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AN INEQUALITY FOR THE PERIMETER OF THE CENTROID BODY IN THE PLANE

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Abstract. Let K be a centrally symmetric convex body in the plane. In this paper we prove an inequality relating the perimeter of the centroid body of K to the perimeter of K, establishing a new Busemann-Petty type inequality.

1. Introduction

Let K be a convex body in $\mathbb{R}^2$, i.e., a compact and convex set with non-empty interior in the plane. We say that K is centrally symmetric if it is O-symmetric, that is, if $K = -K$. We say that a line ℓ is a supporting line of K if $\ell \cap K \neq \emptyset$ and K is contained in one of the half-planes determined by ℓ. Given a unit vector $u \in \mathbb{S}^1$, the support function of K, h_K, is defined as $h_K(u) = \max_{x \in K} \langle u, x \rangle$. The width function of K, $w_K : \mathbb{S}^1 \to \mathbb{R}$, is defined as $w_K(u) = h_K(u) + h_K(-u)$, that is, the distance between the two supporting lines of K which are orthogonal to u. We can associate with K some interesting convex bodies which share some properties with K. Here we are interested in the so called *centroid body* introduced by C. M. Petty in [6]. The centroid body, denoted by ΓK, is defined as the convex body whose support function is

$$h_{\Gamma K}(u) = \frac{1}{A(K)} \int_K |\langle u, x \rangle| dA,$$

where $A(\cdot)$ denotes the area functional, i.e., the 2-dimensional Lebesgue measure on the plane. The name centroid body is justified by the fact that every boundary point of ΓK is the centroid of a half of K, when K is a centrally symmetric body. Recall that the centroid of K is the point $c \in \mathbb{R}^2$ of the form

$$c = \frac{1}{A(K)} \int_K x \, dA.$$

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Also in [6], Petty proved (as a particular case) the following inequality between the areas of K and ΓK:

$$\frac{A(\Gamma K)}{A(K)} \geqslant \left(\frac{4}{3\pi}\right)^2,$$

where equality holds if and only if K is an ellipse centered at the origin.

In the opposite direction, it was proved by T. Bisztriczky and K. Böröczky the following in [1]: let K be a convex body containing the origin O, then

$$\frac{A(\Gamma K)}{A(K)} \leqslant \frac{16}{27},$$

where equality holds if and only if K is a triangle with O as a vertex.

However, with the assumption that K has center of symmetry, they proved more:

$$\frac{A(\Gamma K)}{A(K)} \leqslant \frac{5}{27},$$

with equality if and only if K is a parallelogram.

With respect to the perimeter (Minkowsky content) $L(\cdot)$ of ΓK, it was proved in [4] that for every convex body K of area 1,

$$L(\Gamma K) \geqslant \frac{8}{3\sqrt{\pi}},$$

with equality if and only if K is a Euclidean disk with center at O.

We would like to find a bound without any restriction on the area of K. In this article we show (see Theorem 1) that if K is a centrally symmetric convex body, then

$$\frac{1}{3} \leqslant \frac{L(\Gamma K)}{L(K)} \leqslant \frac{1}{2}.$$

Equalities on the left and right sides are not possible for convex bodies; however the quotient comes arbitrarily close to these bounds by proper choices of K. For the left side we proceed as follows: consider a very thin rhombus $\mathscr{P}$ centred at the origin and with diagonals of length 2 and 2ε, as shown in Figure 1. Each diagonal divides the rhombus into two triangles, obtaining in this way four triangles whose centroids are p, q, r, and s, as shown in the figure. Since $\mathscr{P}$ is a centrally symmetric convex set, the points p, q, r, and s, are in the boundary of $\Gamma\mathscr{P}$. The lines through these points which are perpendicular to the segments $[O,p]$, $[O,q]$, $[O,r]$, and $[O,s]$, respectively, are support lines of $\Gamma\mathscr{P}$. It follows that the perimeter of $\Gamma\mathscr{P}$ is smaller than or equal to $\frac{4}{3}(1+\varepsilon)$. We also have that $L(\mathscr{P}) = 4\sqrt{\varepsilon^2+1}$, then

$$\frac{L(\Gamma\mathscr{P})}{L(\mathscr{P})} \leqslant \frac{\frac{4}{3}(1+\varepsilon)}{4\sqrt{\varepsilon^2+1}} = \frac{(1+\varepsilon)}{3\sqrt{\varepsilon^2+1}}.$$

