Mathematical
Inequalities
& Applications
Volume 24, Number 2 (2021), 477–490 doi:10.7153/mia-2021-24-33

# POSITIVE DEFINITENESS ON PRODUCTS VIA GENERALIZED STIELTJES AND OTHER FUNCTIONS

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(*Communicated by I. Pinelis*)

*Abstract.* Let $(X,\rho)$ and $(Y,\sigma)$ be quasi-metric spaces and $\lambda$ a fixed positive real number. This paper establishes the positive definiteness of functions of the form

$$G_r(t,u)=\frac{1}{h(u)^r}f\left(\frac{g(t)}{h(u)}\right),\ (t,u)\in\{\rho(x,x'):x,x'\in X\}\times\{\sigma(y,y'):y,y'\in Y\},$$

on $X\times Y$, where $r\geqslant\lambda$, $f$ belongs to the convex cone of all generalized Stieltjes functions of order $\lambda$, and $g$ and $h$ are positive valued conditionally negative definite functions on $(X,\rho)$ and $(Y,\sigma)$, respectively. As a bypass, it establishes the positive definiteness of functions of the form

$$H_r(t,u)=\frac{1}{g(t)^r}f\left(\frac{g(t)}{h(u)}\right),\ (t,u)\in\{\rho(x,x'):x,x'\in X\}\times\{\sigma(y,y'):y,y'\in Y\},$$

for a generalized complete Bernstein function $f$ of order $\lambda$, under the same assumptions on $r$, $g$ and $h$. The paper also provides necessary and sufficient conditions for the strict positive definiteness of the two models when the spaces involved are metric. The two results yield additional methods to construct positive definite and strictly positive definite functions on a product of metric spaces by integral transforms.

## 1. Introduction

Let $(X,\rho)$ be a quasi-metric space and write $D_X^\rho=\{\rho(x,x'):x,x'\in X\}$. A continuous function $f:D_X^\rho\to\mathbb{R}$ is said to be *positive definite* on $X$ if

$$\sum_{j,k=1}^{n}c_jc_kf(\rho(x_j,x_k))\geqslant 0,$$

for $n\geqslant 1$, real numbers $c_1,\dots,c_n$, and points $x_1,\dots,x_n$ in $X$. If in addition to that, the inequalities are strict whenever the points $x_j$ are distinct and at least one $c_j$ is nonzero, then $f$ is said to be *strictly positive definite* on $X$. If $(Y,\sigma)$ is another quasi-metric space, a continuous function $f:D_X^\rho\times D_Y^\sigma\to\mathbb{R}$ is *positive definite* on $X\times Y$, if

$$\sum_{j,k=1}^{n}c_jc_kf(\rho(x_j,x_k),\sigma(y_j,y_k))\geqslant 0,$$

*Mathematics subject classification* (2010): 42A82, 43A35.
*Keywords and phrases*: Positive definite functions, generalized Stieltjes functions, conditionally negative definite functions, Gneiting's class, complete Bernstein functions.

for $n \geqslant 1$, real numbers $c_1,\dots,c_n$, and points $(x_1,y_1),\dots,(x_n,y_n)$ in $X\times Y$. The positive definite function $f$ is *strictly positive definite* on $X\times Y$, if the inequalities above are strict when the points $(x_j,y_j)$ are distinct and at least one $c_j$ is nonzero. The classes introduced above will be denoted by $PD(X,\rho)$, $SPD(X,\rho)$, $PD(X\times Y,\rho,\sigma)$, and $SPD(X\times Y,\rho,\sigma)$, respectively. The nomenclature introduced so far agrees with that in the classical reference [17] and in the survey paper [1]. In particular, by saying that a pair $(X,\rho)$ is a *quasi-metric space* we mean that $X$ is a nonempty set and $\rho$ is a nonnegative function acting on $X\times X$ so that $\rho(x,x')=\rho(x',x)$ and $\rho(x,x)=0$, for $x,x'\in X$.

This paper is mainly concerned with the construction of positive definite and strictly positive definite functions on a product of quasi-metric spaces, the motivation of which coming from a classical result of T. Gneiting on space-time covariance functions proved in [4]. Gneiting's result establishes that for a fixed bounded and completely monotone function $f$, the formula

$$G_r(t,u)=\frac{1}{h(u)^r}f\left(\frac{g(t)}{h(u)}\right),\quad (t,u)\in D_X^\rho\times D_Y^\sigma, \tag{1}$$

defines a positive definite function on $X\times Y$, whenever $r\geqslant d/2$, $X=\mathbb{R}^d$, $Y=\mathbb{R}^{d'}$, $\rho$ and $\sigma$ are the respective squared distances on $X$ and $Y$, $g(t)=t$, $t\in D_X^\rho$, and $h$ is a positive valued function possessing a completely monotone derivative. Gneiting's paper is highly-cited and the interested reader can access some of these many citations in order to get acquainted with the many developments implied by this important result, not only in mathematics but also in statistics.

