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REMARKS ON THE MONOTONICITY AND CONVEXITY OF JENSEN'S FUNCTION

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Abstract. Let $x_1,x_2,\ldots,x_n$ be nonnegative real numbers. The Jensen function of $\{x_i\}_{i=1}^n$ is defined as $J_s(x)=\left(\sum_{i=1}^n x_i^s\right)^{1/s}$, also known as the L_p-norm. It is well-known that $J_s(x)$ is decreasing on $s\in(0,+\infty)$. Moreover, Beckenbach [Amer. Math. Monthly, 53 (1946), 501–505] proved further that $J_s(x)$ is a convex function on $s\in(0,+\infty)$. The goal of this note is two-fold. We first revisit the skillful treatment of the proof of Beckenbach, and then we simplify the proof slightly. Additionally, we give a new proof of the convexity of $J_s(x)$ by using the Hölder inequality, our proof is more succinct and short. On the other hand, we investigate a Jensen-type inequality that arised from Fourier analysis by Stein and Weiss. As a byproduct, the Hardy-Littlewood-Póya inequality is also included.

1. Introduction

In the field of analysis, inequalities play an important role in many areas, such as the complex analysis and Fourier analysis, we refer to [6] and [13] for more details. A well-known inequality concerning the convex function usually attributes to Jensen, which is frequently used in functional analysis and discrete geometry. Let $x_1,x_2,\ldots,x_n$ be nonnegative real numbers. We associate a function to $\{x_i\}_{i=1}^n$, called the Jensen function, which defined as

$$J_s(x):=\left(\sum_{i=1}^n x_i^s\right)^{1/s}. \tag{1}$$

The Jensen function has been studied in various aspects, and it could be used to derive some significant inequalities; see [4] and [6, pp. 28–30] for more details. It was proved by Pringsheim [12] and independently by Jensen [7, p. 192] (also see, e.g., [3]) that:

THEOREM 1. *The function $J_s(x)$ is decreasing on $s\in(0,+\infty)$.*

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Theorem 1 has been rediscovered several times in the literature; see, e.g., [6, 3]. Although the proof in [3] is short and elegant, we here give an outline of another versatile proof for completeness. It is sufficient to show that $f(s) = (a^s + b^s)^{1/s}$ is decreasing on $s \in (0, +\infty)$. By a direct computation, we could get

$$f'(s) = f(s) \cdot \frac{s(a^s \log a + b^s \log b) - (a^s + b^s)\log(a^s + b^s)}{s^2(a^s + b^s)}.$$

We are going to prove $f'(s) \leqslant 0$, which is equivalent to show that

$$\alpha \log \alpha + \beta \log \beta \leqslant (\alpha + \beta)\log(\alpha + \beta), \tag{2}$$

where $\alpha = a^s$ and $\beta = b^s$ are positive numbers. Since $g(t) = \log(1/t)$ is a convex function on $t \in (0, +\infty)$, by the Jensen inequality of convex function, that is,

$$g(\delta t_1 + (1-\delta)t_2) \leqslant \delta g(t_1) + (1-\delta)g(t_2).$$

The desired inequality (2) immediately follows by setting $\delta = \alpha/(\alpha + \beta)$, $t_1 = 1/\alpha$ and $t_2 = 1/\beta$ in the above inequality.

Theorem 1 says equivalently that $J_s(x) \leqslant J_r(x)$ for $s \geqslant r > 0$. We remark that this inequality is also valid for $r \leqslant s < 0$ and $s < 0 < r$ by a self-improved proof; see [14] for more details. Additionally, the Jensen function and Jensen-type inequality have been extensively studied over the years; see, e.g., [1, 2, 5, 10, 11]. In particular, Beckenbach [3] further proved a more interesting result, which states that $J_s(x)$ is a convex function on the exponent $s > 0$. For convenience, we rewrite this result as the following theorem.

THEOREM 2. (see [3]) *The function* $J_s(x)$ *is convex on* $s \in (0, +\infty)$.