Taking the limit when ε approximates to 0 we have

$$\lim_{\varepsilon\to 0^+} \frac{L(\Gamma\mathscr{P})}{L(\mathscr{P})} \leqslant \lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)}{3\sqrt{\varepsilon^2+1}} = \frac{1}{3}.$$

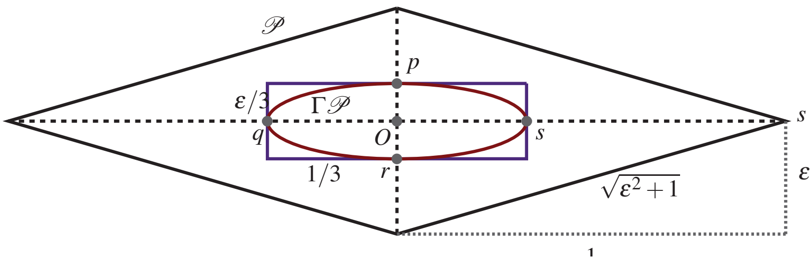

Figure 1: *Perimeter of the centroid body for a thin rhombus*

However, as established in the inequality $\frac{L(\Gamma\mathscr{P})}{L(\mathscr{P})} \geqslant \frac{1}{3}$, hence $\lim_{\varepsilon\to 0^+} \frac{L(\Gamma\mathscr{P})}{L(\mathscr{P})} = \frac{1}{3}$.

For the equality in the right side the procedure is analogous. We consider a very thin rectangle centred at the origin. Let $\mathscr{R}$ be a rectangle centred at O with sides of length 1 and ε. Let p, q, r, and s be the centroids of four of the half parts of $\mathscr{R}$ obtained by division of $\mathscr{R}$ by lines through the origin (see Figure 2). We know that p, q, r, and s belong to the boundary of $\Gamma\mathscr{R}$ and since $\Gamma\mathscr{R}$ is a convex set then the rhombus $pqrs$ is contained in $\Gamma\mathscr{R}$. The perimeter of $\Gamma\mathscr{R}$ is bigger than or equal to $\sqrt{\varepsilon^2+1}$ and so

$$\frac{L(\Gamma\mathscr{R})}{L(\mathscr{R})} \geqslant \frac{\sqrt{\varepsilon^2+1}}{2+2\varepsilon}.$$

Taking the limit when ε approximates to 0 we have

$$\lim_{\varepsilon\to 0^+} \frac{\sqrt{\varepsilon^2+1}}{2+2\varepsilon} = \frac{1}{2},$$

and since $\frac{L(\Gamma\mathscr{R})}{L(\mathscr{R})} \leqslant \frac{1}{2}$ for every $\varepsilon > 0$, we have that

$$\lim_{\varepsilon\to 0^+} \frac{L(\Gamma\mathscr{R})}{L(\mathscr{R})} = \frac{1}{2}.$$

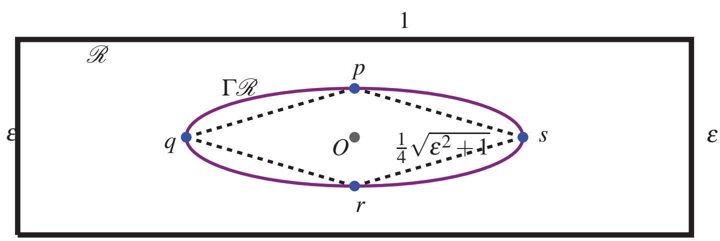

Figure 2: *Perimeter of the centroid body for a thin rectangle*

2. Some auxiliary results

In this section K is a convex body, not necessarily centrally symmetric, whose centroid is at the origin. For every $u \in \mathbb{S}^1$, let $K_u^+ = \{x \in K : \langle x, u \rangle \geqslant 0\}$ and $K_u^- = \overline{K \setminus K_u^+}$, where $\overline{A}$ represents the closure of a set A. Denote the centroids of K_u^+ and K_u^- by c_u^+ and c_u^-, respectively.

