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APPROXIMATE ω–ORTHOGONALITY AND ω–DERIVATION

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Abstract. We introduce the notion of approximate ω-orthogonality (referring to the numerical radius ω) and investigate its significant properties. Let $T,S\in\mathbb{B}(\mathscr{H})$ and $\varepsilon\in[0,1)$. We say that T is approximate ω-orthogonality to S and we write $T\perp_\omega^\varepsilon S$ if

$$\omega^2(T+\lambda S)\geqslant\omega^2(T)-2\varepsilon\omega(T)\omega(\lambda S),\qquad\text{for all }\lambda\in\mathbb{C}.$$

We show that $T\perp_\omega^\varepsilon S$ if and only if $\inf_{\theta\in[0,2\pi)}D_\omega^\theta(T,S)\geqslant-\varepsilon\omega(T)\omega(S)$ in which $D_\omega^\theta(T,S)=\lim_{r\to0^+}\dfrac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{2r}$; and this occurs if and only if for every $\theta\in[0,2\pi)$, there exists a sequence $\{x_n^\theta\}$ of unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty}|\langle Tx_n^\theta,x_n^\theta\rangle|=\omega(T)\text{ and }\lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}\geqslant-\varepsilon\omega(T)\omega(S),$$

where $\omega(T)$ is the numerical radius of T.

1. Introduction

The notion of orthogonality can be defined in many ways for normed spaces without using the inner product structure. One of the most important types of orthogonality in the setting of normed spaces is the Birkhoff–James orthogonality. Let $(X,\|\cdot\|)$ be a linear normed space and $x,y\in X$. Then x is called Birkhoff–James orthogonal to y, written as $x\perp_B y$, if $\|x+\lambda y\|\geqslant\|x\|$ for every $\lambda\in\mathbb{C}$.

Many mathematicians generalized the notion of Birkhoff–James orthogonality in the setup of normed spaces. Dragomir [5] introduced the notion of ε-Birkhoff–James orthogonality in a real normed space X as follows.

Let $x,y\in X$ and $\varepsilon\in[0,1)$. We say that x is ε-Birkhoff–James orthogonal to y if

$$\|x+\lambda y\|\geqslant(1-\varepsilon)\|x\|$$

for all $\lambda\in\mathbb{R}$.

Chmieliński [2] introduced another notion of ε-Birkhoff–James orthogonality in the setting of normed spaces, for $\varepsilon\in[0,1)$ a vector x is said to be approximately Birkhoff–James orthogonal to a vector y, written as $x\perp_B^\varepsilon y$, if

$$\|x+\lambda y\|^2\geqslant\|x\|^2-2\varepsilon\|x\|\|\lambda y\|$$

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for all $\lambda \in \mathbb{R}$. He also proved that in an inner product space $x \perp_B^{\varepsilon} y$ if and only if $|\langle x,y\rangle| \leqslant \varepsilon\|x\|\|y\|$. The notion of approximate orthogonality has been developed in several settings; see e.g. [3, 11, 13].

Throughout the paper, let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$ with the identity I. A capital letter denotes a bounded linear operator. The numerical radius of T is defined by

$$\omega(T) = \sup\{|\langle Tx,x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

We need some formulas for calculating the numerical radius. We state them in the following lemmas.

LEMMA 1. [14, Theorem 3] *Let* $T = \begin{bmatrix} \alpha I & B \\ 0 & \beta I \end{bmatrix} \in \mathbb{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$, *where* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| = |\beta|$. *Then* $\omega(T) = \begin{cases} \frac{|\alpha|\sqrt{|\alpha-\beta|^2+\|B\|^2}}{|\alpha-\beta|}, & |\alpha-\beta|^2 > \|B\||\alpha+\beta| \\ \frac{1}{2}(|\alpha+\beta| + \|B\|), & |\alpha-\beta|^2 \leqslant \|B\||\alpha+\beta|. \end{cases}$

LEMMA 2. [8, Theorem 3.7] *Let* $T,S,U,V \in \mathbb{B}(\mathscr{H})$. *Then*

$$\omega\left(\begin{bmatrix} T & S \\ U & V \end{bmatrix}\right) \geqslant \max\left(\omega(T), \omega(V), \frac{\omega(S+U)}{2}, \frac{\omega(S-U)}{2}\right),$$

and

$$\omega\left(\begin{bmatrix} T & S \\ U & V \end{bmatrix}\right) \leqslant \max(\omega(T), \omega(V)) + \frac{\omega(S+U)+\omega(S-U)}{2}.$$

LEMMA 3. [12, Theorem 2.3] *Suppose that* $U \in M_{r,n-r}(\mathbb{C})$ *and* $T = \begin{bmatrix} rI_r & U \\ 0 & sI_{n-r} \end{bmatrix}$ *for all* $r,s \in \mathbb{R}$. *Then*

$$\omega(T) = \frac{1}{2}|r+s| + \frac{1}{2}\sqrt{(r-s)^2 + \|U\|^2}. \tag{1}$$

Recently, Mal, Paul, and Sen [10] introduced the notion of ω-orthogonality for operators in $\mathbb{B}(\mathscr{H})$. For $T,S \in \mathbb{B}(\mathscr{H})$, we say T to be ω-orthogonality to S, denoted by $T \perp_{\omega} S$ if

$$\omega(T+\lambda S) \geqslant \omega(T) \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

We introduce an approximate counterpart of the above notion and present some of its characterizations. The paper is organized as follows.

In section 2, we introduce the notion of approximate ω-orthogonality "$\perp_{\omega}^{\varepsilon}$" and prove that for operators $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$, it holds that $T \perp_{\omega}^{\varepsilon} S$ if and only if for every $\theta \in [0,2\pi)$ there exists a sequence $\{x_n^{\theta}\}$ of unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty} |\langle Tx_n^{\theta}, x_n^{\theta}\rangle| = \omega(T), \text{ and } \lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^{\theta}, x_n^{\theta}\rangle\overline{\langle Sx_n^{\theta}, x_n^{\theta}\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S).$$

In section 3, we introduce the notion of ω-derivation and study its connection with the approximate ω-orthogonality by showing that $T \perp_{\omega}^{\varepsilon} S$ if and only if $\inf_{\theta\in[0,2\pi)} D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$, where $D_{\omega}^{\theta}(T,S) = \lim_{r\to 0^+} \dfrac{\omega^2(T+re^{i\theta}S) - \omega^2(T)}{2r}$.

