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ON STEVIĆ–SHARMA OPERATOR FROM THE MIXED–NORM SPACES TO ZYGMUND–TYPE SPACES

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Abstract. Let φ be an analytic self-map of the unit disk $\mathbb{D}$, $\mathscr{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$. The boundedness and compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}f = \psi_1 \cdot f\circ\varphi + \psi_2 \cdot f'\circ\varphi$ from the mixed-norm space $H(p,q,\phi)$ to Zygmund-type space $\mathscr{Z}^{\mu}$ and little Zygmund-type space $\mathscr{Z}_0^{\mu}$ are investigated in this paper.

1. Introduction

Let $\mathscr{H}(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. Denote by $\mathbb{N}_0$ the set of nonnegative integers.

A positive continuous function ϕ on [0,1) is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0,1)$ such that (see[18])

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \ \lim_{r\to 1}\frac{\mu(r)}{(1-r)^s} = 0;$$
$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \ \lim_{r\to 1}\frac{\mu(r)}{(1-r)^t} = \infty.$$

For $p,q \in (0,\infty)$ and a normal function ϕ, the mixed-norm space $H(p,q,\phi)$ consists of $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{\frac{1}{p}} < \infty,$$

where the integral means $M_q(f,r)$ are defined by

$$M_q(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q d\theta\right)^{\frac{1}{q}}, \quad 0 \leqslant r < 1.$$

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For $1 \leqslant p,q < \infty$, $H(p,q,\phi)$ equipped with the norm $\|\cdot\|_{H(p,q,\phi)}$ is a Banach space. While for the other vales of p and q, $\|\cdot\|_{H(p,q,\phi)}$ is a quasinorm, $H(p,q,\phi)$ is a Fréchet space but not a Banach space. Note that if $\phi(r) = (1-r)^{\frac{\alpha+1}{p}}$, then $H(p,p,\phi)$ is equivalent to the weighted Bergman space $A^p_\alpha(\mathbb{D})$ defined for $0 < p < \infty$ and $\alpha > -1$, as the spaces of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} r dr d\theta$ is the normalized Lebesgue area measure on $\mathbb{D}$. Recently, many researchers have studied various concrete operators from or to mixed-norm spaces on $\mathbb{D}$ (see, for example, [3, 6, 12, 13, 20, 21, 25, 30]), for the case of the unit ball see [9, 11, 15, 24, 28, 31], while some results in the setting of the polydisk can be found in [19, 22].

Let $\mu : \mathbb{D} \to (0,+\infty)$ be a function that is normal and radial, i.e., $\mu(z) = \mu(|z|)$. A function $f \in \mathscr{H}(\mathbb{D})$ belongs to Zygmund-type space $\mathscr{Z}^\mu$ if

$$b_{\mathscr{Z}^\mu}(f) := \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

The quantity $b_{\mathscr{Z}^\mu}(f)$ is a seminorm on $\mathscr{Z}^\mu$ and a norm on $\mathscr{Z}^\mu/\mathbb{P}_1$, where $\mathbb{P}_1$ is the set of linear complex polynomials. $\mathscr{Z}^\mu$ becomes a Banach space normed by

$$\|f\|_{\mathscr{Z}^\mu} = |f(0)| + |f'(0)| + b_{\mathscr{Z}^\mu}(f).$$

The little Zygmund-type space $\mathscr{Z}^\mu_0$ consists of those functions f in $\mathscr{Z}^\mu$ satisfying

$$\lim_{|z|\to 1} \mu(z)|f''(z)| = 0,$$

and it is easily seen that $\mathscr{Z}^\mu_0$ is a closed subspace of $\mathscr{Z}^\mu$. When $\mu(z) = 1-|z|^2$, the induced spaces $\mathscr{Z}^\mu$ and $\mathscr{Z}^\mu_0$ become the classical Zygmund space and little Zygmund space respectively. For some results on the Zygmund-type spaces and operators on them see, for example, [3, 5, 7, 8, 10, 12, 15, 29, 32, 35, 37].

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in \mathscr{H}(\mathbb{D}).$$

For $\psi \in \mathscr{H}(\mathbb{D})$ the multiplication operator M_ψ is defined by

$$(M_\psi f)(z) = \psi(z)f(z), \quad f \in \mathscr{H}(\mathbb{D}).$$

The product $W_{\psi,\varphi} := M_\psi C_\varphi$ of these two operators is known as the weighted composition operator, which has been extensively studied recently.

The differentiation operator D is defined by $(Df)(z) = f'(z)$, for $f \in \mathscr{H}(\mathbb{D})$. Note that the product operator DM_u is a special case of the so-called first-order differential operator, i.e.,

$$(T_{\psi_1,\psi_2} f)(z) = \psi_1(z)f(z) + \psi_2(z)f'(z), \quad f \in \mathscr{H}(\mathbb{D}).$$

where $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$.