Some extensions and generalizations of Gneiting's result deserve to be mentioned at once: Zastavnyi and Porcu results in [19] apply to the very same model but replacing $(\mathbb{R}^{d'},\rho)$ with a normed linear space $(E,\|\cdot\|)$ and the Bernstein function with a positive valued function $h$ for which the functions $y\in[0,\infty)\mapsto \exp(-sh(y))$, $s>0$, belong to $PD(E,\|\cdot\|)$. However, the reader is advised that [19] deals with a setting different from the one considered here, once the notion of positive definiteness on $E$ used there is in the group sense. In [18], White and Porcu presented a contribution in the case in which $(\mathbb{R}^d,\rho)$ is replaced with the unit sphere $S^d$ endowed with its natural geodesic distance, keeping all the rest the same. Very recently, [12] presented not only a new proof of the original Gneiting's model but also a considerable improvement, by replacing $(\mathbb{R}^{d'},\sigma)$ with a quite general quasi-metric space $(Y,\sigma)$ and the Bernstein function $h$ with a positive valued conditionally negative definite function on $(Y,\sigma)$ as done in [19]. We recall that a continuous function $f:D_Y^\sigma\to\mathbb{R}$ belongs to $CND(Y,\sigma)$ (the triplet *CND* stands for *conditionally negative definite*) if for $n\geqslant 1$ and points $y_1,\dots,y_n$ in $Y$, it holds

$$\sum_{j,k=1}^n c_jc_kf(\rho(y_j,y_k))\leqslant 0,$$

for all real numbers $c_1,\dots,c_n$ satisfying $\sum_{j=1}^n c_j=0$. Conditionally negative definite functions are studied in [2, 6] and references quoted there while characterizations in some specific spaces can be found in [11, 17]. The most popular method to create examples of conditionally negative definite functions uses this known result: if $H$ belongs to $PD(X,\sigma)$, then $-h$ belongs to $CND(X,\sigma)$. Additional constructions can be extracted from [2].

In [10] the focus changed considerably. By fixing $f$ in an specific convex cone $S_\lambda^b$, $\lambda>0$, of bounded completely monotone functions, it was shown that $G_r$ belongs to $PD(X\times Y,\rho,\sigma)$ for $r\geqslant\lambda$, whenever $(X,\rho)$ and $(Y,\sigma)$ are general quasi-metric spaces, $g$ is a nonnegative valued function in $CND(X,\rho)$ and $h$ is a positive valued function in $CND(Y,\sigma)$. So, by restricting $f$ to a smaller class of functions, the result in [10] promoted more generality on the spaces and also on the permissible functions $g$ and $h$ in Gneiting's model. In addition to that, the result also presented necessary and sufficient conditions for the strict positive definiteness of the resulting functions $G_r$ in the case in which $(X,\rho)$ and $(Y,\sigma)$ are metric spaces. This was quite an achievement once the desirable strict positive definiteness property was never analyzed/obtained before within Gneiting's model and its extensions and generalizations.

The results to be described in this paper enhance and complement those proved in [10]. After introducing a few technical results in Section 2, we show in Section 3 that if $f$ is a (not necessarily bounded) generalized Stieltjes function of order $\lambda$, then the formula (1) defines an element in $PD(X\times Y,\rho,\sigma)$ whenever $r\geqslant\lambda$, $(X,\rho)$ and $(Y,\sigma)$ are quasi-metric spaces and $g$ and $h$ are positive valued functions in $CND(X,\rho)$ and $CND(Y,\sigma)$, respectively. Further, if $(X,\rho)$ and $(Y,\sigma)$ are metric spaces, we provide necessary and sufficient conditions for the strict positive definiteness of the model in most cases. In Section 4, we present modified versions of the main results proved in Section 3 that hold in the case when $f$ comes from a set we call the class of generalized complete Bernstein functions of order $\lambda$. In Section 5, we include a list of variations of the models based on certain stability properties of the standard complete Bernstein functions of order $\lambda=1$. A short conclusion in Section 6 closes the paper.

## 2. Technical results

In this section we quote two known auxiliary lemmas to be required in Section 3 along with an independent result on separable positive definite functions on a product $X\times Y$ of quasi-metric spaces.

Let us begin with a lemma that provides information on the positive semi-definiteness of the Schur exponential of matrices of negative type (see [14]). Recall that a real symmetric matrix $A=[A_{jk}]_{j,k=1}^n$ of order $n$ is of *negative type* if

$$\sum_{j,k=1}^n c_jc_kA_{jk}\leqslant 0,$$

for all real numbers satisfying $\sum_{j=1}^n c_j=0$.

LEMMA 1. *Let $A$ be a matrix of negative type of order $n$. The following assertions hold for the Schur exponential $B:=[e^{-A_{jk}}]_{j,k=1}^n$ of $-A$:*

(*i*) *$B$ is positive semi-definite.*

(*ii*) *$B$ is positive definite if and only if $-A$ is diagonally dominant in the sense that $A_{jj}+A_{kk}<2A_{jk}$, for $j\neq k$.*

Lemma 2 below is technical and extends the following well-known property: If $g$ is a function in $CND(X,\rho)$, then $t\in D_X^\rho \mapsto \exp(-sg(t))$ belongs to $PD(X,\rho)$ for $s>0$. Assertion $(i)$ is proved in [2, p. 74] while Assertion $(ii)$ is a consequence of Assertion $(i)$, the Bernstein-Widder Theorem on integral representations for completely monotone functions in [15, p. 3], and Lemma 1. A proof is sketched in [10]. Recall that a function $f:(0,\infty)\to\mathbb{R}$ is completely monotone if it is $C^\infty$ and $(-1)^n f^{(n)}(t)\geqslant 0$ for $n=0,1,\dots$ and $t\in(0,\infty)$.