2. Approximate numerical radius orthogonality

In this section, we introduce the notion of approximate ω-orthogonality and state some of its basic properties.

DEFINITION 1. Let $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$. We say that T is approximately ω-orthogonal to S and we write $T \perp_\omega^\varepsilon S$ if

$$\omega^2(T+\lambda S) \geqslant \omega^2(T) - 2\varepsilon\omega(T)\omega(\lambda S) \text{ for all } \lambda \in \mathbb{C}.$$

It is easy to see that $T \perp_\omega^\varepsilon S$ and $\alpha T \perp_\omega^\varepsilon \beta S\,(\alpha,\beta \in \mathbb{C})$ are equivalent. The following example shows that the relation $\perp_\omega^\varepsilon$ is not symmetric, in general.

EXAMPLE 1. Suppose that $T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ are in $\mathbb{M}_2(\mathbb{C})$ and $\varepsilon \in [0,0.7)$. Lemma 1 and Lemma 3 show that $\omega(T) = 1$ and $\omega(S) = \dfrac{1+\sqrt{2}}{2}$, respectively. Further, it follows from Lemma 2 that

$$\omega(T+\lambda S) = \omega\left(\begin{bmatrix} i & \lambda \\ 0 & i-\lambda \end{bmatrix}\right) \geqslant \max\left\{|i-\lambda|, 1, \frac{|\lambda|}{2}\right\}.$$

Hence, $\omega^2(T+\lambda S) \geqslant \omega^2(T) - 2\varepsilon\omega(T)\omega(\lambda S)$, that is $T \perp_\omega^\varepsilon S$.

For $\lambda = \frac{-i}{2}$, we get

$$\omega(S - \frac{i}{2}T) = \omega\left(\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}\right) = \frac{\sqrt{2}}{2} \approx 0.707,$$

whence $\omega^2(S - \frac{i}{2}T) \approx 0.499$ and $\omega^2(S) \approx 1.457$. Hence, $\omega^2(S - \frac{i}{2}T) < \omega^2(S) - 2\varepsilon\frac{1}{2}\omega(S)\omega(T)$. Thus, $S \not\perp_\omega^\varepsilon T$.

The following proposition yields some relations between the approximate Birkhoff–James orthogonality $\perp_B^\varepsilon$ and the approximate ω-orthogonality $\perp_\omega^\varepsilon$ under some mild conditions.

PROPOSITION 1. Let $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$.
(i) If $T = T^*$, then $T \perp_\omega^\varepsilon S$ implies that $T \perp_B^\varepsilon S$.
(ii) If $T^2 = 0$, then $T \perp_B^\varepsilon S$ entails that $T \perp_\omega^\varepsilon S$.

Proof. (i) Let $T = T^*$ and $T \perp_\omega^\varepsilon S$. Then $\omega(T) = \|T\|$ and for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}\|T+\lambda S\|^2 &\geqslant \omega^2(T+\lambda S) \geqslant \omega^2(T) - 2\varepsilon\omega(T)\omega(\lambda S) \\ &= \|T\|^2 - 2\varepsilon\|T\|\omega(\lambda S) \geqslant \|T\|^2 - 2\varepsilon\|T\|\|\lambda S\|.\end{aligned}$$

Thus $T \perp_B^\varepsilon S$.

(ii) Let $T^2 = 0$ and $T \perp_B^\varepsilon S$. Then $\omega(T) = \frac{1}{2}\|T\|$ and

$$\omega^2(T+\lambda S) \geqslant \frac{1}{4}\|T+\lambda S\|^2 \geqslant \frac{1}{4}(\|T\|^2 - 2\varepsilon\|T\|\|\lambda S\|) = \omega^2(T) - \frac{1}{2}\varepsilon\|T\|\|\lambda S\|$$
$$= \omega^2(T) - \varepsilon\omega(T)\|\lambda S\| \geqslant \omega^2(T) - 2\varepsilon\omega(T)\omega(\lambda S),$$

which yields the required result. □

The following example shows that $T \perp_B^\varepsilon S$ does not entail $T \perp_\omega^\varepsilon S$, in general.

EXAMPLE 2. Suppose that $T = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are in $\mathbb{M}_2(\mathbb{C})$ and $\varepsilon \in [0, 0.01)$. Then $\|T\| = \sqrt{2}$ and $\|S\| = 1$ and for every $\lambda \in \mathbb{C}$, we have

$$\|T+\lambda S\|^2 = \frac{2+|\lambda|^2+\sqrt{4+|\lambda|^4}}{2}.$$

Hence $\|T+\lambda S\|^2 \geqslant 2 \geqslant 2 - 2\varepsilon|\lambda| = \|T\|^2 - \sqrt{2}\varepsilon\|T\|\|\lambda S\| \geqslant \|T\|^2 - 2\varepsilon\|T\|\|\lambda S\|$. Thus $T \perp_B^\varepsilon S$.

In addition, by Lemma 3, we have $\omega(T) = \frac{1+\sqrt{2}}{2}$, $\omega(S) = 1$, and for $\lambda = 1$,

$$\omega(T+S) = \omega\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right) = \frac{\sqrt{5}}{2}.$$

We therefore get $1.25 = \omega^2(T+S) < \omega^2(T) - 2\varepsilon\omega(T)\omega(S) \approx 1.43$ for $\varepsilon = 0.01$. Hence for $\varepsilon \in [0, 0.01)$, we reach $T \not\perp_\omega^\varepsilon S$.

We give an example of two operators T and S such that $T \not\perp_\omega S$, while $T \perp_\omega^\varepsilon S$.