The products of C_φ, M_ψ, D can be obtained in six ways [17]:

$$\begin{aligned}
&(M_\psi C_\varphi D f)(z) = \psi(z) f'(\varphi(z)),\\
&(M_\psi D C_\varphi f)(z) = \psi(z)\varphi'(z) f'(\varphi(z)),\\
&(C_\varphi M_\psi D f)(z) = \psi(\varphi(z)) f'(\varphi(z)),\\
&(D M_\psi C_\varphi f)(z) = \psi'(z) f(\varphi(z)) + \psi(z)\varphi'(z) f'(\varphi(z)),\\
&(C_\varphi D M_\psi f)(z) = \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),\\
&(D C_\varphi M_\psi f)(z) = \psi'(\varphi(z))\varphi'(z) f(\varphi(z)) + \psi(\varphi(z))\varphi'(z) f'(\varphi(z))
\end{aligned} \tag{1}$$

for $z \in \mathbb{D}$ and $f \in \mathscr{H}(\mathbb{D})$. Recently, the research of these product-type operators between analytic function spaces has aroused the interest of experts (see, for example, [1, 4, 5, 11, 12, 14, 15, 17, 25, 30, 33, 34, 35, 36, 37] and also related references therein). In order to treat these operators above in a unified manner, Stević and Sharma [33, 34] introduced the following so-called Stević-Sharma operator:

$$(T_{\psi_1,\psi_2,\varphi} f)(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in \mathscr{H}(\mathbb{D}).$$

where $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$.

Clearly, $T_{\psi_1,\psi_2} = T_{\psi_1,\psi_2,id}$, where id denotes the identity map. We can also easily obtain the product-type operators in (1) by taking some specific choices of the involving symbols:

$$\begin{aligned}
&M_\psi C_\varphi D = T_{0,\psi,\varphi}, \quad M_\psi D C_\varphi = T_{0,\psi\varphi',\varphi}, \quad C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi},\\
&D M_\psi C_\varphi = T_{\psi',\psi\varphi',\varphi}, \quad C_\varphi D M_\psi = T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi}, \quad D C_\varphi M_\psi = T_{\varphi'\psi'\circ\varphi,\varphi'\psi\circ\varphi,\varphi}.
\end{aligned}$$

During recent years, there has been a great interest in studying Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ and characterizing the boundedness and compactness between various analytic function spaces. For instance, Stević et al. in [34] discussed the boundedness of $T_{\psi_1,\psi_2,\varphi}$ on the standard weighted Bergman spaces on the unit disk, where the conditions for boundedness were stated in terms of various suprema and pull-back measures. Jiang in [5] provided necessary and sufficient conditions for $T_{\psi_1,\psi_2,\varphi}$ to be bounded or compact when considered as an operator from Zygmund space to Bloch-Orlicz space. Zhang and Liu in [37] investigated the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from Hardy space to Zygmund-type space. Subsequently, the first author et al. [4] extended their results for the case of Stević weighted space, which was introduced by Stević in [23] (see also [30]). Liu et al. in [11] studied the boundedness and compactness of the extension of a Stević-Sharma operator $T_{\psi_1,\psi_2,\psi_3,\varphi} f = \psi_1 \cdot f\circ\varphi + \psi_2 \cdot \mathscr{R} f\circ\varphi + \psi_3 \cdot \mathscr{R}(f\circ\varphi)$, where $\mathscr{R} f$ is the radial derivative of f, from the mixed-norm space to weighted-type spaces on the unit ball. Some more related results can be found (see, e.g.,[1, 14, 36] and the references therein).

Inspired by the above results, the purpose of the paper is to study the boundedness and compactness of the Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from the mixed-norm space $H(p,q,\phi)$ to Zygmund-type space $\mathscr{Z}^\mu$ and little Zygmund-type space $\mathscr{Z}_0^\mu$.

Throughout the paper we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y whenever there is a positive constant C, which is inessential, such that $X \leqslant CY$. If both $X \lesssim Y$ and $Y \lesssim X$ hold, we write $X \approx Y$.

2. Auxiliary results

In this section, we state several auxiliary results which will be used in the proofs of the main results. The first lemma characterizes compactness in terms of sequential convergence. Since the proof is standard, it is omitted here (see Proposition 3.11 in [2]).

LEMMA 1. *Suppose that* $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$, φ *is an analytic self-map of* $\mathbb{D}$, $1 \leqslant p, q < \infty$ *and* ϕ *is normal. Then the operator* $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ *is compact if and only if* $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ *is bounded and for each sequence* $\{f_n\}_{n\in\mathbb{N}}$ *which is bounded in* $H(p,q,\phi)$ *and converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $n \to \infty$, *we have* $\|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{Z}^\mu} \to 0$ *as* $n \to \infty$.

The following lemma can be found in [25].

LEMMA 2. *Assume that* $1 \leqslant p, q < \infty$, ϕ *is normal and* $f \in H(p,q,\phi)$. *Then for every* $n \in \mathbb{N}_0$, *there is a positive constant* C *independent of* f *such that*

$$|f^{(n)}(z)| \leqslant \frac{C\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n}}.$$

LEMMA 3. *A closed set* K *in* $\mathscr{Z}_0^\mu$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|f''(z)| = 0.$$

The proof of Lemma 3 is similar to that of [16], so we omit the details.

3. Main results

We first consider the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$.