LEMMA 2. *Let $(X,\rho)$ be a quasi-metric space, $f$ a completely monotone function and $g$ a positive valued function in $CND(X,\rho)$. The following assertions hold:*

(*i*) *$f\circ g$ belongs to $PD(X,\rho)$.*

(*ii*) *If $f$ is nonconstant, then $f\circ g$ belongs to $SPD(X,\rho)$ if and only if $g(t)>g(0)$, $t\in D_X^\rho\setminus\{0\}$.*

Next, we establish positive definiteness results for special positive definite functions on $X\times Y$ defined by separated variables. Hereafter, a *trivial quasi-metric space* will mean a quasi-metric space having just one point.

PROPOSITION 1. *Let $(X,\rho)$ and $(Y,\sigma)$ be quasi-metric spaces and $f_1$ and $f_2$ completely monotone functions. If $g$ and $h$ are positive valued functions in $CND(X,\rho)$ and $CND(Y,\sigma)$, respectively, then the following assertions hold for the function $F$ given by*

$$F(t,u)=f_1(g(t))f_2(h(u)),\quad (t,u)\in D_X^\rho\times D_Y^\sigma.$$

(*i*) *$F$ belongs to $PD(X\times Y,\rho,\sigma)$.*

(*ii*) *If $F$ belongs to $SPD(X\times Y,\rho,\sigma)$, then $g(t)>g(0)$, for $t\in D_X^\rho\setminus\{0\}$ and $h(u)>h(0)$, for $u\in D_Y^\sigma\setminus\{0\}$.*

(*iii*) *If $(X,\rho)$ (respect. $(Y,\sigma)$) is nontrivial and $F$ belongs to $SPD(X\times Y,\rho,\sigma)$, then $f_1$ (respectively, $f_2$) is nonconstant.*

*Proof.* Assertion $(i)$ follows from Lemma 2-$(i)$ and the Schur Product Theorem [5, p. 477] combined. If $g(t)=g(0)$ for some $t\in D_X^\rho\setminus\{0\}$, we can pick two distinct points $(x_1,y_1)$ and $(x_2,y_2)$ in $X\times Y$ with $\rho(x_1,x_2)=t$ and $y_1=y_2$ in order to obtain the singular matrix

$$[F(\rho(x_j,x_k),\sigma(y_j,y_k))]_{j,k=1}^2=[f_1(g(0))f_2(h(0))]_{j,k=1}^2.$$

A similar reasoning can be implemented if $h(u)=h(0)$ for some $u\in D_Y^\sigma\setminus\{0\}$. In either case, $F$ cannot belong to $SPD(X\times X,\rho,\sigma)$ and $(ii)$ follows.

If $(X,\rho)$ is nontrivial and $f_1$ is constant, say $C$, we can pick two distinct points $(x_1,y_1)$ and $(x_2,y_2)$ in $X\times Y$ with $y_1=y_2$ in order to obtain the singular matrix

$$[F(\rho(x_j,x_k),\sigma(y_j,y_k))]_{j,k=1}^2=[Cf_2(h(0))]_{j,k=1}^2.$$

Thus, $F$ cannot belong to $SPD(X\times X,\rho,\sigma)$ and $(iii)$ follows. $\square$

Regarding strict positive definiteness, the following result also holds.

THEOREM 1. *Let $(X,\rho)$ and $(Y,\sigma)$ be metric spaces, $f_1$, $f_2$ nonconstant completely monotone functions and $g$ and $h$ positive valued functions in $CND(X,\rho)$ and $CND(Y,\sigma)$, respectively. The following assertions are equivalent for the function $F$ introduced in Proposition 1:*

$(i)$ *$F$ belongs to $SPD(X\times Y,\rho,\sigma)$*

$(ii)$ *$g(t)>g(0)$, for $t\in D_X^\rho\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^\sigma\setminus\{0\}$.*

*Proof.* In view of Proposition 1-$(ii)$, only the implication $(ii)\Rightarrow(i)$ needs to be proved. We know already that $F\in PD(X\times X,\rho,\sigma)$ by Proposition 1-$(i)$. In order to handle the strict positive definiteness of $F$ under the assumptions of the theorem and the two conditions in $(ii)$, we invoke the Bernstein-Widder Theorem to write

$$f_i(t)=\int_{[0,\infty)}e^{-st}d\mu_i(s),\quad t\geqslant 0;\ i=1,2,$$

for some (not necessarily finite) positive measures $\mu_1$ and $\mu_2$ on $[0,\infty)$. Hence,

$$\begin{aligned} f_1(g(t))f_2(h(u)) &= \int_{[0,\infty)}e^{-g(t)s}d\mu_1(s)\int_{[0,\infty)}e^{-h(u)s'}d\mu_2(s') \\ &= \int_{[0,\infty)}\left[\int_{[0,\infty)}e^{-g(t)s-h(u)s'}d\mu_1(s)\right]d\mu_2(s'),\quad (t,u)\in D_X^\rho\times D_Y^\sigma. \end{aligned}$$

The Schur Product Theorem and Lemma 2-$(i)$ imply that the functions

$$(t,u)\in D_X^\rho\times D_Y^\sigma\mapsto\int_{[0,\infty)}e^{-g(t)s-h(u)s'}d\mu_1(s),\quad s'>0, \tag{2}$$

belong to $PD(X\times Y,\rho,\sigma)$. If $f_2$ is nonconstant, $F$ will belong to $SPD(X\times Y,\rho,\sigma)$ as long as we can show that the functions in (2) belong to $SPD(X\times Y,\rho,\sigma)$. However, if $f_1$ is nonconstant, it is promptly seen that the strict positive definiteness of $F$ on $X\times Y$ will follow as long as we can show that the functions $(t,u)\in D_X^\rho\times D_Y^\sigma\mapsto e^{-g(t)s-h(u)s'}$, $s,s'>0$, belong to $SPD(X\times Y,\rho,\sigma)$. In other words, we need to prove that for $s,s'>0$ and distinct points $(x_1,y_1),\ldots,(x_n,y_n)$ in $X\times Y$, the matrix