EXAMPLE 3. Suppose that $T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are in $\mathbb{M}_2(\mathbb{C})$ and $\varepsilon \in [\frac{2}{3}, 1)$. Straightforward computations give us $\omega(T) = 2$, $\omega(S) = \frac{3}{2}$. If $\lambda = -1$, then

$$\omega(T-S) = \omega\left(\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\right) = \frac{\sqrt{5}}{2} < 2.$$

Hence, $T \not\perp_\omega S$. From Lemma 2, we also have

$$\omega(T+\lambda S) = \omega\left(\begin{bmatrix} 2+\lambda & \lambda \\ 0 & \lambda \end{bmatrix}\right) \geqslant \max\{|\lambda|, |\lambda+2|\}.$$

Thus, $\omega^2(T+\lambda S) \geqslant \max\{|\lambda|^2, |2+\lambda|^2\} \geqslant 4 - 6\varepsilon|\lambda| = \omega^2(T) - 2\varepsilon|\lambda|\omega(T)\omega(S)$. Therefore, $T \perp_\omega^\varepsilon S$.

Mal et al. [10, Theorem 2.3] characterized the ω-orthogonality of bounded linear operators acting on a Hilbert space. In [16], the authors investigated some aspects of the ω-orthogonality. Inspired by these papers, we characterize the approximate ω-orthogonality of operators in $\mathbb{B}(\mathscr{H})$.

THEOREM 1. *Let $T,S\in\mathbb{B}(\mathscr{H})$ and $\varepsilon\in[0,1)$. The relation $T\perp_\omega^\varepsilon S$ holds if and only if for every $\theta\in[0,2\pi)$, there exists a sequence $\{x_n^\theta\}_{n\in\mathbb{N}}$ of unit vectors in $\mathscr{H}$ such that the following two conditions hold:*

(i) $\lim\limits_{n\to\infty}|\langle Tx_n^\theta,x_n^\theta\rangle|=\omega(T)$,

(ii) $\lim\limits_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}\geqslant-\varepsilon\omega(T)\omega(S)$.

Proof. ($\Longleftarrow$) Let $\lambda\in\mathbb{C}$. Then $\lambda=|\lambda|e^{i\theta}$ for some $\theta\in[0,2\pi)$. By the assumption, there exists a sequence $\{x_n^\theta\}_{n\in\mathbb{N}}$ of unit vectors in $\mathscr{H}$ such that (i) and (ii) hold. Thus,

$$\begin{aligned}\omega^2(T+\lambda S)&\geqslant\lim_{n\to\infty}|\langle(T+\lambda S)x_n^\theta,x_n^\theta\rangle|^2\\&=\lim_{n\to\infty}\Big(|\langle Tx_n^\theta,x_n^\theta\rangle|^2+2|\lambda|\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}+|\lambda|^2|\langle Sx_n^\theta,x_n^\theta\rangle|^2\Big)\\&\geqslant\lim_{n\to\infty}\Big(|\langle Tx_n^\theta,x_n^\theta\rangle|^2+2|\lambda|\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}\Big)\\&\geqslant\omega^2(T)-2\varepsilon\omega(T)\omega(\lambda S)\end{aligned}$$

Thus, $T\perp_\omega^\varepsilon S$.

($\Longrightarrow$) Let $\theta\in[0,2\pi)$. We derive from $T\perp_\omega^\varepsilon S$ that $\omega^2(T+\lambda S)\geqslant\omega^2(T)-2\varepsilon\omega(T)\omega(\lambda S)$ for all $\lambda\in\mathbb{C}$. Hence, $\omega^2\left(T+\frac{e^{i\theta}}{n}S\right)\geqslant\omega^2(T)-2\varepsilon\omega(T)\omega\left(\frac{e^{i\theta}}{n}S\right)$ for all $n\in\mathbb{N}$.

For every $n\in\mathbb{N}$ there exists x_n^θ with $\|x_n^\theta\|=1$ such that

$$\omega^2\left(T+\frac{e^i\theta}{n}S\right)-\frac{1}{n^2}<\left|\left\langle\left(T+\frac{e^{i\theta}}{n}S\right)x_n^\theta,x_n^\theta\right\rangle\right|^2,$$

whence

$$\begin{aligned}&\omega^2(T)-\frac{2\varepsilon}{n}\omega(T)\omega(S)-\frac{1}{n^2}\\&\quad\leqslant\omega^2\left(T+\frac{e^{i\theta}}{n}S\right)-\frac{1}{n^2}\\&\quad<\left|\left\langle\left(T+\frac{e^{i\theta}}{n}S\right)x_n^\theta,x_n^\theta\right\rangle\right|^2\\&\quad=|\langle Tx_n^\theta,x_n^\theta\rangle|^2+\frac{2}{n}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}+\frac{1}{n^2}|\langle Sx_n^\theta,x_n^\theta\rangle|^2.\end{aligned}\tag{2}$$

Therefore,

$$\begin{aligned}\frac{n}{2}(\omega^2(T)-|\langle Tx_n^\theta,x_n^\theta\rangle|^2)&<\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}\\&\qquad+\frac{1}{2n}\omega^2(S)+\frac{1}{2n}+\varepsilon\omega(T)\omega(S)\qquad(n\in\mathbb{N}),\end{aligned}$$

and hence

$$0\leqslant\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta,x_n^\theta\rangle\overline{\langle Sx_n^\theta,x_n^\theta\rangle}\}+\frac{1}{2n}\omega^2(S)+\frac{1}{2n}+\varepsilon\omega(T)\omega(S)\qquad(n\in\mathbb{N}).\tag{3}$$

Note that $\{\langle Tx_n^\theta, x_n^\theta\rangle\}$ and $\{\langle Sx_n^\theta, x_n^\theta\rangle\}$ are two bounded sequences in $\mathbb{C}$. Therefore, by passing to subsequences of $\{x_n^\theta\}_{n\in\mathbb{N}}$, if necessary, we can assume that these two sequences are convergent. Now, inequality (3) implies that

$$\lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S).$$

Thus, (ii) is valid.