THEOREM 1. *Let* $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$, φ *be an analytic self-map of* $\mathbb{D}$, $1 \leqslant p, q < \infty$ *and* ϕ *is normal. Then* $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ *is bounded if and only if*

$$k_0 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}} < \infty, \tag{2}$$

$$k_1 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+1}} < \infty, \tag{3}$$

$$k_2 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+2}} < \infty, \tag{4}$$

$$k_3 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+3}} < \infty. \tag{5}$$

Moreover, if the operator $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}^{\mu}$ *is nonzero and bounded, then*

$$\|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^{\mu}/\mathbb{P}_1} \approx k_0+k_1+k_2+k_3. \tag{6}$$

Proof. Assume that (2), (3), (4), (5) hold. From Lemma 2, we see that

$$|f(\varphi(z))| \lesssim \frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}},$$
$$|f'(\varphi(z))| \lesssim \frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+1}},$$
$$|f''(\varphi(z))| \lesssim \frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+2}},$$
$$|f'''(\varphi(z))| \lesssim \frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+3}}.$$

Then for each $f \in H(p,q,\phi)$, we have

$$\begin{aligned}
&\mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)|\\
=&\mu(z)|\psi_1''(z)f(\varphi(z))+(2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z))f'(\varphi(z))\\
&+(\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z))f''(\varphi(z))+\psi_2(z)\varphi'(z)^2f'''(\varphi(z))|\\
\leqslant&\mu(z)|\psi_1''(z)||f(\varphi(z))|+\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)||f'(\varphi(z))|\\
&+\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)||f''(\varphi(z))|\\
&+\mu(z)|\psi_2(z)||\varphi'(z)|^2|f'''(\varphi(z))|\\
\lesssim&\|f\|_{H(p,q,\phi)}(k_0+k_1+k_2+k_3).
\end{aligned} \tag{7}$$

We also have

$$\begin{aligned}
&|(T_{\psi_1,\psi_2,\varphi}f)(0)|+|(T_{\psi_1,\psi_2,\varphi}f)'(0)|\\
\leqslant&|\psi_1(0)+\psi_1'(0)||f(\varphi(0))|\\
&+|\psi_2(0)+\psi_1(0)\varphi'(0)+\psi_2'(0)||f'(\varphi(0))|+|\psi_2(0)\varphi'(0)||f''(\varphi(0))|\\
\lesssim&\sum_{i=0}^{2}\frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(0)|)(1-|\varphi(0)|^2)^{\frac{1}{q}+i}}.
\end{aligned}$$

Therefore the Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}^{\mu}$ is bounded. Moreover, if we consider the space $\mathscr{Z}^{\mu}/\mathbb{P}_1$, we have that (see, for example, [26, 27])

$$\|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^{\mu}/\mathbb{P}_1} \lesssim k_0+k_1+k_2+k_3. \tag{8}$$

Conversely, suppose that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}^{\mu}$ is bounded. Taking the function $f(z)=1 \in H(p,q,\phi)$, then we obtain

$$\sup_{z\in\mathbb{D}}\mu(z)|\psi_1''(z)| \leqslant \|T_{\psi_1,\psi_2,\varphi}1\|_{\mathscr{Z}^{\mu}} \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^{\mu}} < \infty. \tag{9}$$

By taking the function $f(z) = z \in H(p,q,\phi)$, we have

$$\begin{aligned}&\sup_{z\in\mathbb{D}} \mu(z)|\psi_1''(z)\varphi(z) + 2\psi_1'(z)\varphi'(z) + \psi_1(z)\varphi''(z) + \psi_2''(z)| \\ \lesssim &\|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} < \infty,\end{aligned}$$

which along with (9), the triangle inequality and the boundedness of φ implies that

$$\sup_{z\in\mathbb{D}} \mu(z)|2\psi_1'(z)\varphi'(z) + \psi_1(z)\varphi''(z) + \psi_2''(z)| \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} < \infty. \tag{10}$$

By using the function $f(z) = \frac{z^2}{2}$ and $f(z) = \frac{z^3}{6} \in H(p,q,\phi)$, in the same manner we can see that

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi'(z)^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)| \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} < \infty, \tag{11}$$

and

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)||\varphi'(z)|^2 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} < \infty. \tag{12}$$

For a fixed $w \in \mathbb{D}$, set

$$\begin{aligned}f_{\varphi(w)}(z) = &\frac{A(1-|\varphi(w)|^2)^{t+1}}{\phi(|\varphi(w)|)(1-\overline{\varphi(w)}z)^{\frac{1}{q}+t+1}} + \frac{B(1-|\varphi(w)|^2)^{t+2}}{\phi(|\varphi(w)|)(1-\overline{\varphi(w)}z)^{\frac{1}{q}+t+2}} \\ &+ \frac{C(1-|\varphi(w)|^2)^{t+3}}{\phi(|\varphi(w)|)(1-\overline{\varphi(w)}z)^{\frac{1}{q}+t+3}} + \frac{(1-|\varphi(w)|^2)^{t+4}}{\phi(|\varphi(w)|)(1-\overline{\varphi(w)}z)^{\frac{1}{q}+t+4}},\end{aligned} \tag{13}$$

where the constant t is from the definition of the normality of the function ϕ.

Then $\sup_{w\in\mathbb{D}} \|f_{\varphi(w)}\|_{H(p,q,\phi)} < \infty$ (see [25]), and we have

$$\begin{aligned}f_{\varphi(w)}(\varphi(w)) &= \frac{A+B+C+1}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}}}, \\ f'_{\varphi(w)}(\varphi(w)) &= \frac{(AX_1+BX_2+CX_3+X_4)\overline{\varphi(w)}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+1}}, \\ f''_{\varphi(w)}(\varphi(w)) &= \frac{(AX_1X_2+BX_2X_3+CX_3X_4+X_4X_5)\overline{\varphi(w)}^2}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+2}}, \\ f'''_{\varphi(w)}(\varphi(w)) &= \frac{(AX_1X_2X_3+BX_2X_3X_4+CX_3X_4X_5+X_4X_5X_6)\overline{\varphi(w)}^3}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+3}},\end{aligned}$$

where $X_i = \frac{1}{q} + t + i$, $i = 1,2,3,4,5,6$.

