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A NOTE ON WEIGHTED ESTIMATES FOR BILINEAR FRACTIONAL INTEGRAL OPERATORS

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Abstract. De Napoli, Drelichman and Durán (2011) proved weighted estimates for the fractional integral operators. Komori-Furuya and Sato (2020) proved weighted estimates for bilinear fractional integral operators. We show that their results are optimal by giving counterexamples.

1. Introduction

Consider the fractional integral operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \qquad 0<\alpha<n.$$

Stein and Weiss [6] proved weighted estimates for this operator.

THEOREM A. ([6]) *If* $0<\alpha<n$, $1<p\leqslant q<\infty$, $A<n/p'$, $B<n/q$,

$$A+B\geqslant 0 \quad \textit{and} \quad \frac{1}{q}=\frac{1}{p}-\frac{\alpha-A-B}{n},$$

then

$$\big\||x|^{-B} I_\alpha f\big\|_{L^q} \leqslant C\big\||x|^{A} f\big\|_{L^p}.$$

When $A=B=0$ this is the Hardy-Littlewood-Sobolev inequality and the condition $A+B\geqslant 0$ is optimal. However, restricted to radial functions, De Napoli, Drelichman and Durán [2] improved this inequality as follows; see also [5].

THEOREM B. ([2]) *If* $0<\alpha<n$, $1<p\leqslant q<\infty$, $A<n/p'$, $B<n/q$,

$$A+B\geqslant (n-1)\left(\frac{1}{q}-\frac{1}{p}\right) \quad \textit{and} \quad \frac{1}{q}=\frac{1}{p}-\frac{\alpha-A-B}{n},$$

then

$$\big\||x|^{-B} I_\alpha f\big\|_{L^q} \leqslant C\big\||x|^{A} f\big\|_{L^p} \qquad \textit{for all radial functions } f.$$

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Y. Komori-Furuya and E. Sato [4] considered bilinear fractional integrals.

DEFINITION 1.

$$I_\alpha(f,g)(x) := \iint_{\mathbb{R}^{2n}} \frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}}\,dydz, \qquad 0<\alpha<2n.$$

THEOREM C . ([4]) *Assume that* $0<\alpha<2n$, $1<p_1,p_2<\infty$, $0<1/q<1/p_1+1/p_2$, $A_1<n/p_1'$, $A_2<n/p_2'$, $B<n/q$,

$$A_1+A_2+B \geqslant (n-1)\left(\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}\right) \quad \textit{and} \quad \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha-A_1-A_2-B}{n}.$$

Then for all radial functions f *and* g,

$$\big\||x|^{-B}I_\alpha(f,g)\big\|_{L^q} \leqslant C\big\||x|^{A_1}f\big\|_{L^{p_1}}\big\||x|^{A_2}g\big\|_{L^{p_2}}.$$

By using this theorem, authors got improvements of the bilinear Caffarelli-Kohn-Nirenberg's inequality.

For Theorem B, De Napoli et al. [2] showed that the condition

$$A+B \geqslant (n-1)\left(\frac{1}{q}-\frac{1}{p}\right)$$

is optimal by giving a counterexample for $n=3$; see also [1]. We show the following. For Theorem C, the condition

$$A_1+A_2+B \geqslant (n-1)\left(\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}\right)$$

is optimal for all $n \geqslant 2$.

Our proof is elementary and by the same argument we can give counterexamples for Theorem B where $n \geqslant 2$.

The following notation is used: $X \lesssim Y$ and $X \gtrsim Y$ denote the statement that $X \leqslant CY$ and $X \geqslant CY$ respectively, where C is a positive constant which is independent of essential parameters. $X \approx Y$ denotes the statement that $X \lesssim Y$ and $X \gtrsim Y$. The characteristic function of a set E is denoted by χ_E.