We shall prove (i). It follows from (2) that

$$\begin{aligned}
&\omega^2(T) - \frac{2\varepsilon}{n}\omega(T)\omega(S) - \frac{1}{n^2}\\
&\quad\leqslant |\langle Tx_n^\theta, x_n^\theta\rangle|^2 + \frac{2}{n}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} + \frac{1}{n^2}|\langle Sx_n^\theta, x_n^\theta\rangle|^2\\
&\quad\leqslant |\langle Tx_n^\theta, x_n^\theta\rangle|^2 + \frac{2}{n}|\langle Tx_n^\theta, x_n^\theta\rangle||\langle Sx_n^\theta, x_n^\theta\rangle| + \frac{1}{n^2}|\langle Sx_n^\theta, x_n^\theta\rangle|^2\\
&\quad\leqslant |\langle Tx_n^\theta, x_n^\theta\rangle|^2 + \frac{2}{n}\omega(T)\|S\| + \frac{1}{n^2}\omega^2(S),
\end{aligned}$$

for all $n\in\mathbb{N}$. Hence

$$\omega^2(T) \geqslant |\langle Tx_n^\theta, x_n^\theta\rangle|^2 \geqslant \omega^2(T) - \frac{2\varepsilon}{n}\omega(T)\omega(S) - \frac{1}{n^2} - \frac{2}{n}\omega(T)\|S\| - \frac{1}{n^2}\omega^2(S),$$

for all $n\in\mathbb{N}$. Therefore, $\lim_{n\to\infty}|\langle Tx_n^\theta, x_n^\theta\rangle| = \omega(T)$. $\square$

REMARK 1. Due to the homogeneity of the relation $\perp_\omega^\varepsilon$, without loss of generality, we may assume that $\omega(T) = \omega(S) = 1$. Then $T \perp_\omega^\varepsilon S$ if and only if for every $\theta\in[0,2\pi)$ there exists a sequence $\{x_n^\theta\}_{n\in\mathbb{N}}$ of unit vectors in $\mathscr{H}$ such that the following two conditions hold:

(i) $\lim_{n\to\infty}|\langle Tx_n^\theta, x_n^\theta\rangle| = 1$,

(ii) $\lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} \geqslant -\varepsilon$.

Given an operator $T\in\mathbb{B}(\mathscr{H})$, the set of all sequences in the closed unit ball of $\mathscr{H}$ at which T attains its numerical radius in limits is denoted by

$$M^*_{\omega(T)} = \{\{x_n\} : \ \|x_n\| = 1, \lim_{n\to\infty}|\langle Tx_n, x_n\rangle| = \omega(T)\}.$$

In the following result, we show that under some mild conditions, $\perp_\omega^\varepsilon$ behaves like a symmetric relation. Recall that the Crawford number of an operator $T\in\mathbb{B}(\mathscr{H})$ is defined by

$$c(T) = \inf\{|\langle Tx, x\rangle| : \|x\| = 1\}.$$

PROPOSITION 2. Let $T,S\in\mathbb{B}(\mathscr{H})$ and $c(T)\neq 0$ and $\varepsilon\in[0,1)$. If $T\perp_\omega^\varepsilon S$ and $M^*_{\omega(S)}\cap M^*_{\omega(T+\lambda S)}\neq\emptyset$ for all $\lambda\in\mathbb{C}$, then $S\perp_\omega^\varepsilon T$.

Proof. Let $\lambda\in\mathbb{C}$. Put $\beta := \dfrac{\omega(S)}{c(T)}$. Since $\perp_\omega^\varepsilon$ is homogeneous, we have $\beta T\perp_\omega^\varepsilon S$. Hence $\omega^2(\beta T + \overline{\lambda}S) \geqslant \omega^2(\beta T) - 2\varepsilon\omega(\beta T)\omega(\overline{\lambda}S)$. Let $\{x_n\}\in M^*_{\omega(S)}\cap M^*_{\omega(T+\frac{\overline{\lambda}}{\beta}S)}$.

We have

$$\begin{aligned}\omega^2(\beta T) - 2\varepsilon\omega(\beta T)\omega(\overline{\lambda}S) &\leqslant \omega^2(\beta T + \overline{\lambda}S) \\ &= \lim_{n\to\infty} |\langle(\beta T + \overline{\lambda}S)x_n, x_n\rangle|^2 \\ &= \lim_{n\to\infty}\Big(|\langle\beta T x_n, x_n\rangle|^2 + 2\mathrm{Re}\lambda\langle\beta T x_n, x_n\rangle\overline{\langle Sx_n, x_n\rangle} + |\lambda|^2|\langle Sx_n, x_n\rangle|^2\Big).\end{aligned}$$

From $\lim_{n\to\infty}|\langle\beta T x_n, x_n\rangle|^2 \leqslant \omega^2(\beta T)$, we infer that

$$\lim_{n\to\infty}\Big(2\mathrm{Re}\lambda\langle\beta T x_n, x_n\rangle\overline{\langle Sx_n, x_n\rangle} + |\lambda|^2|\langle Sx_n, x_n\rangle|^2\Big) \geqslant -2\varepsilon\omega(\beta T)\omega(\overline{\lambda}S). \tag{4}$$

From $\beta = \dfrac{\omega(S)}{c(T)}$, we conclude that $\lim_{n\to\infty}(\beta^2|\langle Tx_n, x_n\rangle|^2 - |\langle Sx_n, x_n\rangle|^2) \geqslant 0$. Thus

$$\begin{aligned}\omega^2(S + \lambda\beta T) &\geqslant \lim_{n\to\infty}|\langle(S + \lambda\beta T)x_n, x_n\rangle|^2 \\ &= \lim_{n\to\infty}\Big(|\langle Sx_n, x_n\rangle|^2 + 2\mathrm{Re}\lambda\langle\beta T x_n, x_n\rangle\overline{\langle Sx_n, x_n\rangle} + |\lambda|^2|\langle\beta T x_n, x_n\rangle|^2\Big) \\ &\geqslant \lim_{n\to\infty}\Big(|\langle Sx_n, x_n\rangle|^2 + 2\mathrm{Re}\lambda\langle\beta T x_n, x_n\rangle\overline{\langle Sx_n, x_n\rangle} + |\lambda|^2|\langle Sx_n, x_n\rangle|^2\Big) \\ &\geqslant \lim_{n\to\infty}|\langle Sx_n, x_n\rangle|^2 - 2\varepsilon\omega(\beta T)\omega(\overline{\lambda}S) \qquad \text{(by (4))}.\end{aligned}$$

It follows from the assumption that $\lim_{n\to\infty}|\langle Sx_n, x_n\rangle| = \omega(S)$. Therefore

$$\omega^2(S + \lambda\beta T) \geqslant \omega^2(S) - 2\varepsilon\omega(S)\omega(\beta\lambda T),$$

since $\omega(\mu S) = |\mu|\omega(S)$ for each $\mu \in \mathbb{C}$. Thus $S \perp_\omega^\varepsilon T$. □

For compact operators, in particular in the case where $\mathscr{H}$ is finite dimensional, Theorem 1 yields the following result.