$$\left[e^{-g(\rho(x_j,x_k))s-h(\sigma(y_j,y_k))s'}\right]_{j,k=1}^n$$

is positive definite. If $n=1$, there is nothing to be proved. If $n \geqslant 2$, according to Lemma 1, the positive definiteness holds if and only if

$$g(0)s+h(0)s' < g(\rho(x_j,x_k))s+h(\sigma(y_j,y_k))s', \quad j \neq k.$$

But, for $j \neq k$, the fact that the points used above are distinct yields that either $x_j \neq x_k$ or $y_j \neq y_k$. Since the spaces are metric, we have that either $\rho(x_j,x_k)>0$ or $\sigma(y_j,y_k)>0$. This fact along with the assumptions in $(ii)$ imply that either $g(\rho(x_j,x_k))>g(0)$ or $h(\sigma(y_j,y_k))>h(0)$. This closes the proof. □

## 3. Main results

Let us begin with the formal definition of a *generalized Stieltjes function of order* $\lambda>0$. It is a function $f:(0,\infty)\to[0,\infty)$ that has a representation in the form

$$f(x)=C_f+\frac{D_f}{x^\lambda}+\int_{(0,\infty)}\frac{1}{(x+s)^\lambda}d\mu_f(s), \quad x>0, \tag{3}$$

where $C_f=\lim_{x\to\infty}f(x)$, $D_f \geqslant 0$, and $\mu_f$ is a positive measure on $(0,\infty)$ such that

$$\int_{(0,\infty)}\frac{1}{(1+s)^\lambda}d\mu_f(s)<\infty.$$

They were exploited in [8, 9, 16] and other references quoted in there where many examples can be found. We will write $\mathscr{S}_\lambda$ to indicate the set of all generalized Stieltjes functions of order $\lambda$. The elementary identity

$$\frac{\Gamma(\lambda)}{(s+t)^\lambda}=\int_0^\infty e^{-sv}e^{-tv}v^{\lambda-1}\mathrm{d}v, \quad s,t>0, \tag{4}$$

along with the Bernstein-Widder Theorem show that every function in $\mathscr{S}_\lambda$ is completely monotone.

The main results in this section are given in the next three theorems.

THEOREM 2. *Let $(X,\rho)$ and $(Y,\sigma)$ be quasi-metric spaces. Assume $f$ belongs to $\mathscr{S}_\lambda$, $g$ is a positive valued function in $CND(X,\rho)$ and $h$ is a positive valued function in $CND(Y,\sigma)$. For $r \geqslant \lambda$, set $G_r$ as in (1). The following assertions hold:*

- $(i)$ *$G_r$ belongs to $PD(X\times Y,\rho,\sigma)$.*
- $(ii)$ *If $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, then $g(t)>g(0)$, for $t\in D_X^\rho\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^\sigma\setminus\{0\}$.*

*Further, in the case in which $(X,\rho)$ is nontrivial, the following additional assertion holds:*

- $(iii)$ *If $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, then either $D_f>0$ or $\mu_f$ is not the zero measure.*

*Proof.* Direct calculation reveals that

$$G_r(t,u) = \frac{C_f}{h(u)^r} + \frac{D_f}{g(t)^\lambda h(u)^{r-\lambda}} + \frac{1}{h(u)^{r-\lambda}} \int_{(0,\infty)} \frac{1}{[g(t)+sh(u)]^\lambda} d\mu_f(s). \qquad (5)$$

The functions $t \in (0,\infty) \mapsto t^{-\alpha}$, $\alpha = \lambda, r, r-\lambda$, are completely monotone. Thus, Lemma 2-$(i)$ and Proposition 1-$(i)$ yields that $(t,u) \in D_X^\rho \times D_Y^\sigma \mapsto D_f g(t)^{-\lambda} h(u)^{-r+\lambda}$ belongs to $PD(X \times Y, \rho, \sigma)$. On the other hand, it is obvious that the same is true of $(t,u) \in D_X^\rho \times D_Y^\sigma \mapsto h(u)^{-\alpha}$, $\alpha = r, r-\lambda$. A similar reasoning reveals that $(t,u) \in D_X^\rho \times D_Y^\sigma \mapsto \exp(-vg(t) - vh(u))$ belongs to $PD(X \times Y, \rho, \sigma)$ for $v > 0$. Since integration with respect to an independent parameter does not affect positive definiteness, it follows from (4) that all the functions

$$(t,u) \in D_X^\rho \times D_Y^\sigma \mapsto \frac{1}{[g(t)+sh(u)]^\lambda}, \quad s > 0,$$

belong to $PD(X \times Y, \rho, \sigma)$. Furthermore, the cone $PD(X \times Y, \rho, \sigma)$ being closed under products, we have that

$$(t,u) \in D_X^\rho \times D_Y^\sigma \mapsto \frac{1}{h(u)^{r-\lambda}} \int_{(0,\infty)} \frac{1}{[g(t)+sh(u)]^\lambda} d\mu_f(s)$$

belongs to $PD(X \times Y, \rho, \sigma)$ as well. Assertion $(i)$ follows from the fact that $PD(X \times Y, \rho, \sigma)$ is a convex cone.