In this note, we are mainly concentrated on the convexity of $J_s(x)$. The note is organized as follows. In Section 2, we give a half-page revisting of Beckenbach's proof of Theorem 2 and then we present a slightly simplification. Moreover, we present a quite different method to prove the convexity of Jensen's function $J_s(x)$. Our treatment is only based on the Hölder inequality. In Section 3, we study a Jensen-type function for two sequences, which was first introduced by E. M. Stein and G. Weiss in Fourier Analysis. Finally, we conclude with the Hardy-Littlewood-Póya inequality as a corollary.

2. Revisiting and new proof

In this section, we will revisit Beckenbach's reduction in proving Theorem 2. Our presentation here is just slightly different from that in [3]. Beckenbach's proof could be roughly decomposed as the following four steps:

(i) $\log\big(J_s(x)\big)^s$ is a convex function on $s \in (-\infty, +\infty)$ for every $x_1, \dots, x_n \geqslant 0$.

(ii) If $f(s)$ and $g(s)$ are positive non-increasing convex functions for $b < s < c$, then the function $\phi(s) = f(s)g(s)$ is convex on the interval (b, c).

(iii) $\log J_s(a)$ is a convex function for $s \in (0,+\infty)$.

(iv) If $p(s)$ is positive and $\log p(s)$ is convex, then also $p(s)$ is convex.

The treatment of Beckenbach is technical and skillful. Next, we review the above steps briefly. First of all, we provide another way to prove the first step (i). We define $h(s) = \log\big(J_s(x)\big)^s$, a direct computation leads to

$$h''(s) = \frac{\left(\sum_{i=1}^n x_i^s (\log x_i)^2\right)\left(\sum_{i=1}^n x_i^s\right) - \left(\sum_{i=1}^n x_i^s \log x_i\right)^2}{\left(\sum_{i=1}^n x_i^t\right)^2}.$$

By Cauchy-Schwartz's inequality, we could get $h''(s) \geqslant 0$, which means that $h(s)$ is a convex function. Among the four steps above, the proof of (ii) is straightforward, it can be found in [3], as well as the proof of (iv) that is also easy to verify. We omit the details and leave it to interested readers.

Among the above steps in Beckenbach's proof, the key part is the third step: $\log J_s(x)$ is a convex function on $s \in (0,+\infty)$, which together with (iv) yields the required result. However, the proof of (iii) is tricky and mystified, it based on the previous steps (i) and (ii). For more details, we refer to [3].

In what follows, we will overcome the difficulty and provide a completely different method to prove the convexity of $J_s(x)$, we start from another point of view instead of proving (iii). More precisely, we give a new short proof of the convexity of $J_s(x)$ by using Hölder's inequality only. Our proof is quite different from that in [3]. To proceed, we next introduce the following lemma, known as the Hölder inequality [6, p. 17].

LEMMA 3. (Hölder's inequality) *Let* a_i, b_i $(i = 1,2,\ldots,n)$ *be nonnegative real numbers and* $r,s > 1$ *such that* $1/r + 1/s = 1$. *Then*

$$\sum_{i=1}^n a_i b_i \leqslant \left(\sum_{i=1}^n a_i^r\right)^{1/r} \left(\sum_{i=1}^n b_i^s\right)^{1/s}.$$

We are now ready for a completely new proof of Theorem 2. We shall prove a slightly more general result. First of all, we define a weighted version [14] of Jensen's function as

$$J_s(x,w) := \left(\sum_{i=1}^n w_i x_i^s\right)^{1/s},$$

where x_i and w_i are nonnegative such that $w_i \geqslant 1$ for $1 \leqslant i \leqslant n$.

Next, we show that $J_s(x,w)$ is convex on $s \in (0,+\infty)$.

New proof. Since $J_s(x)$ is continuous on $s > 0$, it suffices to show that

$$J_{\frac{p+q}{2}}(x,w) \leqslant \frac{1}{2} J_p(x,w) + \frac{1}{2} J_q(x,w)$$

for all real numbers $p,q>0$. Without loss of generality, we may assume by scaling that $\max_{1\leqslant i\leqslant n}\{x_i\}=1$. By AM-GM inequality, we get $J_p(x,w)+J_q(x,w)=$

$$\left(\sum_{i=1}^{n} w_i x_i^p\right)^{1/p}+\left(\sum_{i=1}^{n} w_i x_i^q\right)^{1/q}\geqslant 2\left(\sum_{i=1}^{n} w_i x_i^p\right)^{1/2p}\left(\sum_{i=1}^{n} w_i x_i^q\right)^{1/2q}.$$

