_k$ is a projection onto $\overline{\mathscr{R}(P)}$, we have

$$\langle PU_\lambda x, x\rangle = \left\langle \left(\sum_{k=1}^{d} V_k^* V_k\right) PU_\lambda x, x\right\rangle = \sum_{k=1}^{d} \langle V_k PU_\lambda x, V_k x\rangle.$$

Moreover, by Lemma 3, we see that

$$V_k PU_\lambda = V_k P\left(\sum_{j=1}^{d} \lambda_j V_j\right) = \left(\sum_{j=1}^{d} \lambda_j V_j\right) PV_k = U_\lambda PV_k.$$

Then, we obtain

$$\langle PU_\lambda x, x\rangle = \sum_{k=1}^{d} \langle V_k PU_\lambda x, V_k x\rangle = \sum_{k=1}^{d} \langle U_\lambda PV_k x, V_k x\rangle.$$

Put $y_k = \frac{V_k x}{\|V_k x\|}$. Since $\sum_{k=1}^{d} V_k^* V_k$ is a projection onto $\overline{\mathscr{R}(P)}$, we have

$$\begin{aligned}
|\langle PU_\lambda x, x\rangle| &= \left|\sum_{k=1}^{d} \|V_k x\|^2 \langle U_\lambda P y_k, y_k\rangle\right| \\
&\leqslant \sum_{k=1}^{d} \|V_k x\|^2 \left|\langle U_\lambda P y_k, y_k\rangle\right| \\
&\leqslant \sum_{k=1}^{d} \|V_k x\|^2 \omega(U_\lambda P) \\
&= \left\langle \sum_{k=1}^{d} V_k^* V_k x, x\right\rangle \omega(U_\lambda P) \leqslant \omega(U_\lambda P).
\end{aligned}$$

So, we get (12) as required. Thus, the proof is finished by taking the supremum over all $(\lambda_1,\ldots,\lambda_d) \in \overline{\mathbb{B}}_d$ in (12) and then using (9) together with (10).

QUESTION 1. It would be interesting to know whether or not the inequalities $\omega(\mathbf{T}^D) \leqslant \omega(\mathbf{T})$ and $\omega(\widehat{\mathbf{T}}) \leqslant \omega(\mathbf{T})$ hold for non-commuting d-tuples of operators?

3. Precise estimation of joint numerical radius

In this section, we shall give a precise estimation of joint numerical radius.

THEOREM 4. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$ *be a* d*-tuple of operators. Then,*

$$\omega(\mathbf{T}) \leqslant \frac{1}{2}\|\mathbf{T}\| + \frac{1}{2}\omega(\widehat{\mathbf{T}}).$$

REMARK 1. By letting $d = 1$ in Theorem 4, we get the well-known result proved by the second author in [36] asserting that

$$\omega(T) \leqslant \frac{1}{2}\|T\| + \frac{1}{2}\omega(\widetilde{T}),$$

for every $T \in \mathscr{B}(\mathscr{H})$.

Proof. By (9), we see that

$$\omega(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\overline{\mathbb{B}}_d} \omega(U_\lambda P),$$

where $U_\lambda = \lambda_1 V_1 + \ldots + \lambda_d V_d$. Now, let $x \in \mathscr{H}$ be such that $\|x\| = 1$. By the generalized polarization identity (see [36]), we see that

$$\begin{aligned}
\langle e^{i\theta} U_\lambda P x, x\rangle &= \langle e^{i\theta} P x, U_\lambda^* x\rangle \\
&= \frac{1}{4}\big(\langle P(e^{i\theta} + U_\lambda^*)x, (e^{i\theta} + U_\lambda^*)x\rangle - \langle P(e^{i\theta} - U_\lambda^*)x, (e^{i\theta} - U_\lambda^*)x\rangle\big) \\
&\quad + \frac{i}{4}\big(\langle P(e^{i\theta} + iU_\lambda^*)x, (e^{i\theta} + iU_\lambda^*)x\rangle - \langle P(e^{i\theta} - iU_\lambda^*)x, (e^{i\theta} - iU_\lambda^*)x\rangle\big).
\end{aligned}$$