If $g(t) = g(0)$, for some $t \in D_X^\rho \setminus \{0\}$, by picking two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ in $X \times Y$ such that $\rho(x_1, x_2) = t$ and $y_1 = y_2$, we obtain the singular matrix

$$[G_r(\rho(x_j,x_k), \sigma(y_j,y_k))]_{j,k=1}^2 = \left[\frac{1}{h(0)^r} f\left(\frac{g(0)}{h(0)}\right)\right]_{j,k=1}^2$$

If $h(u) = h(0)$, for some $u \in D_Y^\sigma \setminus \{0\}$, we can take two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ in $X \times Y$ such that $x_1 = x_2$ and $\sigma(y_1, y_2) = u$ to obtain the very same matrix. Thus, $(ii)$ holds.

Next, assume $(X, \rho)$ is nontrivial. If $D_f = 0$ and $\mu_f$ is the zero measure, then we can take two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ in $X \times Y$ such that $y_1 = y_2$ and reach the singular matrix

$$[G_r(\rho(x_j,x_k), \sigma(y_j,y_k))]_{j,k=1}^2 = \left[\frac{C_f}{h(0)^r}\right]_{j,k=1}^2.$$

Thus, $G_r$ cannot belong to $SPD(X \times Y, \rho, \sigma)$ and $(iii)$ is proved. □

THEOREM 3. *Let $(X, \rho)$ and $(Y, \sigma)$ be metric spaces. Assume $f$ belongs to $\mathscr{S}_\lambda$, $g$ is a positive valued function in $CND(X, \rho)$ and $h$ is a positive valued function in $CND(Y, \sigma)$. If $D_f > 0$, and $r > \lambda$, then the following assertions for $G_r$ as in (1) are equivalent:*

(i) *$G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$.*

(ii) *$g(t)>g(0)$, for $t\in D_X^\rho\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^\sigma\setminus\{0\}$.*

*Proof.* One implication follows from Theorem 2-$(ii)$. As for the converse, assume $D_f>0$, $r>\lambda$, and also the two assumptions on $g$ and $h$ quoted in $(ii)$. Theorem 1 reveals that

$$(t,u)\in D_X^\rho\times D_Y^\sigma\mapsto\frac{D_f}{g(t)^\lambda h(u)^{r-\lambda}}$$

belongs to $SPD(X\times Y,\rho,\sigma)$. Thus, since the other two summands in (5) define functions in $PD(X\times Y,\rho,\sigma)$, we may infer that $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$. □

THEOREM 4. *Let $(X,\rho)$ and $(Y,\sigma)$ be metric spaces. Assume $f$ belongs to $\mathscr{S}_\lambda$, $g$ is a positive valued function in $CND(X,\rho)$ and $h$ is a positive valued function in $CND(Y,\sigma)$. If $(X,\rho)$ is nontrivial, $D_f=0$, and $r\geqslant\lambda$, then the following assertions for $G_r$ as in (1) are equivalent:*

(i) *$G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$.*

(ii) *$f$ is nonconstant, $g(t)>g(0)$, for $t\in D_X^\rho\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^\sigma\setminus\{0\}$.*

*Proof.* If $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, $(X,\rho)$ is nontrivial, and $D_f=0$, Theorem 2-$(iii)$ yields that $\mu_f$ is nonzero. In particular, $f$ is nonconstant. This along with Theorem 2-$(ii)$ shows that $(ii)$ holds. Conversely, if $f$ is nonconstant and $D_f=0$, we know already via Theorem 2-$(iii)$ that $\mu_f$ is not the zero measure. Hence, in order to show $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, it suffices to verify that

$$(t,u)\in D_X^\rho\times D_Y^\sigma\mapsto\int_{(0,\infty)}\frac{1}{[g(t)+sh(u)]^\lambda}d\mu_f(s)$$

belongs to $SPD(X\times Y,\rho,\sigma)$. Indeed, the strict positive definiteness of $G_r$ will follow from Oppenheim's inequality ([5, p.509]) and the elementary properties of the classes $PD(X\times Y,\rho,\sigma)$ and $SPD(X\times Y,\rho,\sigma)$. However, to show that, it suffices to prove that

$$(t,u)\in D_X^\rho\times D_Y^\sigma\mapsto\frac{1}{[g(t)+sh(u)]^\lambda}$$

belongs to $SPD(X\times Y,\rho,\sigma)$, for $s>0$. But, from equality (4), it is promptly seen that it is then sufficient to show that the functions

$$(t,u)\in D_X^\rho\times D_Y^\sigma\mapsto e^{-g(t)v-h(u)sv},\quad v,s>0,$$

belong to $SPD(X\times Y,\rho,\sigma)$. Repeating the arguments developed in the second half of the proof of Theorem 1 closes the proof. □

An obvious simplification of Theorem 4 is as follows.

THEOREM 5. *Let $(X,\rho)$ and $(Y,\sigma)$ be metric spaces. Assume $f$ belongs to $\mathscr{S}_\lambda$, $g$ is a positive valued function in $CND(X,\rho)$ and $h$ is a positive valued function in $CND(Y,\sigma)$. If $(X,\rho)$ is nontrivial, $f$ is nonconstant, $D_f = 0$, and $r \geqslant \lambda$, then the following assertions for $G_r$ as in (1) are equivalent:*

(*i*) *$G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$.*

(*ii*) *$g(t) > g(0)$, for $t \in D_X^\rho \setminus \{0\}$, and $h(u) > h(0)$, for $u \in D_Y^\sigma \setminus \{0\}$.*

Under the setting of Theorem 4 the assumptions on $g$ may be relaxed depending upon the additional assumptions $f$ carries. Indeed, if $D_f = 0$ and the integral $\int_{(0,\infty)} s^{-\lambda} d\mu_f(s)$ is finite, is easy to see that $f$ is bounded. In particular, it has a continuous extension to $[0,\infty)$. Thus, the function $g$ does not need to be positive valued, that is, it may assume the value 0. This particular situation belongs to the setting adopted in [10].