Now, let us first prove (2). Choose the corresponding function in (13) with

$$A = \frac{2X_3X_4 - 2X_4^2X_5 + 2X_4X_5X_6}{4X_3 - X_4X_5 + X_2X_3},$$
$$B = \frac{3X_4^2X_5 - 3X_4X_5X_6}{4X_2X_3 - X_2X_4X_5 + X_2^2X_3},$$
$$C = \frac{6X_3X_4 - X_4X_5X_6 + X_2X_3X_4}{X_3X_4X_5 - X_2X_3^2 - 4X_3^2},$$

and denote it by $g_{\varphi(w)}$, then we can calculate that

$$g_{\varphi(w)}(\varphi(w)) = \frac{P}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}}},$$
$$g'_{\varphi(w)}(\varphi(w)) = g''_{\varphi(w)}(\varphi(w)) = g'''_{\varphi(w)}(\varphi(w)) = 0, \tag{14}$$

where

$$P = \frac{2X_3X_4 - 2X_4^2X_5 + 2X_4X_5X_6}{4X_3 - X_4X_5 + X_2X_3} + \frac{3X_4^2X_5 - 3X_4X_5X_6}{4X_2X_3 - X_2X_4X_5 + X_2^2X_3} + \frac{6X_3X_4 - X_4X_5X_6 + X_2X_3X_4}{X_3X_4X_5 - X_2X_3^2 - 4X_3^2} + 1.$$

Since $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ is bounded, we conclude that

$$\infty > \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} \gtrsim \|T_{\psi_1,\psi_2,\varphi}g_{\varphi(w)}\|_{\mathscr{Z}^\mu}$$
$$\geqslant \sup_{w\in\mathbb{D}} \mu(w)|(T_{\psi_1,\psi_2,\varphi}g_{\varphi(w)})''(w)| = \sup_{w\in\mathbb{D}} \frac{P\mu(w)|\psi_1''(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}}}, \tag{15}$$

from which we can get (2).

For the proof of (3), we choose the corresponding function in (13) with

$$A = \frac{-X_2X_3^2X_4 + 4X_2X_3X_4 + X_3X_4^2X_5 + 6X_2X_4X_5 + 2X_1X_2X_3X_4 - 2X_2X_4X_5X_6}{X_2X_3^2X_4 - X_1X_2^2X_3 - 4X_2X_3X_4},$$
$$B = \frac{3X_1X_2X_4X_5 - X_3X_4^2X_5 - 2X_1X_2X_3X_4}{X_2X_3^2X_4 - X_1X_2^2X_3 - 4X_2X_3X_4},$$
$$C = \frac{2X_4X_5X_6 - 3X_3X_4X_5 + X_1X_2X_3}{X_3^2X_4 - X_1X_2X_3 - 4X_3X_4},$$

and denote it by $h_{\varphi(w)}$, then we have

$$h'_{\varphi(w)}(\varphi(w)) = \frac{E\overline{\varphi(w)}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+1}},$$
$$h_{\varphi(w)}(\varphi(w)) = h''_{\varphi(w)}(\varphi(w)) = h'''_{\varphi(w)}(\varphi(w)) = 0, \tag{16}$$

where

$$E = \frac{-X_1X_2X_3^2X_4 + 4X_1X_2X_3X_4 + X_1X_3X_4^2X_5 + 6X_1X_2X_4X_5 + 2X_1^2X_2X_3X_4 - 2X_1X_2X_4X_5X_6}{X_2X_3^2X_4 - X_1X_2^2X_3 - 4X_2X_3X_4}$$
$$+ \frac{3X_1X_2X_4X_5 - X_3X_4^2X_5 - 2X_1X_2X_3X_4}{X_3^2X_4 - X_1X_2X_3 - 4X_3X_4} + \frac{2X_4X_5X_6 - 3X_3X_4X_5 + X_1X_2X_3}{X_3X_4 - X_1X_2 - 4X_4} + X_4.$$

By the boundedness of $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$, we have

$$\begin{aligned} \infty > \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi} h_{\varphi(w)}\|_{\mathscr{Z}^\mu} \\ &\geqslant \sup_{w\in\mathbb{D}} \mu(w)|(T_{\psi_1,\psi_2,\varphi} h_{\varphi(w)})''(w)| \\ &= \sup_{w\in\mathbb{D}} \frac{E\mu(w)|\varphi(w)||2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+1}}. \end{aligned}$$

It follows that

$$\sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{\phi(|\varphi(w)|)(1-|\psi(w)|^2)^{\frac{1}{q}+1}} \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu}. \tag{17}$$

Since ϕ is normal, and using (10), we have

$$\sup_{|\varphi(w)|\leqslant\frac{1}{2}} \frac{\mu(w)|2\psi_1'(w)\varphi'(w) + \psi_1(w)\varphi''(w) + \psi_2''(w)|}{\phi(|\varphi(w)|)(1-|\psi(w)|^2)^{\frac{1}{q}+1}} \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu}. \tag{18}$$

From (17) and (18), it may be concluded that (3) holds.