Noting that all inner products of the terminal side are all positive since $P \geqslant 0$. Hence, one observes that

$$\begin{aligned}
\langle \Re(e^{i\theta} U_\lambda P)x, x\rangle &= \Re(\langle e^{i\theta} U_\lambda P x, x\rangle) \\
&= \frac{1}{4}\big(\langle (e^{i\theta} + U_\lambda^*)^* P(e^{i\theta} + U_\lambda^*)x, x\rangle - \langle (e^{i\theta} - U_\lambda^*)^* P(e^{i\theta} - U_\lambda^*)x, x\rangle\big) \\
&\leqslant \frac{1}{4}\langle (e^{i\theta} + U_\lambda^*)^* P(e^{i\theta} + U_\lambda^*)x, x\rangle \\
&\leqslant \frac{1}{4}\left\|(e^{i\theta} + U_\lambda^*)^* P(e^{i\theta} + U_\lambda^*)\right\| \\
&= \frac{1}{4}\left\|P^{\frac{1}{2}}(e^{i\theta} + U_\lambda^*)(e^{-i\theta} + U_\lambda)P^{\frac{1}{2}}\right\| \quad (\text{by } \|X^*X\| = \|XX^*\|) \\
&= \frac{1}{4}\left\|P + P^{\frac{1}{2}}U_\lambda^* U_\lambda P^{\frac{1}{2}} + 2\Re(e^{i\theta} P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})\right\| \\
&\leqslant \frac{1}{4}\|P\| + \frac{1}{4}\|P\|\,\|U_\lambda^* U_\lambda\| + \frac{1}{2}\left\|\Re(e^{i\theta} P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})\right\| \\
&\leqslant \frac{1}{4}\|P\| + \frac{1}{4}\|P\|\,\|U_\lambda^* U_\lambda\| + \frac{1}{2}\omega\left(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}\right) \quad (\text{by } (2)).
\end{aligned}$$

So, by taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality and then using (2) we get

$$\begin{aligned}\omega(U_\lambda P) &\leqslant \frac{1}{4}\|P\| + \frac{1}{4}\|P\|\left\|U_\lambda^* U_\lambda\right\| + \frac{1}{2}\omega\left(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}\right) \\ &\leqslant \frac{1}{4}\|P\| + \frac{1}{4}\|P\|\left\|U_\lambda^* U_\lambda\right\| + \frac{1}{2}\omega(\widehat{\mathbf{T}}) \quad (\text{by } (3)).\end{aligned} \tag{13}$$

On the other hand, let $x \in \mathscr{H}$ with $\|x\| = 1$ and $(\lambda_1, \ldots, \lambda_d) \in \overline{\mathbb{B}}_d$. By applying the Cauchy-Schwarz inequality and making elementary calculations we see that

$$\begin{aligned}\langle U_\lambda^* U_\lambda x, x\rangle &= \sum_{j=1}^{d}\sum_{k=1}^{d} \overline{\lambda_j}\lambda_k \langle V_k x, V_j x\rangle \leqslant \sum_{j=1}^{d}\sum_{k=1}^{d} |\lambda_j| \cdot |\lambda_k| \cdot \|V_k x\| \cdot \|V_j x\| \\ &= \left(\sum_{k=1}^{d} |\lambda_k| \cdot \|V_k x\|\right)^2 \leqslant \left(\sum_{j=1}^{d} |\lambda_j|^2\right)\left(\sum_{j=1}^{d} \|V_j x\|^2\right) \\ &= \left(\sum_{j=1}^{d} |\lambda_j|^2\right)\left(\sum_{j=1}^{d} \langle V_j^* V_j x, x\rangle\right) \leqslant \left(\sum_{j=1}^{d} |\lambda_j|^2\right)\left\|\sum_{i=1}^{d} V_i^* V_i\right\| \leqslant 1.\end{aligned}$$

So, by taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we obtain $\left\|U_\lambda^* U_\lambda\right\| \leqslant 1$. This yields, by using (13), that

$$\omega(U_\lambda P) \leqslant \frac{1}{2}\|P\| + \frac{1}{2}\omega(\widehat{\mathbf{T}}).$$

Thus, by taking the supremum over all $(\lambda_1, \ldots, \lambda_d) \in \overline{\mathbb{B}}_d$ in the above inequality and then using (9), we obtain

$$\omega(\mathbf{T}) \leqslant \frac{1}{2}\|P\| + \frac{1}{2}\omega(\widehat{\mathbf{T}}).$$

Therefore, we get the desired result since $\|P\| = \|\mathbf{T}\|$. $\square$

4. Joint spectral radius

In this section, we shall characterize the joint spectral radius via spherical Aluthge transform.