THEOREM 2. *Let $T, S \in \mathbb{B}(\mathscr{H})$ be two compact operators and $\varepsilon \in [0,1)$. Then $T \perp_\omega^\varepsilon S$ holds if and only if for every $\theta \in [0, 2\pi)$, there exists a unit vector $x^\theta \in \mathscr{H}$ such that $|\langle Tx^\theta, x^\theta\rangle| = \omega(T)$ and $\mathrm{Re}\{e^{-i\theta}\langle Tx^\theta, x^\theta\rangle\overline{\langle Sx^\theta, x^\theta\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S)$.*

Proof. ($\Longleftarrow$) It is obvious by Theorem 1.

($\Longrightarrow$) Let $\theta \in [0, 2\pi)$. It follows from Theorem 1 that there exists a sequence $\{x_n^\theta\}_{n\in\mathbb{N}}$ of unit vectors in $\mathscr{H}$ such that both (i) $\lim_{n\to\infty}|\langle Tx_n^\theta, x_n^\theta\rangle| = \omega(T)$ and (ii) $\lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S)$ hold.

Since the closed unit ball of $\mathscr{H}$ is weakly compact, $\{x_n^\theta\}$ has a weakly convergent subsequence. Without loss of generality, we assume that $\{x_n^\theta\}$ weakly converges, say to x^θ. Hence, $\langle x_n^\theta - x^\theta, T^*y\rangle \to 0$ as $n \to \infty$ for all $y \in \mathscr{H}$. Therefore $\{Tx_n^\theta\}$ weakly converges to Tx^θ. Similarly $\{Sx_n^\theta\}$ weakly converges to Sx^θ.

On the other hand, since $\{x_n^\theta\}$ is norm-bounded and the operators T and S are compact, by passing to subsequences, we can assume that $\{Tx_n^\theta\}$ and $\{Sx_n^\theta\}$ are norm-convergent. Thus, $\lim_{n\to\infty}Tx_n^\theta = Tx^\theta$ and $\lim_{n\to\infty}Sx_n^\theta = Sx^\theta$ in the norm topology.

Therefore, $\lim_{n\to\infty}\langle Tx_n^\theta, x_n^\theta\rangle = \langle Tx^\theta, x^\theta\rangle$ and $\lim_{n\to\infty}\langle Sx_n^\theta, x_n^\theta\rangle = \langle Sx^\theta, x^\theta\rangle$. Now by considering (i) and (ii), the proof is completed. □

EXAMPLE 4. Suppose that $x, y \in \mathscr{H}$ are unit vectors and $x\otimes y$ denotes the rank one operator defined by $(x\otimes y)(z) := \langle z, y\rangle x,\ z \in \mathscr{H}$.

The authors in [7, Lemma 3.2] proved that $\omega(x\otimes y) = \frac{1}{2}(|\langle x,y\rangle| + \|x\otimes y\|)$ for all $x, y \in \mathscr{H}$. Hence, for the compact operator $x\otimes x$, we get $\omega(x\otimes x) = \|x\|^2$. Let $\varepsilon \in [0,1)$. From Theorem 2, $x\otimes x \perp_\omega^\varepsilon y\otimes y$ if and only if for every $\theta \in [0,2\pi)$, there exists a unit vector $x^\theta \in \mathscr{H}$ such that $|\langle (x\otimes x)(x^\theta), x^\theta\rangle| = |\langle x^\theta, x\rangle|^2 = 1 = \|x\|^2 = \omega(x\otimes x)$ and $\mathrm{Re}\{e^{-i\theta}|\langle x^\theta, x\rangle|^2|\langle x^\theta, y\rangle|^2\} = \cos\theta\,|\langle x^\theta, x\rangle|^2|\langle x^\theta, y\rangle|^2 \geqslant -\varepsilon$.

From equality case in the Cauchy–Schwarz inequality and $1 = |\langle x^\theta, x\rangle|$ we infer that $x^\theta = x$. If x and y are orthogonal, then $\langle x, y\rangle = 0$ and above discussion shows that $x\otimes x \perp_\omega^\varepsilon y\otimes y$, since $|\langle (x\otimes x)(x), x\rangle| = \|x\|^2 = \omega(x\otimes x)$ and $\cos\theta\,|\langle x,x\rangle|^2|\langle x,y\rangle|^2 = 0 \geqslant -\varepsilon$.

Moreover, if $\varepsilon > 0$ is given and unit vectors $x_\varepsilon, y_\varepsilon \in \mathscr{H}$ are such that $\varepsilon < |\langle x_\varepsilon, y_\varepsilon\rangle|^2$ and $\theta_\varepsilon \in [0,\pi)$ is such that $-1 < \cos\theta_\varepsilon < \frac{-\varepsilon}{|\langle x_\varepsilon, y_\varepsilon\rangle|^2}$, then the inequality $\cos\theta_\varepsilon|\langle x_\varepsilon, y_\varepsilon\rangle|^2 < -\varepsilon$ ensures that $x\otimes x \not\perp_\omega^\varepsilon y\otimes y$.

PROPOSITION 3. Let $T, S \in \mathbb{B}(\mathscr{H})$, the operator T be positive, and $T \perp_\omega^\varepsilon S$ and $\varepsilon \in [0,1)$. Then $(T+I) \perp_\omega^\varepsilon S$.