2. A counterexample

Let $n \geqslant 2$. If

$$1/q = 1/p_1+1/p_2-\alpha/n+(A_1+A_2+B)/n \tag{1}$$

and

$$A_1+A_2+B < (n-1)\left(\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}\right), \tag{2}$$

then substituting (2) for (1), we have

$$\frac{1}{q} < \frac{1}{p_1} + \frac{1}{p_2} - \alpha.$$

We take $0 < \varepsilon < 1$ such that

$$\frac{1}{q} < \frac{1}{p_1} + \frac{1}{p_2} - \alpha - \varepsilon\left(\frac{1}{p_1} + \frac{1}{p_2}\right).$$

Let

$$f_0(t) := \frac{1}{|1-t|^{(1-\varepsilon)/p_1}}\chi_{(1/2,2)}(t) \quad \text{and} \quad g_0(t) := \frac{1}{|1-t|^{(1-\varepsilon)/p_2}}\chi_{(1/2,2)}(t),$$

and define

$$f(x) := f_0(|x|) \quad \text{and} \quad g(x) := g_0(|x|).$$

Then $|x|^{A_1} f(x) \in L^{p_1}(\mathbb{R}^n)$ and $|x|^{A_2} g(x) \in L^{p_2}(\mathbb{R}^n)$. However we shall show the following:

COUNTEREXAMPLE.

$$|x|^{-B} I_\alpha(f,g)(x) \notin L^q(1 < |x| < 2).$$

Note that this is equivalent to

$$I_\alpha(f,g)(x) \notin L^q(1 < |x| < 2). \tag{3}$$

For the proof we need the following lemmas.

LEMMA 1. ([3] p. 442 or [2] Lemma 4.1) *Let $n \geqslant 2$ and $h \in L^1(\mathbb{R}^1)$. Then*

$$\int_{S^{n-1}} h(x\cdot\omega)\,d\sigma(\omega) = \frac{2\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \int_{-1}^{1} h(|x|t)(1-t^2)^{(n-3)/2}\,dt,$$

where $d\sigma$ is the Lebesgue measure on S^{n-1}.

LEMMA 2. *Let $\frac{1}{2} < r < 1$, $e_0 = (1,0,\dots,0)$ and*

$$S_r := \{\omega = (\omega_1,\dots,\omega_n) \in S^{n-1}; \omega_1 > r(2-r)\}.$$

$$\text{If} \quad \omega \in S_r, \quad \text{then} \quad |e_0 - r\omega| \leqslant \sqrt{3}(1-r). \tag{4}$$

$$\sigma(S_r) \approx (1-r)^{n-1}. \tag{5}$$

Proof of (4).

$$|e_0 - r\omega|^2 = (1 - r\omega_1)^2 + \sum_{k=2}^{n} (r\omega_k)^2 = r^2 - 2r\omega_1 + 1 \leqslant r^2 - 2r^2(2-r) + 1$$
$$= (2r+1)(1-r)^2 \leqslant 3(1-r)^2.$$

Proof of (5). Let $h(t) = \chi_{(r(2-r),1]}(t)$. Then $h(e_0 \cdot \omega) = \chi_{S_r}(\omega)$ where $\omega \in S^{n-1}$. By Lemma 1,

$$\sigma(S_r) \approx \int_{r(2-r)}^{1} (1-t^2)^{(n-3)/2}\, dt \approx \int_{r(2-r)}^{1} (1-t)^{(n-3)/2}\, dt \approx (1-r)^{n-1}.$$

LEMMA 3. *Let* $\frac{1}{2} < r, s < 1$ *and* $e_0 = (1, 0, \ldots, 0)$. *Then*

$$J(r,s,e_0) := \int_{S^{n-1}} \int_{S^{n-1}} \frac{d\sigma(\omega) d\sigma(\omega')}{(|e_0 - r\omega| + |e_0 - s\omega'|)^{2n-\alpha}} \gtrsim \frac{(1-r)^{n-1}(1-s)^{n-1}}{\big((1-r)+(1-s)\big)^{2n-\alpha}}.$$

Proof. By Lemma 2,

$$J(r,s,e_0) \geqslant \int_{S_r} \int_{S_s} \frac{d\sigma(\omega) d\sigma(\omega')}{(|e_0 - r\omega| + |e_0 - s\omega'|)^{2n-\alpha}}$$
$$\gtrsim \frac{\sigma(S_r) \cdot \sigma(S_s)}{\big((1-r)+(1-s)\big)^{2n-\alpha}} \approx \frac{(1-r)^{n-1}(1-s)^{n-1}}{\big((1-r)+(1-s)\big)^{2n-\alpha}}. \quad \square$$