THEOREM 5. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$ *be a commuting* d*-tuple of operators. Then*

$$\lim_{n\to\infty} \|\widehat{\mathbf{T}}_n\| = r(\mathbf{T}),$$

where $\widehat{\mathbf{T}}_n$ *means the* n*-th iteration of spherical Aluthge transform, i.e.,* $\widehat{\mathbf{T}}_n := \widehat{\widehat{\mathbf{T}}_{n-1}}$, *and* $\widehat{\mathbf{T}}_0 := \mathbf{T}$ *for a non-negative integer* n.

We will prove this by similar arguments as in [35]. In order to achieve the goals of the present section, we need the following results.

THEOREM C. ([3]) Let $A,B,X \in \mathscr{B}(\mathscr{H})$. Then

$$\|A^*XB\|^2 \leqslant \|A^*AX\| \, \|XBB^*\|.$$

THEOREM D. ([22]) Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive, and $X \in \mathscr{B}(\mathscr{H})$. Then

$$\|A^{\alpha}XB^{\alpha}\| \leqslant \|AXB\|^{\alpha}\|X\|^{1-\alpha}$$

for all $0 \leqslant \alpha \leqslant 1$.

LEMMA 4. *Let $\mathbf{T} = (T_1,\dots,T_d) \in \mathscr{B}(\mathscr{H})^d$ be a commuting d-tuple of operators. Then the spherical Aluthge transform $\widehat{\mathbf{T}}$ is also a commuting d-tuple of operators.*

Proof. Let $T_k = V_kP$. Then $\widehat{\mathbf{T}} = (\widehat{T}_1,\dots,\widehat{T}_d) = (P^{\frac{1}{2}}V_1P^{\frac{1}{2}},\dots,P^{\frac{1}{2}}V_dP^{\frac{1}{2}})$. By Lemma 3, we have $V_jPV_k = V_kPV_j$ for all $j,k = 1,\dots,d$. Hence we have

$$\widehat{T}_j\widehat{T}_k = P^{\frac{1}{2}}V_jPV_kP^{\frac{1}{2}} = P^{\frac{1}{2}}V_kPV_jP^{\frac{1}{2}} = \widehat{T}_k\widehat{T}_j. \quad \square$$

LEMMA 5. *There is an $s \geqslant r(\mathbf{T})$ for which $\lim_{n\to\infty} \|\widehat{\mathbf{T}}_n\| = s$.*

Proof. By Theorem 1, a sequence $\{\|\widehat{\mathbf{T}}_n\|\}_{n=0}^{\infty}$ is decreasing, and

$$\|\widehat{\mathbf{T}}_n\| \geqslant r(\widehat{\mathbf{T}}_n) = r(\mathbf{T})$$

for all non-negative integer n, where the last equation is shown in [6]. Hence there exists a limit point s of $\{\|\widehat{\mathbf{T}}_n\|\}_{n=0}^{\infty}$ such that $s \geqslant r(\mathbf{T})$. $\square$

LEMMA 6. *For any positive integer k and non-negative integer n,*

$$\left\|\widehat{\mathbf{T}}_{n+1}^k\right\| \leqslant \left\|\widehat{\mathbf{T}}_n^k\right\|.$$

Proof. Since $\widehat{\mathbf{T}}_{n+1} = \widehat{\widehat{\mathbf{T}}_n}$, we only prove $\left\|\widehat{\mathbf{T}}^k\right\| \leqslant \left\|\mathbf{T}^k\right\|$. We notice that by Lemma 1, $\|\mathbf{T}^k\|$ is given as follows:

$$\left\|\mathbf{T}^k\right\|^2 = \left\|\sum_{i_1,\dots,i_k=1}^{d} T_{i_1}^*\cdots T_{i_k}^*T_{i_k}\cdots T_{i_1}\right\|.$$