REMARK 1. Taking into account the proof of Theorem 5, it is easily seen that the equivalence in Theorem 3 still holds in the case in which $D_f > 0$, $r = \lambda$, and $\mu_f$ is nonzero. So, the only missing case in Theorems 2 and 4 is that where $D_f > 0$, $r = \lambda$ and $\mu_f$ is the zero measure. However, a procedure analogous to that employed in the proof of Theorem 2-$(iii)$, shows that if $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, $D_f > 0$ and $r = \lambda$, then either $C_f > 0$ or $\mu_f$ is not the zero measure. In other words, the only case really missing in Theorems 2 and 4 is that where $C_f D_f > 0$, $r = \lambda$ and $\mu_f$ is the zero measure, that is, the case in which

$$G_r(t,u) = \frac{C_f}{h(u)^r} + \frac{D_f}{g(t)^r}, \quad (t,u) \in D_X^\rho \times D_Y^\sigma,$$

and $C_f D_f > 0$. So far, the strict positive definiteness of this model remains elusive.

We now look at some simple examples in order to illustrate the results.

EXAMPLE 1. If $f$ belongs to $\mathscr{S}_\alpha$, then Theorem 1 in [7] reveals that the same is true of the function $F$ given by

$$F(x) = x^{-\lambda} f\left(\frac{1}{x}\right), \quad x > 0.$$

So, Theorems 2, 3 and 4 hold for the model

$$G_r(t,u) = \frac{1}{g(t)^\lambda h(u)^{r-\lambda}} f\left(\frac{h(u)}{g(t)}\right), \quad (t,u) \in D_X^\rho \times D_Y^\sigma;\ r \geqslant \lambda,$$

as well. However, in the statement of Theorems 4 and 5, we need to replace "$f$ is nonconstant" with "$f$ is not of the form $Cx^{-\lambda}$ for some $C \geqslant 0$".

EXAMPLE 2. If $f$ belongs to $\mathscr{S}_\lambda$ and $\alpha > 1$, Theorem 15 in [7] asserts that $x \in (0,\infty) \mapsto f(x^{1/\alpha})$ belongs to $\mathscr{S}_1$. Thus, Theorems 2, 3 and 4 hold for the model

$$G_r(t,u) = \frac{1}{h(u)^r} f\left(\frac{g(t)^{1/\alpha}}{h(u)^{1/\alpha}}\right), \quad (t,u) \in D_X^\rho \times D_Y^\sigma;\ r \geqslant 1.$$

By Example 1, they also hold for the model

$$G_r(t,u) = \frac{1}{g(t)h(u)^{r-1}} f\left(\frac{h(u)^{1/\alpha}}{g(t)^{1/\alpha}}\right), \quad (t,u) \in D_X^\rho \times D_Y^\sigma;\ r \geqslant 1.$$

However, in the statement of Theorems 4 and Theorem 5, we need to replace "$f$ is nonconstant" with "$f$ is not of the form $Cx^{-1/\alpha}$ for some $C \geqslant 0$".

## 4. Models based on generalized complete Bernstein functions

In this section we will establish versions of the previous theorems for the class $\mathscr{CB}_\lambda$ of all functions $f : (0,\infty) \to \mathbb{R}$ that can be represented in the form

$$f(x) = a_f + b_f x^\lambda + \int_{(0,\infty)} \left(1 - \frac{s}{x+s}\right)^\lambda d\mu_f(s), \quad x > 0,$$

where $a_f, b_f \geqslant 0$ and $\mu_f$ is a positive measure on $(0,\infty)$ for which

$$\int_{(0,\infty)} \frac{1}{(1+s)^\lambda} d\mu_f(s) < \infty.$$

The functions in $\mathscr{CB}_\lambda$ will be called *generalized complete Bernstein functions of order* $\lambda > 0$ once $\mathscr{CB}_1$ coincides with the class of complete Bernstein functions studied in [15]. But one should notice that for $\lambda > 1$, the functions in $\mathscr{CB}_\lambda$ are not necessarily Bernstein functions so the name may be not appropriated for some readers. Clearly, a function $f$ belongs to $\mathscr{CB}_\lambda$ if and only if $x \in (0,\infty) \mapsto x^{-\lambda} f(x)$ belongs to $\mathscr{S}_\lambda$. In particular, the following modified results are direct consequences of Theorems 2, 3 and 4.