We next turn to proving (4). Choose the corresponding function in (13) with

$$\begin{aligned} A &= \frac{2X_3X_4X_5 - 2X_2X_3^2 - 2X_2X_3 - X_4X_5X_6 + X_2X_3X_4}{X_3X_4X_5 - X_2X_3^2 - 4X_2X_3}, \\ B &= \frac{3X_2X_3^2 - 3X_3X_4X_5 + 2X_4X_5X_6 - 2X_2X_3X_4}{X_3X_4X_5 - X_2X_3^2 - 4X_2X_3}, \\ C &= \frac{6X_2X_3 - X_4X_5X_6 + X_2X_3X_4}{X_3X_4X_5 - X_2X_3^2 - 4X_2X_3}, \end{aligned}$$

and denote it by $r_{\varphi(w)}$, then we can obtain

$$\begin{aligned} r''_{\varphi(w)}(\varphi(w)) &= \frac{F\overline{\varphi(w)}^2}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+2}}, \\ r_{\varphi(w)}(\varphi(w)) &= r'_{\varphi(w)}(\varphi(w)) = r'''_{\varphi(w)}(\varphi(w)) = 0, \end{aligned} \tag{19}$$

where

$$F = \frac{2X_1X_2X_3X_4X_5 - 2X_1X_2^2X_3^2 - 2X_1X_2^2X_3 - X_1X_2X_4X_5X_6 + X_1X_2^2X_3X_4 - 3X_2X_3^2X_4X_5}{X_3X_4X_5 - X_2X_3^2 - 4X_2X_3}$$
$$+ \frac{3X_2^2X_3^3 + 2X_2X_3X_4X_5X_6 - 2X_2^2X_3^2X_4 + 6X_2X_3^2X_4 - X_3X_4^2X_5 - X_2X_3^2X_4 - 4X_2X_3X_4X_5}{X_3X_4X_5 - X_2X_3^2 - 4X_2X_3}.$$

Using the fact that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ is bounded, we have

$$\begin{aligned}
\infty &> \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} \gtrsim \|T_{\psi_1,\psi_2,\varphi} r_{\varphi(w)}\|_{\mathscr{Z}^\mu} \\
&\geqslant \sup_{w\in\mathbb{D}} \mu(w)|(T_{\psi_1,\psi_2,\varphi} r_{\varphi(w)})''(w)| \\
&= \sup_{w\in\mathbb{D}} \frac{F\mu(w)|\varphi(w)|^2|\psi_1(w)\varphi'(w)^2 + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+2}}.
\end{aligned}$$

This gives

$$\begin{aligned}
&\sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|\psi_1(w)\varphi'(w)^2 + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+2}} \\
&\lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu}. \qquad (20)
\end{aligned}$$

Because ϕ is normal, and using (11), we have

$$\sup_{|\varphi(w)|\leqslant\frac{1}{2}} \frac{\mu(w)|\psi_1(w)\varphi'(w)^2 + 2\psi_2'(w)\varphi'(w) + \psi_2(w)\varphi''(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+2}} \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu}. \qquad (21)$$

From (20) and (21), we can get (4).

It remains to verify (5). For this, we choose the corresponding function in (13) with

$$A = -1, \quad B = 3, \quad C = -3,$$

and denote it by $s_{\varphi(w)}$. An easy computation shows that

$$\begin{aligned}
s'''_{\varphi(w)}(\varphi(w)) &= \frac{12\overline{\varphi(w)}^3}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+3}}, \\
s_{\varphi(w)}(\varphi(w)) &= s'_{\varphi(w)}(\varphi(w)) = s''_{\varphi(w)}(\varphi(w)) = 0. \qquad (22)
\end{aligned}$$

Once again, we can use the boundedness of $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ and get

$$\begin{aligned}
\infty &> \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} \gtrsim \|T_{\psi_1,\psi_2,\varphi} s_{\varphi(w)}\|_{\mathscr{Z}^\mu} \\
&\geqslant \sup_{w\in\mathbb{D}} \mu(w)|(T_{\psi_1,\psi_2,\varphi} s_{\varphi(w)})''(w)| \\
&= \sup_{w\in\mathbb{D}} \frac{12\mu(w)|\varphi(w)|^3|\psi_2(w)||\varphi'(w)|^2}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+3}},
\end{aligned}$$

which yields

$$\sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|\psi_2(w)||\varphi'(w)|^2}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+3}} \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu}. \tag{23}$$

On the other hand, we also have

$$\sup_{|\varphi(w)|\leqslant\frac{1}{2}} \frac{\mu(w)|\psi_2(w)||\varphi'(w)|^2}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{1}{q}+2}} \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu}. \tag{24}$$

Hence, (5) follows from (23) and (24).

By using (15), (17), (18), (20), (21), (23), (24), we see at once that

$$\|T_{\psi_1,\psi_2,\varphi}\|_{H(p,q,\phi)\to\mathscr{Z}^\mu} \gtrsim k_0+k_1+k_2+k_3. \tag{25}$$

From (8) and (25) it follows that the asymptotic expression (6) holds. □

THEOREM 2. *Let $\psi_1,\psi_2\in\mathscr{H}(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, $1\leqslant p,q<\infty$ and ϕ is normal. Then $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}^\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}^\mu$ is bounded and*

$$\lim_{|\varphi(z)|\to1} \frac{\mu(z)|\psi_1''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}}=0, \tag{26}$$

$$\lim_{|\varphi(z)|\to1} \frac{\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+1}}=0, \tag{27}$$

$$\lim_{|\varphi(z)|\to1} \frac{\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+2}}=0, \tag{28}$$

$$\lim_{|\varphi(z)|\to1} \frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+3}}=0. \tag{29}$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}^\mu$ is compact, and hence bounded. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)|\to1$ as $n\to\infty$. Set

$$g_n(z)=g_{\varphi(z_n)}(z),\quad h_n(z)=h_{\varphi(z_n)}(z),\quad r_n(z)=r_{\varphi(z_n)}(z),\quad s_n(z)=s_{\varphi(z_n)}(z).$$