It suffices to show that

$$\left(\sum_{i=1}^{n} w_i x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} w_i x_i^q\right)^{1/q}\geqslant\left(\sum_{i=1}^{n} w_i x_i^{(p+q)/2}\right)^{4/(p+q)}. \tag{3}$$

Since $x_i\leqslant 1, 2pq/(p+q)\leqslant (p+q)/2$ and $4/(p+q)\leqslant 1/p+1/q$, we have

$$x_i^{(p+q)/2}\leqslant x_i^{2pq/(p+q)},\quad 1\leqslant i\leqslant n.$$

Moreover, bearing in mind $\max_{1\leqslant i\leqslant n}\{x_i\}=1$ and $w_i\geqslant 1$, we can get

$$\left(\sum_{i=1}^{n} w_i x_i^{(p+q)/2}\right)^{4/(p+q)}\leqslant\left(\sum_{i=1}^{n} w_i x_i^{2pq/(p+q)}\right)^{4/(p+q)}\leqslant\left(\sum_{i=1}^{n} w_i x_i^{2pq/(p+q)}\right)^{1/p+1/q}. \tag{4}$$

Now we set

$$a_i=w_i^{q/(p+q)}x_i^{pq/(p+q)},\quad b_i=w_i^{p/(p+q)}x_i^{pq/(p+q)},\quad r=1+p/q,\quad s=1+q/p.$$

Then $1/r+1/s=1$. By the Hölder inequality, we obtain

$$\begin{aligned}\left(\sum_{i=1}^{n} w_i x_i^{2pq/(p+q)}\right)^{1/p+1/q}&=\left(\sum_{i=1}^{n} a_i b_i\right)^{1/p+1/q}\\&\leqslant\left(\left(\sum_{i=1}^{n} a_i^r\right)^{1/r}\left(\sum_{i=1}^{n} b_i^s\right)^{1/s}\right)^{1/p+1/q}\\&\leqslant\left(\sum_{i=1}^{n} w_i x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} w_i x_i^q\right)^{1/q}.\end{aligned} \tag{5}$$

Combining (4) and (5), the desired inequality (3) immediately follows. □

3. A Jensen-type function from Fourier analysis

In this section, we present a Jensen-type inequality, which was first introduced by E. M. Stein and G. Weiss to obtain the inclusion relation of the space $L(p,q_1)\subset L(p,q_2)$ for $q_1\leqslant q_2$. For conciseness, we here won't explain the background of Fourier analysis;

see [13] for more details. Let $a_1 > a_2 > \cdots > a_n$ and $b_n > b_{n-1} > \cdots > b_1$ be positive real numbers and denote $b_0 = 0$. We define

$$S_t(a,b) := \left(\sum_{i=1}^{n} a_i^t \left(b_i^t - b_{i-1}^t\right)\right)^{1/t},$$

where $t \in (0,+\infty)$. To some extend, the function $S_t(a,b)$ shares some similar properties to Jensen's function $J_s(x)$. For instance, it is implicitly shown in [13, pp. 193–194] that:

THEOREM 4. (see [13]) *The fuction $S_t(a,b)$ is decreasing on $t \in (0,+\infty)$.*

The original proof of Stein and Weiss is by induction on n, which is dexterous and proficient. In this section, we provide an alternative proof of Theorem 4. To state our proof clearly, let us start with the following lemma, which is essentially a direct consequence of majorization theory (see [15, p. 342]). We provide an elementary proof for completeness.

LEMMA 5. *Let a_1,a_2,b_1,b_2 be nonnegative numbers such that* $\max\{a_1,a_2\} \leqslant \max\{b_1,b_2\}$ *and $a_1+a_2 \leqslant b_1+b_2$. Then $a_1^p + a_2^p \leqslant b_1^p + b_2^p$ for any $p \geqslant 1$.*