Let $A_k := \operatorname{diag}(P,\dots,P)$ be a d^k–by–d^k operator matrix, and let

$$X_k = \begin{pmatrix} V_1P\cdots PV_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ V_dP\cdots PV_d & 0 & \cdots & 0 \end{pmatrix}$$

be a d^k–by–d^k operator matrix, where the 1 st column contains $V_{i_1}PV_{i_2}P\cdots PV_{i_k}$ for all $i_1,\ldots,i_k=1,2,\ldots,d$. Then by Theorem C,

$$\begin{aligned}\left\|\widehat{\mathbf{T}}^k\right\|^2 &= \left\|\sum_{i_1,\ldots,i_k=1}^{d}\widehat{T_{i_1}}^*\cdots\widehat{T_{i_k}}^*\widehat{T_{i_k}}\cdots\widehat{T_{i_1}}\right\| \\ &= \left\|\sum_{i_1,\ldots,i_k=1}^{d}P^{\frac{1}{2}}V_{i_1}^*P\cdots PV_{i_k}^*PV_{i_k}P\cdots PV_{i_1}P^{\frac{1}{2}}\right\| \\ &= \left\|A_k^{\frac{1}{2}}X_k^*A_kX_kA_k^{\frac{1}{2}}\right\| = \left\|A_k^{\frac{1}{2}}X_kA_k^{\frac{1}{2}}\right\|^2 \leqslant \|A_kX_k\|\|X_kA_k\|. \qquad (14)\end{aligned}$$

Now, it can be seen that

$$\begin{aligned}\|A_kX_k\| &= \|X_k^*A_k^2X_k\|^{\frac{1}{2}} \\ &= \left\|\sum_{i_1,\ldots,i_k=1}^{d}V_{i_1}^*P\cdots PV_{i_k}^*P^2V_{i_k}P\cdots PV_{i_1}\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{i_1,\ldots,i_k=1}^{d}V_{i_1}^*P\cdots PV_{i_k}^*P\left(\sum_{i_{k+1}=1}^{d}V_{i_{k+1}}^*V_{i_{k+1}}\right)PV_{i_k}P\cdots PV_{i_1}\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{i_1=1}^{d}V_{i_1}^*\left(\sum_{i_2,\ldots,i_{k+1}=1}^{d}PV_{i_2}^*P\cdots PV_{i_k}^*PV_{i_{k+1}}^*V_{i_{k+1}}PV_{i_k}P\cdots PV_{i_2}P\right)V_{i_1}\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{i_1=1}^{d}V_{i_1}^*\left(\sum_{i_2,\ldots,i_{k+1}=1}^{d}T_{i_2}^*\cdots T_{i_k}^*T_{i_{k+1}}^*T_{i_{k+1}}\cdots T_{i_2}\right)V_{i_1}\right\|^{\frac{1}{2}} \\ &\leqslant \left\|\sum_{i_1=1}^{d}V_{i_1}^*V_{i_1}\right\|^{\frac{1}{2}}\left\|\sum_{i_2,\ldots,i_{k+1}=1}^{d}T_{i_2}^*\cdots T_{i_{k+1}}^*T_{i_{k+1}}\cdots T_{i_2}\right\|^{\frac{1}{2}} = \|\mathbf{T}^k\|, \qquad (15)\end{aligned}$$

where the last inequality follows from Lemma 2 and the fact that $\sum_{k=1}^{d}V_k^*V_k$ is a projection onto $\overline{\mathscr{R}(P)}$. Moreover

$$\begin{aligned}\|X_kA_k\| &= \|A_kX_k^*X_kA_k\|^{\frac{1}{2}} \\ &= \left\|\sum_{i_1,\ldots,i_k=1}^{d}PV_{i_1}^*P\cdots PV_{i_k}^*V_{i_k}P\cdots PV_{i_1}P\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{i_1,\ldots,i_k=1}^{d}T_{i_1}^*\cdots T_{i_k}^*T_{i_k}\cdots T_{i_1}\right\|^{\frac{1}{2}} = \|\mathbf{T}^k\|.\end{aligned}$$