LEMMA 1. *For every $u \in \mathbb{S}^1$ we have that $h_{\Gamma K}(u)$ is the harmonic mean of the distances from c_u^+ and c_u^- to the line $u^\perp$.*

Proof. From the definition of centroid, it is easy to see that c_u^+, O, and c_u^- are aligned and that

$$\frac{\|c_u^+\|}{\|c_u^-\|} = \frac{A(K_u^-)}{A(K_u^+)}. \tag{1}$$

Now, for the support function of the centroid body of K we have that

$$\begin{aligned}
h_{\Gamma K}(u) &= \frac{1}{A(K)} \int_K |\langle u, x \rangle| dA \\
&= \frac{1}{A(K)} \left[\int_{K_u^+} \langle x, u \rangle dA - \int_{K_u^-} \langle x, u \rangle dA \right] \\
&= \frac{1}{A(K)} \left[\frac{A(K_u^+)}{A(K_u^+)} \int_{K_u^+} \langle x, u \rangle dA - \frac{A(K_u^-)}{A(K_u^-)} \int_{K_u^-} \langle x, u \rangle dA \right] \\
&= \frac{A(K_u^+)}{A(K)} \left\langle \frac{1}{A(K_u^+)} \int_{K_u^+} x dA, u \right\rangle - \frac{A(K_u^-)}{A(K)} \left\langle \frac{1}{A(K_u^-)} \int_{K_u^-} x dA, u \right\rangle \\
&= \left\langle \frac{A(K_u^+)}{A(K)} c_u^+, u \right\rangle - \left\langle \frac{A(K_u^-)}{A(K)} c_u^-, u \right\rangle \\
&= \left\langle \frac{A(K_u^+)}{A(K)} c_u^+ - \frac{A(K_u^-)}{A(K)} c_u^-, u \right\rangle.
\end{aligned}$$

It follows that $h_{\Gamma K}(u)$ is the projection of the vector obtained as the convex combination of c_u^+ and $-c_u^-$ given by

$$\frac{A(K_u^+)}{A(K)} c_u^+ + \frac{A(K_u^-)}{A(K)} (-c_u^-) \tag{2}$$

over the vector u. Let $q_u = \lambda_0 u$ be the point of intersection between the segment $[c_u^+, p_u]$ with the ray $\{\lambda u : \lambda \geqslant 0\}$, where p_u denotes the reflection of c_u^- along the line $u^\perp$. Suppose that the u-coordinates of c_u^+ and c_u^- are given by y_u^+ and $-y_u^-$, respectively. By the similarity of the triangles $\triangle c_u^+ O q_u$ and $\triangle c_u^+ c_u^- p_u$ (see Figure 3) we have that

$$\frac{\lambda_0}{2y_u^-} = \frac{\|c_u^+\|}{\|c_u^+\| + \|c_u^-\|} = \frac{y_u^+}{y_u^+ + y_u^-}, \tag{3}$$

and thus

$$\lambda_0 = \frac{y_u^-}{y_u^- + y_u^+} y_u^+ + \frac{y_u^+}{y_u^- + y_u^+} y_u^-.$$

Moreover, from (1) we get $y_u^+/y_u^- = A(K_u^-)/A(K_u^+)$ and hence

$$\lambda_0 = \frac{A(K_u^+)}{A(K)} y_u^+ + \frac{A(K_u^-)}{A(K)} y_u^-.$$

Comparing with (2) we conclude that $\lambda_0 = h_{\Gamma K}(u)$. The assertion now follows from (3), since

$$\lambda_0 = \left(\frac{1}{2}(y_u^+)^{-1} + \frac{1}{2}(y_u^-)^{-1}\right)^{-1}. \quad \square$$