Proof. Let $\theta \in [0,2\pi)$. By the assumption, there exists a sequence $\{x_n^\theta\}$ of unit vectors such that

$$\lim_{n\to\infty}|\langle Tx_n^\theta, x_n^\theta\rangle| = \omega(T), \text{ and } \lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S).$$

Since T is positive, $\omega(T+I) = \omega(T)+1$ and $\lim_{n\to\infty}\mathrm{Re}\langle Tx_n^\theta, x_n^\theta\rangle = \lim_{n\to\infty}\langle Tx_n^\theta, x_n^\theta\rangle$.

Hence,

$$\lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} \geqslant -\varepsilon\omega(S)$$

and

$$\begin{aligned}\lim_{n\to\infty}|\langle (T+I)x_n^\theta, x_n^\theta\rangle|^2 &= \lim_{n\to\infty}\left(|\langle Tx_n^\theta, x_n^\theta\rangle|^2 + |\langle Ix_n^\theta, x_n^\theta\rangle|^2 + 2\mathrm{Re}\langle Tx_n^\theta, x_n^\theta\rangle\right)\\ &= \omega^2(T) + 1 + 2\omega(T) = \omega^2(T+I).\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle (T+I)x_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\}\\ &\quad= \lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\} + \lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Ix_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\}\\ &\quad\geqslant -\varepsilon\omega(T)\omega(S) - \varepsilon\omega(S)\\ &\quad\geqslant -\varepsilon\omega(S)\omega(T+I).\end{aligned}$$

Therefore, $(T+I) \perp_\omega^\varepsilon S$. □

Our last result of this section reads as follows.

PROPOSITION 4. Let $S, K \in \mathbb{B}(\mathscr{H})$ be positive operators of norm one, $K \leqslant S$, and $\varepsilon \in [0,1)$. If $T \perp_{\omega}^{\varepsilon} S$, then $T \perp_{\omega}^{2\varepsilon} S+K$.

Proof. Let $\theta \in [0,2\pi)$. There exists a sequence $\{x_n^\theta\}_{n\in\mathbb{N}}$ of unit vectors in $\mathscr{H}$ such that $\lim\limits_{n\to\infty} |\langle Tx_n^\theta, x_n^\theta\rangle| = \omega(T)$ and $\lim\limits_{n\to\infty} \langle Sx_n^\theta, x_n^\theta\rangle \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\} \geqslant -\varepsilon\omega(T)$ hold. We may assume that $\lim_{n\to\infty}\langle Sx_n^\theta, x_n^\theta\rangle > 0$. We have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\langle Kx_n^\theta, x_n^\theta\rangle\} &= \lim_{n\to\infty}\langle Kx_n^\theta, x_n^\theta\rangle \lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\} \\
&\geqslant \lim_{n\to\infty}\langle Kx_n^\theta, x_n^\theta\rangle \frac{-\varepsilon\omega(T)}{\lim_{n\to\infty}\langle Sx_n^\theta, x_n^\theta\rangle} \\
&\geqslant -\varepsilon\omega(T) = -\varepsilon\omega(T)\omega(K).
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle (S+K)x_n^\theta, x_n^\theta\rangle}\} &= \lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\langle Sx_n^\theta, x_n^\theta\rangle\} \\
&\quad + \lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\langle Kx_n^\theta, x_n^\theta\rangle\} \\
&\geqslant -\varepsilon\omega(T)\omega(S) - \varepsilon\omega(T)\omega(K) \\
&\geqslant -2\varepsilon\omega(T)\omega(S+K) \quad (\text{as } 0 \leqslant S, K \leqslant S+K).
\end{aligned}$$

Hence $T \perp_{\omega}^{2\varepsilon} S+K$. $\square$

3. Numerical radius derivation

In this section, we introduce the notion of ω-derivation and provide a characterization of $\perp_{\omega}^{\varepsilon}$ by employing this notion.

Let $\theta \in [0,2\pi)$. For given operators $T, S \in \mathbb{B}(\mathscr{H})$, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(r) = \omega^2(T + re^{i\theta}S)$ is convex. To show this, let $r, s \in \mathbb{R}$ and $\alpha \in [0,1]$. By the convexity of the real function $g(r) = r^2$, we have

$$\begin{aligned}
f(\alpha r + (1-\alpha)s) &= \omega^2(T + (\alpha r + (1-\alpha)s)e^{i\theta}S) \\
&= \omega^2(\alpha(T + re^{i\theta}S) + (1-\alpha)(T + se^{i\theta}S)) \\
&\leqslant \Big(\alpha\omega((T + re^{i\theta}S) + (1-\alpha)\omega((T + se^{i\theta}S))\Big)^2 \\
&\leqslant \alpha\omega^2(T + re^{i\theta}S) + (1-\alpha)\omega^2(T + se^{i\theta}S) \\
&= \alpha f(r) + (1-\alpha)f(s).
\end{aligned}$$

Thus, for every $\theta \in [0,2\pi)$ the function $D_\omega^\theta : \mathbb{B}(\mathscr{H}) \times \mathbb{B}(\mathscr{H}) \to \mathbb{R}$ defined by

$$D_\omega^\theta(T,S) := \lim_{r\to 0^+} \frac{\omega^2(T + re^{i\theta}S) - \omega^2(T)}{2r}$$

exists, and we call it ω-derivation.

Furthermore, the functions $f(r) = \omega^2(T + re^{i\theta}S)$ and $g(r) = 2r\varepsilon\omega(T)\omega(S)$, $\varepsilon \in [0,1)$ are convex functions and so is $h(r) = \omega^2(T + re^{i\theta}S) + 2r\varepsilon\omega(T)\omega(S)$.

The following theorem gives a characterization of the approximate ω-orthogonality for operators.