THEOREM 6. *Let $(X,\rho)$ and $(Y,\sigma)$ be quasi-metric spaces. Assume $f$ belongs to $\mathscr{CB}_\lambda$, $g$ is a positive valued function in $CND(X,\rho)$ and $h$ is a positive valued function in $CND(Y,\sigma)$. If $r \geqslant \lambda$, then the following assertions hold for the function $G_r$ given by*

$$G_r(t,u) = \frac{1}{g(t)^\lambda h(u)^{r-\lambda}} f\left(\frac{g(t)}{h(u)}\right), \quad (t,u) \in D_X^\rho \times D_Y^\sigma.$$

(*i*) *$G_r$ belongs to $PD(X \times Y, \rho, \sigma)$.*

(*ii*) *If $G_r$ belongs to $SPD(X \times Y, \rho, \sigma)$, then $g(t) > g(0)$, for $t \in D_X^\rho \setminus \{0\}$, and $h(u) > h(0)$, for $u \in D_Y^\sigma \setminus \{0\}$.*

*If $(X,\rho)$ is nontrivial, the following additional assertion holds:*

*$(iii)$ If $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, then either $a_f>0$ or $\mu_f$ is not the zero measure.*

THEOREM 7. *The following assertions hold under the setting of Theorem 6, if $(X,\rho)$ and $(Y,\sigma)$ are metric spaces:*

*$(i)$ If $a_f>0$, and $r>\lambda$, then $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$ if and only if $g(t)>g(0)$, for $t\in D_X^{\rho}\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^{\sigma}\setminus\{0\}$.*

*$(ii)$ If $(X,\rho)$ is nontrivial, $a_f=0$, and $r\geqslant\lambda$, then $G_r$ belongs to $SPD(X\times Y,\rho,\sigma)$ if and only if $f$ is not of the form $f(x)=Cx^{\lambda}$, for some $C>0$, $g(t)>g(0)$, for $t\in D_X^{\rho}\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^{\sigma}\setminus\{0\}$.*

A version of Theorem 2 employing generating functions from $\mathscr{C}\mathscr{B}_\lambda$ can be obtained by direct calculation.

THEOREM 8. *Let $(X,\rho)$ and $(Y,\sigma)$ be quasi-metric spaces. Assume $f$ belongs to $\mathscr{C}\mathscr{B}_\lambda$, $g$ is a positive valued function in $CND(X,\rho)$ and $h$ is a positive valued function in $CND(Y,\sigma)$. If $r\geqslant\lambda$, then the following assertions hold for the function $H_r$ given by*

$$H_r(t,u)=\frac{1}{g(t)^r}f\left(\frac{g(t)}{h(u)}\right),\quad (t,u)\in D_X^{\rho}\times D_Y^{\sigma}.$$

*$(i)$ $H_r$ belongs to $PD(X\times Y,\rho,\sigma)$.*

*$(ii)$ If $H_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, then $g(t)>g(0)$, for $t\in D_X^{\rho}\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^{\sigma}\setminus\{0\}$.*

*If $(Y,\sigma)$ is nontrivial, the following additional assertion holds:*

*$(iii)$ In the case in which $(Y,\rho)$ is nontrivial, if $H_r$ belongs to $SPD(X\times Y,\rho,\sigma)$, then either $b_f>0$ or $\mu_f$ is not the zero measure.*

*Proof.* It suffices to observe that

$$H_r(t,u)=\frac{a_f}{g(t)^r}+\frac{b_f}{g(t)^{r-\lambda}h(u)^{\lambda}}+\frac{1}{g(t)^{r-\lambda}}\int_{(0,\infty)}\frac{1}{[g(t)+sh(u)]^{\lambda}}d\mu_f(s),$$

and to reproduce the arguments in the proof of Theorem 2 under the current setting. □

As for Theorems 3 and 4, the following counterpart holds.

THEOREM 9. *If $(X,\rho)$ and $(Y,\sigma)$ are metric spaces, the following two additional assertions also hold under the setting of Theorem 8:*

*$(i)$ If $b_f>0$ and $r>\lambda$, then $H_r$ belongs to $SPD(X\times Y,\rho,\sigma)$ if and only if $g(t)>g(0)$, for $t\in D_X^{\rho}\setminus\{0\}$, and $h(u)>h(0)$, for $u\in D_Y^{\sigma}\setminus\{0\}$.*

(ii) *If* $(Y,\sigma)$ *is nontrivial,* $b_f = 0$ *and* $r \geqslant \lambda$, *then* $H_r$ *belongs to* $SPD(X\times Y,\rho,\sigma)$ *if and only if* $f$ *is nonconstant,* $g(t) > g(0)$, *for* $t \in D_X^\rho \setminus \{0\}$, *and* $h(u) > h(0)$, *for* $u \in D_Y^\sigma \setminus \{0\}$.

Theorem 9-$(i)$ covers the case $b_f > 0$ and $r = \lambda$ as long as $\mu_f$ is not the zero measure. As far as we know, Theorem 8 is the first result of Gneiting type that employs a main generating function coming from the class of Bernstein functions. And contrary to some expectations, the resulting function remained positive definite on $X\times Y$.

## 5. Additional constructions

Here, we will derive some consequences of the previous results based on stability properties of the class $\mathscr{CB}_1$ of complete Bernstein functions. We observe that in many circumstances these functions are preferred in the complex variable setting. In that case, they are labeled under different names such as operator monotone functions, Löwner functions, Pick functions, Nevanlinna functions, etc. A list of concrete examples of functions in $\mathscr{CB}_1$ can be found in [15, Chapter 16] while additional examples in both $\mathscr{S}_1$ and $\mathscr{CB}_1$ can be obtained via the mixing properties established in Corollary 7.9 in [15].

Theorem 7.3 in [15] states that within the set of functions that do not vanish in $(0,\infty)$, a function $f$ belongs to $\mathscr{CB}_1$ if and only if $1/f$ belongs to $\mathscr{S}_1$. In view of this, it is promptly seen that Theorems 2, 3, and 4 hold for the model

$$G_r(t,u) = \frac{1}{h(u)^r}\left[f\left(\frac{g(t)}{h(u)}\right)\right]^{-1}, \quad (t,u)\in D_X^\rho \times D_Y^\sigma;\ r\geqslant 1,$$

whenever $f \in \mathscr{CB}_1$ and $f$ does not vanish in $(0,\infty)$. Likewise, Theorems 8 and 9 hold for the model

$$H_r(t,u) = \frac{1}{g(t)^r}\left[f\left(\frac{g(t)}{h(u)}\right)\right]^{-1}, \quad (t,u)\in D_X^\rho \times D_Y^\sigma;\ r\geqslant 1,$$

whenever $f \in \mathscr{S}_1$ and $f$ does not vanish in $(0,\infty)$.