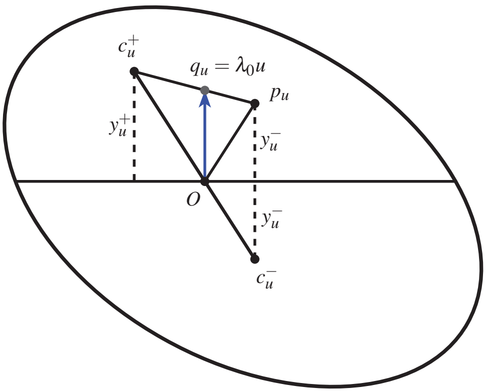

Figure 3: λ_0 *is the harmonic mean of* y_u^+ *and* y_u^-

REMARK 1. Since $\lambda_0 = \frac{2y_u^+ y_u^-}{y_u^+ + y_u^-}$ is the harmonic mean of y_u^+ and y_u^-, and the harmonic mean is smaller than or equal to the arithmetic mean, i.e.,

$$\frac{2y_u^+ y_u^-}{y_u^+ + y_u^-} \leqslant \frac{y_u^+ + y_u^-}{2},$$

we have that $\lambda_0 \leqslant \frac{y_u^+ + y_u^-}{2}$. Now, the width of ΓK in direction u, denoted by $w_{\Gamma K}(u)$, is equal to $2\lambda_0$, and then

$$w_{\Gamma K}(u) \leqslant y_u^+ + y_u^-. \tag{4}$$

Now, let $ABCD$ be an isosceles trapezium of height 1 with bases AB and CD that are parallel to the x axis. Let P and Q be points on AD and BC, respectively, such that PQ is parallel to AB. Then $ABCD$ is divided into two trapeziums, namely $ABQP$ with height h and $PQCD$ with height $1-h$. Suppose that AB, PQ and CD have lengths $2a$, $2b$ and 2, respectively, with $a \leqslant b \leqslant 1$. Then the distance from the centroid of $ABQP$ to the segment PQ is given by (see for instance [5])

$$\frac{b+2a}{3(b+a)}h.$$

Similarly, the distance from the centroid of $PQCD$ to the segment PQ is given by

$$\frac{b+2}{3(b+1)}(1-h).$$

We will prove that for $0 \leqslant a \leqslant 1$ and $\frac{1}{2} \leqslant h \leqslant \frac{2}{3}$ the distance between the centroids of both trapeziums is at most $\frac{1}{2}$. In other words, we will prove the following lemma.

LEMMA 2. *For every* $(a,h) \in D = [0,1] \times [\frac{1}{2}, \frac{2}{3}]$ *we have that*

$$f(a,h) = \frac{(b+2a)}{3(b+a)}h + \frac{(b+2)}{3(b+1)}(1-h) \leqslant \frac{1}{2}.$$

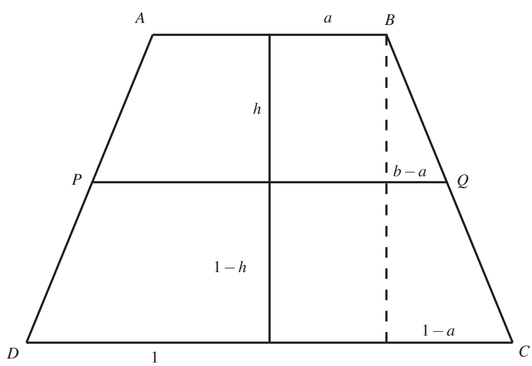

Figure 4: *The circumscribed trapezium*

Proof. By similarity of triangles (see Figure 4) we have that

$$b = (1-a)h + a.$$

Then, we may write

$$f(a,h) = \frac{1}{3} \cdot \frac{(1-a)h+3a}{(1-a)h+2a}h + \frac{1}{3} \cdot \frac{(1-a)h+a+2}{(1-a)h+a+1}(1-h).$$

Now we determine the critical points of $f(a,h)$ in D by solving $f_a(a,h) = 0$ and $f_h(a,h) = 0$, where f_a and f_h denote the partial derivatives of f with respect to a and h, respectively.

Solving $f_h(a,h) = 0$ is equivalent to solve

$$(a-1)(a+h-ah) = 0,$$

which is true when $a = 1$ or $a = \frac{h}{h-1}$. Nonetheless, the equality $a = \frac{h}{h-1}$ is not satisfied in D, since $a < 0$ for $\frac{1}{2} \leqslant h \leqslant \frac{2}{3}$. Then $a = 1$ and $f_h(1,h) = 0$.