On account of the proof of Theorem 1, we have $g_n,\ h_n,\ r_n,\ s_n\in H(p,q,\phi)$. Moreover, we deduce that $\{g_n\},\{h_n\},\{r_n\},\{s_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Since $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}^\mu$ is compact, by Lemma 1, we know that

$$\begin{aligned}\lim_{n\to\infty}\|T_{\psi_1,\psi_2,\varphi}g_n\|_{\mathscr{Z}^\mu} &= \lim_{n\to\infty}\|T_{\psi_1,\psi_2,\varphi}h_n\|_{\mathscr{Z}^\mu}\\ &= \lim_{n\to\infty}\|T_{\psi_1,\psi_2,\varphi}r_n\|_{\mathscr{Z}^\mu} = \lim_{n\to\infty}\|T_{\psi_1,\psi_2,\varphi}s_n\|_{\mathscr{Z}^\mu}=0.\end{aligned}$$

From (14) it follows that

$$g_n(\varphi(z_n)) = \frac{P}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\frac{1}{q}}}, \quad g_n'(\varphi(z_n)) = g_n''(\varphi(z_n)) = g_n'''(\varphi(z_n)) = 0.$$

Consequently

$$\frac{\mu(z_n)|\psi_1''(z_n)|}{\phi(|\varphi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}} \lesssim \|T_{\psi_1,\psi_2,\varphi} g_n\|_{\mathscr{Z}^\mu},$$

which along with $|\varphi(z_n)| \to 1$ as $n \to \infty$ implies that

$$\lim_{|\varphi(z_n)|\to 1} \frac{\mu(z_n)|\psi_1''(z_n)|}{\phi(|\varphi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}} = \lim_{n\to\infty} \frac{\mu(z_n)|\psi_1''(z_n)|}{\phi(|\varphi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}}} = 0,$$

from which (26) holds. By (16) we have

$$h_n'(\varphi(z_n)) = \frac{E\overline{\varphi(z_n)}}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\frac{1}{q}+1}}, \quad h_n(\varphi(z_n)) = h_n''(\varphi(z_n)) = h_n'''(\varphi(z_n)) = 0.$$

Hence

$$\frac{\mu(z_n)|\varphi(z_n)||2\psi_1'(z_n)\varphi'(z_n) + \psi_1(z_n)\varphi''(z_n) + \psi_2''(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\frac{1}{q}+1}} \lesssim \|T_{\psi_1,\psi_2,\varphi} h_n\|_{\mathscr{Z}^\mu}.$$

It follows that

$$\begin{aligned}
&\lim_{|\varphi(z_n)|\to 1} \frac{\mu(z_n)|2\psi_1'(z_n)\varphi'(z_n) + \psi_1(z_n)\varphi''(z_n) + \psi_2''(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\frac{1}{q}+1}} \\
&= \lim_{n\to\infty} \frac{\mu(z_n)|\varphi(z_n)||2\psi_1'(z_n)\varphi'(z_n) + \psi_1(z_n)\varphi''(z_n) + \psi_2''(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\frac{1}{q}+1}} = 0.
\end{aligned}$$

Therefore (27) follows. With the same argument as above one can show that (28) and (29) hold, so we omit the details.

Conversely, suppose that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to \mathscr{Z}^\mu$ is bounded and (26), (27), (28), (29) hold. Then (9), (10), (11), (12) are true by Theorem 1 and for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}} < \varepsilon, \tag{30}$$

$$\frac{\mu(z)|2\psi_1'(z)\varphi'(z) + \psi_1(z)\varphi''(z) + \psi_2''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+1}} < \varepsilon, \tag{31}$$

$$\frac{\mu(z)|\psi_1(z)\varphi'(z)^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+2}} < \varepsilon, \tag{32}$$

$$\frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+3}} < \varepsilon, \tag{33}$$

whenever $\delta < |\varphi(z)| < 1$.

Assume that $\{t_n\}_{n\in\mathbb{N}}$ is a sequence in $H(p,q,\phi)$ such that $\sup_{n\in\mathbb{N}} \|t_n\|_{H(p,q,\phi)} \leqslant L$, and $\{t_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Let $J = \{z \in \mathbb{D} : |\varphi(z)| \leqslant \delta\}$. Then by Lemma 2, (9), (10), (11), (12) and (30), (31), (32), (33) we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi}t_n)''(z)| \\
\leqslant &\sup_{z\in\mathbb{D}} \mu(z)|\psi_1''(z)||t_n(\varphi(z))| + \sup_{z\in\mathbb{D}} \mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)||t_n'(\varphi(z))| \\
&+\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)||t_n''(\varphi(z))| \\
&+\sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)||\varphi'(z)|^2|t_n'''(\varphi(z))| \\
\lesssim &\sup_{z\in J} \mu(z)|\psi_1''(z)||t_n(\varphi(z))| + \sup_{z\in\mathbb{D}\setminus J} \frac{\mu(z)|\psi_1''(z)|\|t_n\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}} \\
&+\sup_{z\in J} \mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)||t_n'(\varphi(z))| \\
&+\sup_{z\in\mathbb{D}\setminus J} \frac{\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|\|t_n\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+1}} \\
&+\sup_{z\in J} \mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)||t_n''(\varphi(z))| \\
&+\sup_{z\in\mathbb{D}\setminus J} \frac{\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|\|t_n\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+2}} \\
&+\sup_{z\in J} \mu(z)|\psi_2(z)||\varphi'(z)|^2|t_n'''(\varphi(z))| + \sup_{z\in\mathbb{D}\setminus J} \frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2\|t_n\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+3}} \\
\lesssim &\sup_{|w|\leqslant\delta} |t_n(w)| + \sup_{|w|\leqslant\delta} |t_n'(w)| + \sup_{|w|\leqslant\delta} |t_n''(w)| + \sup_{|w|\leqslant\delta} |t_n'''(w)| + L\varepsilon.
\end{aligned}$$