Once the function $x \in (0,\infty) \mapsto x^{-1}$ belongs to $\mathscr{S}_1$, Corollary 7.9 in [15] shows that a function $f$ belongs to $\mathscr{S}_1$ (respect. $\mathscr{CB}_1$) if and only if $x\in(0,\infty)\mapsto 1/f(x^{-1})$ belongs to $\mathscr{S}_1$ (respect. $\mathscr{CB}_1$). This fact poses no difficulty at all in seeing that Theorems 2, 3, and 4 hold for the model

$$G_r(t,u) = \frac{1}{h(u)^r}\left[f\left(\frac{h(u)}{g(t)}\right)\right]^{-1}, \quad (t,u)\in D_X^\rho \times D_Y^\sigma;\ r\geqslant 1,$$

whenever $f \in \mathscr{S}_1$ and $f$ does not vanish in $(0,\infty)$. On the other hand, Theorems 8 and 9 hold for the model

$$H_r(t,u) = \frac{1}{g(t)^r}\left[f\left(\frac{h(u)}{g(t)}\right)\right]^{-1}, \quad (t,u)\in D_X^\rho \times D_Y^\sigma;\ r\geqslant 1,$$

whenever $f \in \mathscr{CB}_1$ and $f$ does not vanish in $(0,\infty)$.

Taking into account that $f$ belongs to $\mathscr{CB}_1$ if and only if $x \in (0,\infty) \mapsto xf(x^{-1})$ does (see Proposition 7.1 in [15]), one can see that Theorem 8 holds for the model

$$H_r(t,u) = \frac{1}{g(t)^{r-1}h(u)} f\left(\frac{h(u)}{g(t)}\right), \quad (t,u) \in D_X^\rho \times D_Y^\sigma;\ r \geqslant 1,$$

whenever $f \in \mathscr{CB}_1$. Theorem 9 can be implemented only if $f$ is not of the form $f(x) = cx$ for some $c > 0$.

Similarly, since a function $f$ belongs to $f \in \mathscr{S}_1$ if and only if $x \in (0,\infty) \mapsto x^{-1}[f(x)]^{-1}$ does, Theorems 2, 3, and 4 hold for the model

$$G_r(t,u) = \frac{1}{g(t)h(u)^{r-1}} \left[f\left(\frac{g(t)}{h(u)}\right)\right]^{-1}, \quad (t,u) \in D_X^\rho \times D_Y^\sigma;\ r \geqslant 1,$$

whenever $f \in \mathscr{S}_1$ and $f$ does not vanish in $(0,\infty)$. Theorem 4 can be implemented in this case only if $f$ is not of the form $f(x) = c/x$ for some $c > 0$.

If $c > 0$ and $f$ belongs to $\mathscr{S}_1$, then $f/(c+f)$ belongs to $\mathscr{S}_1$ as shown in [15, p. 96]. Thus, Theorems 2, 3, and 4 hold for the model

$$G_r(t,u) = \frac{1}{h(u)^r} \frac{f(g(t)/h(u))}{c + f(g(t)/h(u))}, \quad (t,u) \in D_X^\rho \times D_Y^\sigma;\ r \geqslant 1,$$

whenever $f \in \mathscr{CB}_1$.

We close the paper by providing an example involving the class $\mathscr{TB}$ of Thorin-Bernstein functions. A function $f : (0,\infty) \to (0,\infty)$ is called a *Thorin-Bernstein function* if it has a representation in the form

$$f(x) = a + bx + \int_{(0,\infty)} \ln\left(1 + \frac{x}{s}\right) d\mu(s),$$

where $a, b \geqslant 0$ and $\mu$ is positive measure on $(0,\infty)$ satisfying

$$-\int_{(0,1)} \ln s \, d\mu(s) + \int_{[1,\infty)} \frac{1}{s} d\mu(s) < \infty.$$

A few examples of Thorin Bernstein functions can be extracted from Section 5 in [3]. It is known that if $f$ belongs to $\mathscr{TB}$, then $f$ belongs to $\mathscr{CB}_1$ while its derivative $f'$ belongs to $\mathscr{S}_1$. In particular, if $f$ belongs to $\mathscr{TB}$, then $f$ can be used to define a valid model $G_r$ in Theorems 2, 3, and 4 while $f'$ can be used to define a valid model in Theorems 8 and 9, all of them holding for $r \geqslant 1$.

## 6. Conclusion

In this paper we have provided large classes of functions that can be used to solve adapted versions to quasi-metric spaces of the so-called Gneiting's method to construct space-time covariance functions. In the case the spaces are metric, we also provided necessary and sufficient conditions in order that the positive definite functions produced by the methods be strictly positive definite, a desirable property in specific applications. One of the classes of functions employed includes a special subclass of the class of bounded completely monotone functions originally employed by Gneiting. Some of the classes used in this paper are easy-to-find in the sense that they are very commonly found in the literature. Although the results in the paper are mainly theoretical, the search for applications of these results is one of our scopes for the future.

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(Received October 1, 2020)

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