LEMMA 4. *If* $1 < |x| < 2$ *and* $1/|x| < r, s < \frac{1+1/|x|}{2}$, *then*

$$J(r,s,e_0) \gtrsim \frac{1}{(|x|-1)^{2-\alpha}}.$$

Proof. If $1/|x| < r, s < (1 + 1/|x|)/2$, then $1/2 < r, s < 1$ and

$$\frac{1}{4}(|x|-1) < 1 - r < |x| - 1$$
$$\frac{1}{4}(|x|-1) < 1 - s < |x| - 1.$$

Therefore we have

$$\frac{(1-r)^{n-1}(1-s)^{n-1}}{\big((1-r)+(1-s)\big)^{2n-\alpha}} = \left(\frac{(1-r)(1-s)}{\big((1-r)+(1-s)\big)^2} \right)^{n-1} \frac{1}{\big((1-r)+(1-s)\big)^{2-\alpha}}$$
$$\approx \frac{1}{\big(|x|-1\big)^{2-\alpha}}. \quad \square$$

Now we we shall prove (3).

Proof of (3). Let $1<|x|<2$. By Lemma 4,

$$I_\alpha(f,g)(x)=\int_{1/2}^2\int_{1/2}^2\int_{S^{n-1}}\int_{S^{n-1}}\frac{f_0(r)r^{n-1}g_0(s)s^{n-1}}{(|x-r\omega|+|x-s\omega'|)^{2n-\alpha}}drdsd\sigma(\omega)d\sigma(\omega')$$

$$\approx\int_{1/2}^2\int_{1/2}^2\int_{S^{n-1}}\int_{S^{n-1}}\frac{f_0(r)g_0(s)}{\left(\left|x/|x|-r\omega/|x|\right|+\left|x/|x|-s\omega'/|x|\right|\right)^{2n-\alpha}}drdsd\sigma(\omega)d\sigma(\omega')$$

$$\approx\int_{1/2|x|}^{2/|x|}\int_{1/2|x|}^{2/|x|}\left(\int_{S^{n-1}}\int_{S^{n-1}}\frac{d\sigma(\omega)d\sigma(\omega')}{(|e_0-r\omega|+|e_0-s\omega'|)^{2n-\alpha}}\right)f_0(|x|r)g_0(|x|s)\,drds$$

$$\geqslant\int_{\frac{1}{|x|}}^{\frac{1+1/|x|}{2}}\int_{\frac{1}{|x|}}^{\frac{1+1/|x|}{2}}J(r,s,e_0)f_0(|x|r)g_0(|x|s)\,drds$$

$$\gtrsim\frac{1}{(|x|-1)^{2-\alpha}}\int_{\frac{1}{|x|}}^{\frac{1+1/|x|}{2}}\frac{1}{(|x|r-1)^{(1-\varepsilon)/p_1}}dr\int_{\frac{1}{|x|}}^{\frac{1+1/|x|}{2}}\frac{1}{(|x|s-1)^{(1-\varepsilon)/p_2}}ds.$$

Since $0<|x|r-1<(|x|-1)/2$, we have

$$\int_{\frac{1}{|x|}}^{\frac{1+1/|x|}{2}}\frac{1}{(|x|r-1)^{(1-\varepsilon)/p_1}}dr\gtrsim\frac{1}{(|x|-1)^{(1-\varepsilon)/p_1}}\left(\frac{1+1/|x|}{2}-\frac{1}{|x|}\right)$$
$$\approx\frac{1}{(|x|-1)^{(1-\varepsilon)/p_1}}(|x|-1),$$

and we obtain

$$I_\alpha(f,g)(x)\geqslant\frac{1}{(|x|-1)^{(1-\varepsilon)(1/p_1+1/p_2)-\alpha}}\geqslant\frac{1}{(|x|-1)^{1/q}}\notin L^q(1<|x|<2).$$

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