Proof. We may assume that $a_1 = \max\{a_1,a_2\}$ and $b_1 = \max\{b_1,b_2\}$. Let

$$b_1 = a_1 + \delta, \quad b_1 + b_2 = a_1 + a_2 + \gamma,$$

for some $\delta \geqslant 0$ and $\gamma \geqslant 0$. If $a_2 < b_2$, the required result immediately follows. If $b_2 \leqslant a_2 \leqslant a_1 \leqslant b_1$, it suffices to show that

$$a_1^p + a_2^p \leqslant (a_1+a_2-b_2)^p + b_2^p \leqslant (a_1+a_2+\gamma-b_2)^p + b_2^p = b_1^p + b_2^p,$$

which follows by the convexity and the monotonicity of $h(x) := x^p$ for $p \geqslant 1$. $\square$

We now in the position to show the monotonicity of $S_t(a,b)$, and will prove $S_p(a,b) \leqslant S_q(a,b)$ for every $p \geqslant q > 0$. By a standard variable substitution $x_i = a_i^q$ and $y_i = b_i^q$, it suffices to show that $S_{p/q}(x,y) \leqslant S_1(x,y)$, which is is equivalent to prove $S_p(x,y) \leqslant S_1(x,y)$ for all $p \geqslant 1$, i.e.,

$$\left(\sum_{i=1}^{n} x_i(y_i - y_{i-1})\right)^p \geqslant \sum_{i=1}^{n} x_i^p (y_i^p - y_{i-1}^p). \tag{6}$$

For each $k = 2,3,\ldots,n$, we set

$$a_1 = \sum_{i=1}^{k} x_i(y_i - y_{i-1}), \quad a_2 = x_k y_{k-1}, \quad b_1 = \sum_{i=1}^{k-1} x_i(y_i - y_{i-1}), \quad b_2 = x_k y_k.$$

Clearly, we have $a_1 \geqslant b_1$ and $a_1 + a_2 = b_1 + b_2$. Lemma 5 yields

$$\left(\sum_{i=1}^{k} x_i(y_i - y_{i-1})\right)^p + (x_k y_{k-1})^p \geqslant \left(\sum_{i=1}^{k-1} x_i(y_i - y_{i-1})\right)^p + (x_k y_k)^p.$$

Theorefore, summing over all k yields

$$\sum_{k=2}^{n}\left(\Big(\sum_{i=1}^{k} x_i(y_i - y_{i-1})\Big)^p + (x_k y_{k-1})^p\right) \geqslant \sum_{k=2}^{n}\left(\Big(\sum_{i=1}^{k-1} x_i(y_i - y_{i-1})\Big)^p + (x_k y_k)^p\right).$$

Upon computation, we can see that

$$\sum_{k=2}^{n-1}\Big(\sum_{i=1}^{k} x_i(y_i - y_{i-1})\Big)^p = \sum_{k=3}^{n}\Big(\sum_{i=1}^{k-1} x_i(y_i - y_{i-1})\Big)^p.$$

Then, we have

$$\Big(\sum_{i=1}^{n} x_i(y_i - y_{i-1})\Big)^p + \sum_{k=2}^{n}(x_k y_{k-1})^p \geqslant \big(x_1(y_1 - y_0)\big)^p + \sum_{k=2}^{n}(x_k y_k)^p.$$

So the disired inequality (6) now follows.

Let $b_k = k$, (6) reduces to Hardy-Littlewood-Póya's inequality [6, p. 100]; for more recent studies, the reader is referred to [8, 9] and references therein.

COROLLARY 6. *Let $a_1 > a_2 > \cdots > a_n$ be positive numbers and $q > 1$. Then*

$$\sum_{i=1}^{n}\left(i^q - (i-1)^q\right)a_i^q \leqslant \left(\sum_{i=1}^{n} a_i\right)^q.$$

Comparing to Theorem 2, it is natrual to ask that whether $S_t(a,b)$ is a convex function on $t \in (0,+\infty)$, which is an interesting problem and may deserve further investigation. At the end of the paper, we write this question as the following conjecture.

CONJECTURE 7. *Let $a_1 > a_2 > \cdots > a_n$ and $b_n > b_{n-1} > \cdots > b_1$ be positive real numbers and denote $b_0 = 0$. For every $t \in (0,+\infty)$, we define*

$$S_t(a,b) := \left(\sum_{i=1}^{n} a_i^t\left(b_i^t - b_{i-1}^t\right)\right)^{1/t}.$$

Then $S_t(a,b)$ is convex on $t \in (0,+\infty)$.

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