Hence we have

$$\left\|\widehat{\mathbf{T}}^k\right\| \leqslant \|A_kX_k\|^{\frac{1}{2}}\|X_kA_k\|^{\frac{1}{2}} \leqslant \left\|\mathbf{T}^k\right\|. \quad \square$$

LEMMA 7. *For any positive integer k,*

$$\left\|\widehat{\mathbf{T}}_{n+1}^{k}\right\| \leqslant \left\|\widehat{\mathbf{T}}_{n}^{k+1}\right\|^{\frac{1}{2}} \left\|\widehat{\mathbf{T}}_{n}^{k-1}\right\|^{\frac{1}{2}}$$

for all $n \geqslant 0$.

Proof. We shall prove $\|\widehat{\mathbf{T}}^{k}\| \leqslant \|\mathbf{T}^{k+1}\|^{\frac{1}{2}}\|\mathbf{T}^{k-1}\|^{\frac{1}{2}}$. Let A_k and X_k be defined in the proof of Lemma 6. Then, by (14) and Theorem D, we have

$$\left\|\widehat{\mathbf{T}}^{k}\right\| = \left\|A_k^{\frac{1}{2}} X_k A_k^{\frac{1}{2}}\right\| \leqslant \|A_k X_k A_k\|^{\frac{1}{2}} \|X_k\|^{\frac{1}{2}}.$$

By taking into consideration the fact that $\sum_{k=1}^{d} V_k^* V_k$ is an orthogonal projection onto $\overline{\mathscr{R}(P)}$, it can be observed that

$$\begin{aligned}
\|A_k X_k A_k\| &= \left\| \sum_{i_1,\ldots,i_k=1}^{d} P V_{i_1}^* P \cdots P V_{i_k}^* P^2 V_{i_k} P \cdots P V_{i_1} P \right\|^{\frac{1}{2}} \\
&= \left\| \sum_{i_1,\ldots,i_k=1}^{d} P V_{i_1}^* P \cdots P V_{i_k}^* P \left(\sum_{i_{k+1}=1}^{d} V_{i_{k+1}}^* V_{i_{k+1}} \right) P V_{i_k} P \cdots P V_{i_1} P \right\|^{\frac{1}{2}} \\
&= \left\| \sum_{i_1,\ldots,i_{k+1}=1}^{d} P V_{i_1}^* P \cdots P V_{i_k}^* P V_{i_{k+1}}^* V_{i_{k+1}} P V_{i_k} P \cdots P V_{i_1} P \right\|^{\frac{1}{2}} \\
&= \left\| \sum_{i_1,\ldots,i_{k+1}=1}^{d} T_{i_1}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_1} \right\|^{\frac{1}{2}} = \left\|\mathbf{T}^{k+1}\right\|.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\|X_k\| &= \left\| \sum_{i_1,\ldots,i_k=1}^{d} V_{i_1}^* P \cdots P V_{i_k}^* V_{i_k} P \cdots P V_{i_1} \right\|^{\frac{1}{2}} \\
&= \left\| \sum_{i_1,\ldots,i_{k-1}=1}^{d} V_{i_1}^* P \cdots P \left(\sum_{i_k=1}^{d} V_{i_k}^* V_{i_k} \right) P \cdots P V_{i_1} \right\|^{\frac{1}{2}} \\
&= \left\| \sum_{i_1,\ldots,i_{k-1}=1}^{d} V_{i_1}^* P \cdots V_{i_{k-1}}^* P^2 V_{i_{k-1}} \cdots P V_{i_1} \right\|^{\frac{1}{2}} \\
&= \left\| X_{k-1}^* A_{k-1}^2 X_{k-1} \right\|^{\frac{1}{2}} \leqslant \left\|\mathbf{T}^{k-1}\right\|,
\end{aligned}$$

where the last inequality follows from (15). Therefore

$$\|\widehat{\mathbf{T}}^{k}\| \leqslant \|A_k X_k A_k\|^{\frac{1}{2}} \|X_k\|^{\frac{1}{2}} \leqslant \|\mathbf{T}^{k+1}\|^{\frac{1}{2}} \|\mathbf{T}^{k-1}\|^{\frac{1}{2}}. \quad \square$$