THEOREM 3. *Let $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$. The relation $T \perp_{\omega}^{\varepsilon} S$ holds if and only if $\inf_{\theta\in[0,2\pi)} D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$.*

Proof. ($\Longrightarrow$) Suppose that $\theta \in [0,2\pi)$. It follows from $T \perp_{\omega}^{\varepsilon} S$ that $\omega^2(T + re^{i\theta}S) \geqslant \omega^2(T) - 2r\varepsilon\omega(T)\omega(S)$ for all $r \in \mathbb{R}^+$. We have

$$\begin{aligned} D_{\omega}^{\theta}(T,S) &= \lim_{r\to 0^+} \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{2r} \\ &= \lim_{r\to 0^+} \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)+2r\varepsilon\omega(T)\omega(S)}{2r} + \lim_{r\to 0^+} \frac{-2r\varepsilon\omega(T)\omega(S)}{2r} \\ &= \lim_{r\to 0^+} \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)+2r\varepsilon\omega(T)\omega(S)}{2r} - \varepsilon\omega(T)\omega(S). \end{aligned}$$

Since $\frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)+2r\varepsilon\omega(T)\omega(S)}{2r} \geqslant 0$, passing to the limit, we get

$$D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S).$$

Thus, $\inf_{\theta} D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$.

($\Longleftarrow$) Let $\inf_{\theta\in[0,2\pi)} D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$. Then for every $\theta \in [0,2\pi)$ we have $D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$. Hence

$$\begin{aligned} -\varepsilon\omega(T)\omega(S) \leqslant D_{\omega}^{\theta}(T,S) &= \lim_{r\to 0^+} \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)+2r\varepsilon\omega(T)\omega(S)}{2r} - \varepsilon\omega(T)\omega(S) \\ &= \frac{1}{2}\lim_{r\to 0^+} \frac{h(r)-h(0)}{r-0} - \varepsilon\omega(T)\omega(S), \end{aligned}$$

whence $h'(0) = \lim_{r\to 0^+} \frac{h(r)-h(0)}{r-0} \geqslant 0$.

Then the convexity of h implies that $h(r)-h(0) \geqslant (r-0)h'(0) \geqslant 0$ and so $h(r) \geqslant h(0)$ for every $r \geqslant 0$. Therefore, $\omega^2(T+re^{i\theta}S) \geqslant \omega^2(T) - 2r\varepsilon\omega(T)\omega(S)$ for every $\theta \in [0,2\pi)$. This entails that $T \perp_{\omega}^{\varepsilon} S$. □

COROLLARY 1. *Let $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$. The following statements are equivalent:*

(i) T is approximately ω-orthogonal to S.

(ii) $\inf_{\theta\in[0,2\pi)} D_{\omega}^{\theta}(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$.

(iii) For every $\theta \in [0,2\pi)$, there exists a sequence $\{x_n^{\theta}\}$ of unit vectors in $\mathscr{H}$ such that $\lim_{n\to\infty} |\langle Tx_n^{\theta}, x_n^{\theta}\rangle| = \omega(T)$ and $\lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^{\theta}, x_n^{\theta}\rangle\overline{\langle Sx_n^{\theta}, x_n^{\theta}\rangle}\} \geqslant -\varepsilon\omega(T)\omega(S)$.

REMARK 2. Let $T \in \mathbb{B}(\mathscr{H})$. Lumer [9, Theorem 11] proved that

$$\lim_{r\to 0^+} \frac{\|I+rT\|-1}{r} = \sup_{\|x\|=1} \mathrm{Re}\langle Tx,x\rangle.$$

Dragomir [4, Theorem 66] proved that

$$\lim_{r\to 0^+} \frac{\omega(I+rT)-1}{r} = \sup_{\|x\|=1} \mathrm{Re}\,\langle Tx,x\rangle.$$

Therefore, for every $\theta \in [0,2\pi)$, we have

$$\begin{aligned} D_\omega^\theta(I,T) &= \lim_{r\to 0^+} \frac{\omega^2(I+re^{i\theta}T)-1}{2r} \\ &= \lim_{r\to 0^+} \frac{\omega(I+re^{i\theta}T)-1}{r} \lim_{r\to 0^+} \frac{\omega(I+re^{i\theta}T)+1}{2} \\ &= \lim_{r\to 0^+} \frac{\omega(I+re^{i\theta}T)-1}{r} \\ &= \sup_{\|x\|=1} \mathrm{Re}\,\langle e^{i\theta}Tx,x\rangle. \end{aligned}$$

REMARK 3. Let $T,S \in \mathbb{B}(\mathscr{H})$ and $\varepsilon \in [0,1)$. We showed that $T \perp_\omega^\varepsilon S$ if and only if $\inf_{\theta\in[0,2\pi)} D_\omega^\theta(T,S) \geqslant -\varepsilon\omega(T)\omega(S)$. In virtue of

$$\begin{aligned} \lim_{r\to 0^+} \frac{\omega(T+re^{i\theta}S)-\omega(T)}{r} &= \lim_{r\to 0^+} \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{r(\omega(T+re^{i\theta}S)+\omega(T))} \\ &= \lim_{r\to 0^+} \left(\frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{r} \cdot \frac{1}{\omega(T+re^{i\theta}S)+\omega(T)}\right) \\ &= \frac{1}{\omega(T)} \lim_{r\to 0^+} \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{2r} \\ &= \frac{1}{\omega(T)} D_\omega^\theta(T,S), \end{aligned} \tag{5}$$

we may say that $T \perp_\omega^\varepsilon S$ if and only if $\lim_{r\to 0^+} \frac{\omega(T+re^{i\theta}S)-\omega(T)}{r} \geqslant -\varepsilon\omega(S)$ for every $\theta \in [0,2\pi)$.

If $\theta = 0$, then Dragomir [4] proved that

$$[S,T] := \lim_{r\to 0^+} \frac{\omega^2(T+rS)-\omega^2(T)}{2r} \qquad (T,S \in \mathbb{B}(\mathscr{H}))$$

gives rise to a semi-inner product-type on $\mathbb{B}(\mathscr{H})$, see also [1].

Now, we list here some properties of the above semi-inner product type.