From the above it follows that

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi}t_n\|_{\mathscr{Z}^\mu} \\
=&|\psi_1(0)t_n(\varphi(0))+\psi_2(0)t_n'(\varphi(0))| \\
&+|\psi_1'(0)t_n(\varphi(0))+(\psi_1(0)\varphi'(0)+\psi_2'(0))t_n'(\varphi(0))+\psi_2(0)\varphi'(0)t_n''(\varphi(0))| \\
&+\sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi}t_n)''(z)| \\
\leqslant&(|\psi_1(0)|+|\psi_1'(0)|)|t_n(\varphi(0))|+(|\psi_2(0)|+|\psi_1(0)\varphi'(0)+\psi_2'(0))|)|t_n'(\varphi(0))| \\
&+|\psi_2(0)\varphi'(0)||t_n''(\varphi(0))|+L\varepsilon \\
&+\sup_{|w|\leqslant\delta} |t_n(w)| + \sup_{|w|\leqslant\delta} |t_n'(w)| + \sup_{|w|\leqslant\delta} |t_n''(w)| + \sup_{|w|\leqslant\delta} |t_n'''(w)|. \qquad (34)
\end{aligned}$$

Since t_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$, Cauchy's estimation gives that t_n', t_n'' and t_n''' also do as $n\to\infty$. In particular, $\{w : |w| \leqslant \delta\}$ and

$\{\varphi(0)\}$ are compact, and then we have

$$\lim_{n\to\infty}(|\psi_1(0)|+|\psi_1'(0)|)|t_n(\varphi(0))|=0,$$
$$\lim_{n\to\infty}(|\psi_2(0)|+|\psi_1(0)\varphi'(0)+\psi_2'(0))|)|t_n'(\varphi(0))|=0,$$
$$\lim_{n\to\infty}|\psi_2(0)\varphi'(0)||t_n''(\varphi(0))|=0,$$
$$\lim_{n\to\infty}\sup_{|w|\leqslant\delta}|t_n(w)|=\lim_{n\to\infty}\sup_{|w|\leqslant\delta}|t_n'(w)|=\lim_{n\to\infty}\sup_{|w|\leqslant\delta}|t_n''(w)|=\lim_{n\to\infty}\sup_{|w|\leqslant\delta}|t_n'''(w)|=0.$$

Letting $n\to\infty$ in (34) yields

$$\limsup_{n\to\infty}\|T_{\psi_1,\psi_2,\varphi}t_n\|_{\mathscr{Z}^\mu}\leqslant L\varepsilon.$$

Since ε is an arbitrary positive number, from the last inequality we can obtain

$$\lim_{n\to\infty}\|T_{\psi_1,\psi_2,\varphi}t_n\|_{\mathscr{Z}^\mu}=0.$$

Finally, we conclude that $T_{\psi_1,\psi_2,\varphi}:H(p,q,\phi)\to\mathscr{Z}^\mu$ is compact by applying Lemma 1. □

We shall now describe the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}:H(p,q,\phi)\to\mathscr{Z}_0^\mu$.

THEOREM 3. *Let $\psi_1,\psi_2\in\mathscr{H}(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, $1\leqslant p,q<\infty$ and ϕ is normal. Then $T_{\psi_1,\psi_2,\varphi}:H(p,q,\phi)\to\mathscr{Z}_0^\mu$ is bounded if and only if $T_{\psi_1,\psi_2,\varphi}:H(p,q,\phi)\to\mathscr{Z}^\mu$ is bounded and*

$$\lim_{|z|\to1}\mu(z)|\psi_1''(z)|=0, \tag{35}$$
$$\lim_{|z|\to1}\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|=0, \tag{36}$$
$$\lim_{|z|\to1}\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|=0, \tag{37}$$
$$\lim_{|z|\to1}\mu(z)|\psi_2(z)||\varphi'(z)|^2=0. \tag{38}$$

Proof. If $T_{\psi_1,\psi_2,\varphi}:H(p,q,\phi)\to\mathscr{Z}_0^\mu$ is bounded, then $T_{\psi_1,\psi_2,\varphi}:H(p,q,\phi)\to\mathscr{Z}^\mu$ is bounded and for $f\in H(p,q,\phi)$, $T_{\psi_1,\psi_2,\varphi}f\in\mathscr{Z}_0^\mu$. Taking $f(z)=1\in H(p,q,\phi)$, we get

$$\lim_{|z|\to1}\mu(z)|(T_{\psi_1,\psi_2,\varphi}1)''(z)|=\lim_{|z|\to1}\mu(z)|\psi_1''(z)|=0,$$

That is, (35) follows. Taking the function $f(z)=z\in H(p,q,\phi)$, we have

$$\lim_{|z|\to1}\mu(z)|\psi_1''(z)\varphi(z)+2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|=0,$$

which along with (35), the triangle inequality, and the boundedness of φ implies that (36) holds. The proof of (37) and (38) can be handled in much the same way by using the function $f(z)=\frac{z^2}{2}$ and $f(z)=\frac{z^3}{6}\in H(p,q,\phi)$ respectively, and the details are omitted.