LEMMA 8. *For each positive integer k, $\|\mathbf{T}^{k+1}\| \leqslant \|\mathbf{T}^k\|\|\mathbf{T}\|$.*

Proof.

$$\begin{aligned}
\|\mathbf{T}^{k+1}\|^2 &= \left\| \sum_{i_1,\dots,i_{k+1}=1}^{d} T_{i_1}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_1} \right\| \\
&= \left\| \sum_{i_1=1}^{d} T_{i_1}^* \left(\sum_{i_2,\dots,i_{k+1}=1}^{d} T_{i_2}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_2} \right) T_{i_1} \right\| \\
&\leqslant \left\| \sum_{i_1=1}^{d} T_{i_1}^* T_{i_1} \right\| \left\| \sum_{i_2,\dots,i_{k+1}=1}^{d} T_{i_2}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_2} \right\| \quad \text{(by Lemma 2)} \\
&= \|\mathbf{T}\|^2 \|\mathbf{T}^k\|^2. \quad \square
\end{aligned}$$

LEMMA 9. *For any positive integer k, $\lim_{n\to\infty} \|\widehat{\mathbf{T}}_n^k\| = s^k$.*

Proof. We will prove the lemma by induction. Since $\lim_{n\to\infty} \|\widehat{\mathbf{T}}_n\| = s$ by Lemma 5, the lemma is proven for $k = 1$. Assume the lemma is proven for $1 \leqslant k \leqslant m$. By Lemmas 7 and 8,

$$\begin{aligned}
\|\widehat{\mathbf{T}}_{n+1}^k\| &\leqslant \|\widehat{\mathbf{T}}_n^{k+1}\|^{\frac{1}{2}} \|\widehat{\mathbf{T}}_n^{k-1}\|^{\frac{1}{2}} \\
&\leqslant \|\widehat{\mathbf{T}}_n^{k}\|^{\frac{1}{2}} \|\widehat{\mathbf{T}}_n\|^{\frac{1}{2}} \|\widehat{\mathbf{T}}_n^{k-1}\|^{\frac{1}{2}}.
\end{aligned} \tag{16}$$

Let $t := \lim_{n\to\infty} \|\widehat{\mathbf{T}}_n^{m+1}\|$. The existence of limit follows from Lemma 6. Taking limits, the induction hypothesis and (16) show that

$$s^m \leqslant t^{\frac{1}{2}} s^{\frac{m-1}{2}} \leqslant s^{\frac{m}{2}} s^{\frac{1}{2}} s^{\frac{m-1}{2}} = s^m.$$

It follows that $t = s^{m+1}$, and the proof is completed. $\square$

Proof of Theorem 5. It follows from Lemmas 6 and 9 that, for each positive integer k, the decreasing sequence $\{\|\widehat{\mathbf{T}}_n^k\|^{\frac{1}{k}}\}_{n=0}^{\infty}$ converges to s. Therefore

$$s \leqslant \|\widehat{\mathbf{T}}_n^k\|^{\frac{1}{k}} \tag{17}$$

for all n and k. Now fix an n. If $r(\mathbf{T}) < s$, then by Lemma 4 and (7),

$$\lim_{k\to\infty} \|\mathbf{T}^k\|^{\frac{1}{k}} = r(\widehat{\mathbf{T}}_n) = r(\mathbf{T})$$

would imply that $\|\widehat{\mathbf{T}}_n^k\|^{\frac{1}{k}} < s$ for sufficiently large k. Clearly this is a contradiction to (17). Therefore, we must have $s = r(\mathbf{T})$, and the result follows from Lemma 5.

REMARK 2. For a d-tuple of operators $\mathbf{T}$ and a natural number n, $\mathbf{T}^n$ is a d^n-tuple of operators. Then we should consider d^n-tuple of operators for $n = 1, 2, \dots$ to use (7). However, since $\widehat{\mathbf{T}}_n$ is also a d-tuple of operators, we only treat d-tuple of operators to get $r(\mathbf{T})$ by Theorem 5.

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