LEMMA 4. *Let $\theta \in [0,2\pi)$ and let $T,S \in \mathbb{B}(\mathscr{H})$. Then the following statements hold:*

(i) $[e^{i\theta}T, e^{i\theta}T] = \omega^2(T)$.
(ii) $[ie^{i\theta}T, e^{i\theta}T] = 0$ *and* $[0,T] = [e^{i\theta}T, 0] = 0$.
(iii) The following Cauchy–Schwarz type inequality holds

$$\left|\left[e^{i\theta}S, T\right]\right| \leqslant \omega(T)\omega(S).$$

(iv) The mapping $[e^{i\theta}S,T]$ *is subadditive in the first variable, that is, for all operators* $R \in \mathbb{B}(\mathscr{H})$*, it holds that*

$$[e^{i\theta}(S+R),T] \leqslant [e^{i\theta}S,T] + [e^{i\theta}R,T].$$

Proof. (i) and (ii) are clear.
(iii) It is easy to see that for any $r > 0$

$$\begin{aligned} -\omega(S) = \frac{\omega(T) - r\omega(S) - \omega(T)}{r} &\leqslant \frac{\omega(T + re^{i\theta}S) - \omega(T)}{r} \\ &\leqslant \frac{\omega(T) + r\omega(S) - \omega(T)}{r} = \omega(S). \end{aligned} \tag{6}$$

It follows from (5) and (6) that

$$[e^{i\theta}S,T] = D_\omega^\theta(T,S) = \omega(T)\lim_{r\to 0^+}\frac{\omega(T + re^{i\theta}S) - \omega(T)}{r} \leqslant \omega(T)\omega(S)$$

.

Similarly, one can show that $[e^{i\theta}S,T] \geqslant -\omega(T)\omega(S)$. Therefore,

$$|[e^{i\theta}S,T]| \leqslant \omega(T)\omega(S).$$

(iv) Since $f(r) = \omega^2(T + re^{i\theta}S)$ is convex, we have

$$\omega^2\left(\frac{2T + re^{i\theta}(S+R)}{2}\right) \leqslant \frac{1}{2}\omega^2(T + re^{i\theta}S) + \frac{1}{2}\omega^2(T + re^{i\theta}R),$$

whence

$$\begin{aligned} &2\left(\omega^2(T + \frac{r}{2}e^{i\theta}(S+R)) - \omega^2(T)\right) \\ &\leqslant \left(\omega^2(T + re^{i\theta}S) - \omega^2(T)\right) + \left(\omega^2(T + re^{i\theta}R) - \omega^2(T)\right). \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{r\to 0^+}\frac{2\left(\omega^2(T + \frac{r}{2}e^{i\theta}(S+R)) - \omega^2(T)\right)}{2r} \\ &\leqslant \lim_{r\to 0^+}\frac{\omega^2(T + re^{i\theta}S) - \omega^2(T)}{2r} + \lim_{r\to 0^+}\frac{\omega^2(T + re^{i\theta}R) - \omega^2(T)}{2r}. \end{aligned}$$

Thus, $[e^{i\theta}(S+R),T] \leqslant [e^{i\theta}S,T] + [e^{i\theta}R,T]$. □

In the following proposition, we show a relation between $D_\omega^\theta(T,S)$ and

$$\mathrm{Re}\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\overline{\langle Sx_n^\theta, x_n^\theta\rangle}\}$$

under some mild conditions.

PROPOSITION 5. Let $\theta \in [0,2\pi)$ be fixed and let $T,S \in \mathbb{B}(\mathscr{H})$. If $\{x_n^r\} \in M^*_{\omega(T)} \cap M^*_{\omega(T+re^{i\theta}S)}$ for all $r \in \mathbb{R}^+$, then

$$[e^{i\theta}S,T] = D^\theta_\omega(T,S) = \lim_{r\to 0^+}\lim_{n\to\infty} \mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\}.$$

Proof. We have

$$\begin{aligned}\omega^2(T+re^{i\theta}S) &= \lim_{n\to\infty}|\langle (T+re^{i\theta}S)x_n^r,x_n^r\rangle|^2\\ &= \lim_{n\to\infty}\left(|\langle Tx_n^r,x_n^r\rangle|^2 + 2r\mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\} + r^2|\langle Sx_n^r,x_n^r\rangle|^2\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{2r}\\ &= \frac{\lim_{n\to\infty}\left(|\langle Tx_n^r,x_n^r\rangle|^2 + 2r\mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\} + r^2|\langle Sx_n^r,x_n^r\rangle|^2\right)-\omega^2(T)}{2r}\\ &= \lim_{n\to\infty}\frac{2r\mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\} + r^2|\langle Sx_n^r,x_n^r\rangle|^2}{2r}\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\} &\leqslant \frac{\omega^2(T+re^{i\theta}S)-\omega^2(T)}{2r}\\ &\leqslant \lim_{n\to\infty}\mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\} + r\omega(S).\end{aligned}$$

Letting $r \to 0^+$, we get the desired equality. □

REMARK 4. The intersection $M^*_{\omega(T)} \cap M^*_{\omega(T+re^{i\theta}S)}$ can be nonempty. Following we provide two examples:

(i) If $\{x_n\} \in M^*_{\omega(T)}$ such that $\langle Sx_n,x_n\rangle = 0$ for each n, then $\omega(T+re^{i\theta}S) = \omega(T)$, that is $\{x_n\} \in M^*_{\omega(T)} \cap M^*_{\omega(T+re^{i\theta}S)}$ for all $r \in \mathbb{R}^+$.

(ii) If $T = I$ and $\{x_n^r\} \in M^*_{\omega(I+re^{i\theta}S)}$ for all $r \in \mathbb{R}^+$, then $\{x_n^r\} \in M^*_{\omega(I)}$. Hence $\{x_n^r\} \in M^*_{\omega(T)} \cap M^*_{\omega(I+re^{i\theta}S)}$ for all $r \in \mathbb{R}^+$.

REMARK 5. Under the conditions of Proposition 5, Remark 3 yields that

$$\begin{aligned}\lim_{r\to 0^+}\frac{\omega(T+re^{i\theta}S)-\omega(T)}{r} &= \frac{1}{\omega(T)}D^\theta_\omega(T,S)\\ &= \frac{1}{\omega(T)}\lim_{r\to 0^+}\lim_{n\to\infty}\left(\mathrm{Re}\{e^{-i\theta}\langle Tx_n^r,x_n^r\rangle\overline{\langle Sx_n^r,x_n^r\rangle}\}\right).\end{aligned}$$

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