On the contrary, assume that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to \mathscr{Z}^{\mu}$ is bounded and (35), (36), (37), (38) hold. For each polynomial Q, we have

$$\begin{aligned}
&\mu(z)|(T_{\psi_1,\psi_2,\varphi}Q)''(z)|\\
=&\mu(z)|\psi_1''(z)Q(\varphi(z))+(2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z))Q'(\varphi(z))\\
&+(\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z))Q''(\varphi(z))+\psi_2(z)\varphi'(z)^2Q'''(\varphi(z))|\\
\leqslant&\mu(z)|\psi_1''(z)|\|Q\|_\infty+\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|\|Q'\|_\infty\\
&+\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|\|Q''\|_\infty\\
&+\mu(z)|\psi_2(z)||\varphi'(z)|^2\|Q'''\|_\infty,
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Letting $|z|\to 1$ in the above inequality and using (35), (36), (37), (38) yields

$$\lim_{|z|\to 1}\mu(z)|(T_{\psi_1,\psi_2,\varphi}Q)''(z)|=0,$$

which says that $T_{\psi_1,\psi_2,\varphi}Q\in\mathscr{Z}_0^{\mu}$. Note that the set of all polynomials is dense in $H(p,q,\phi)$, and consequently for each $f\in H(p,q,\phi)$, there is a sequence of polynomials $\{Q_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\|Q_n-f\|_{H(p,q,\phi)}=0,$$

which along with the boundedness of $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}^{\mu}$ implies that

$$\|T_{\psi_1,\psi_2,\varphi}Q_n-T_{\psi_1,\psi_2,\varphi}f\|_{\mathscr{Z}^{\mu}}\leqslant\|T_{\psi_1,\psi_2,\varphi}\|\cdot\|Q_n-f\|_{H(p,q,\phi)}\to 0$$

as $n\to\infty$. Due to $\mathscr{Z}_0^{\mu}$ is a closed subspace of $\mathscr{Z}^{\mu}$, we know that $T_{\psi_1,\psi_2,\varphi}f\in\mathscr{Z}_0^{\mu}$ and thus $T_{\psi_1,\psi_2,\varphi}\big(H(p,q,\phi)\big)\subseteq\mathscr{Z}_0^{\mu}$. As a consequence, $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}_0^{\mu}$ is bounded. □

THEOREM 4. *Let $\psi_1,\psi_2\in\mathscr{H}(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$, $1\leqslant p,q<\infty$ and ϕ is normal. Then $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi)\to\mathscr{Z}_0^{\mu}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(z)|\psi_1''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}}=0, \tag{39}$$

$$\lim_{|z|\to 1}\frac{\mu(z)|2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+1}}=0, \tag{40}$$

$$\lim_{|z|\to 1}\frac{\mu(z)|\psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+2}}=0, \tag{41}$$

$$\lim_{|z|\to 1}\frac{\mu(z)|\psi_2(z)||\varphi'(z)|^2}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}+3}}=0. \tag{42}$$

Proof. Asumme that (39)–(42) hold. It is evident that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}_0^\mu$ is bounded by Theorem 1. Taking the supremum in inequality (7) over all $f \in H(p,q,\phi)$ such that $\|f\|_{H(p,q,\phi)} \leqslant 1$ and letting $|z| \to 1$, we can obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{H(p,q,\phi)}\leqslant 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)| = 0.$$

From Lemma 3 it follows that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}_0^\mu$ is compact.

Conversely, suppose that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}_0^\mu$ is compact, then $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}^\mu$ is compact by Theorem 2. Moreover, for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that (30), (31), (32), (33) hold for $\delta < |\varphi(z)| < 1$. The compactness of $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}_0^\mu$ implies that $T_{\psi_1,\psi_2,\varphi}: H(p,q,\phi) \to \mathscr{Z}_0^\mu$ is bounded, by Theorem 3, it follows from (35), (36), (37), (38) that for $\varepsilon > 0$, there exists $\eta \in (0,1)$ such that

$$\mu(z)|\psi_1''(z)| < \varepsilon, \tag{43}$$

$$\mu(z)|2\psi_1'(z)\varphi'(z) + \psi_1(z)\varphi''(z) + \psi_2''(z)| < \varepsilon, \tag{44}$$

$$\mu(z)|\psi_1(z)\varphi'(z)^2 + 2\psi_2'(z)\varphi'(z) + \psi_2(z)\varphi''(z)| < \varepsilon, \tag{45}$$

$$\mu(z)|\psi_2(z)||\varphi'(z)|^2 < \varepsilon, \tag{46}$$

for $\eta < |z| < 1$.

By (30), when $\delta < |\varphi(z)| < 1$ and $\eta < |z| < 1$,

$$\frac{\mu(z)|\psi_1''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}} < \varepsilon. \tag{47}$$

When $|\varphi(z)| \leqslant \delta$ and $\eta < |z| < 1$, by using (43) and the normality of ϕ we can obtain

$$\frac{\mu(z)|\psi_1''(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{q}}} \leqslant \frac{\mu(z)|\psi_1''(z)|}{\inf_{r\in[0,\delta]}\phi(r)(1-\delta^2)^{\frac{1}{q}}} \lesssim \varepsilon. \tag{48}$$

Along with (47) and (48), we can see that (39) follows. Employing (31) and (44), (32) and (45), (33) and (46), with the similar argument, one can obtain (40), (41), (42), respectively. This finishes the proof. □

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