Solving $f_a(a,h) = 0$ is equivalent to solve

$$(a(h-1)-h)(a(h-1)^2-h^2) = 0.$$

This equality holds when $a = \frac{h}{h-1}$ or $a = \frac{h^2}{(h-1)^2}$. By the comment above, we conclude that $a = \frac{h^2}{(h-1)^2}$. Since we know that $a = 1$, we have $h = \frac{1}{2}$.

We conclude that f has only one critical point given by $(1, \frac{1}{2})$, and it lies on the boundary of D. It follows that f attains its maximum at the boundary of D.

Since $f_a(a, \frac{2}{3}) \neq 0$ for every $0 \leqslant a \leqslant 1$ then $f(a, \frac{2}{3})$ achieves its maximum when $a = 0$ or $a = 1$. By a simple calculation this maximum is equal to $\frac{1}{2}$ and occurs at $a = 1$. Similarly, we can see that $f(a, \frac{1}{2})$ reaches its maximum $\frac{1}{2}$ at $a = 1$, and $f(0,h)$ reaches its maximum $\frac{4}{9}$ at $h = \frac{1}{2}$. On the other side, $f(1,h) \leqslant \frac{1}{2}$ for every $\frac{1}{2} \leqslant h \leqslant \frac{2}{3}$. This completes the proof of the lemma. □

Now, consider K is a convex body enclosed by the interval $[-b,b]$ and the convex arc (symmetric with respect to the y-axis) from the point $(b,0)$ to the point $(-b,0)$ in the upper half-plane. Let T be the isosceles trapezium with base $[-b,b]$, altitude equal to the width of K in the vertical direction, and with the same area as K (see Figure 5).

LEMMA 3. *Let y_K and y_T be the y-coordinates of the centroids of K and T, respectively. Then $y_K \leqslant y_T$ with equality if and only if $K = T$.*

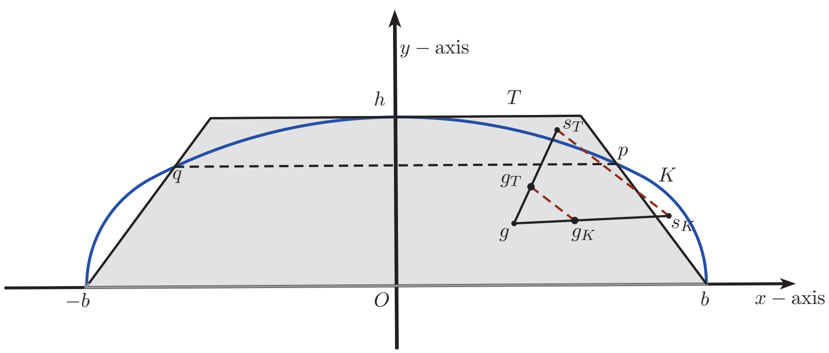

Figure 5: *The centroids of T and K*

Proof. Since K and T have equal area, the boundary of K must cross the boundary of T in two points p and q, as shown in Figure 5. By the symmetry of K and T with respect to the y-axis, it is sufficient to prove the assertion of the lemma for the parts of them contained in the first quadrant, namely K^* and T^*. Let g, g_T, g_K, s_T and s_K, be the centroids of $K^* \cap T^*$, T^*, K^*, $T^* \setminus K^*$ and $K^* \setminus T^*$, respectively. Since all points of $T^* \setminus K^*$ are above the line pq and all points of $K^* \setminus T^*$ are below, we have that the y-coordinate of s_T is larger than the y-coordinate of s_K. Since $T^* \setminus K^*$ and

$K^* \setminus T^*$ have equal area, the points g_T and g_K divide the segments $[g, s_T]$ and $[g, s_K]$ in the same ratio. It follows that the y-coordinate of g_T is larger than or equal to the y-coordinate of g_K and equality is only possible if $T^* = K^*$. Therefore, $y_K \leqslant y_T$ with equality if and only if $K = T$. □

3. Proof of the main result

THEOREM 1. *Let K be a centrally symmetric planar convex body. Then*

$$\frac{1}{3} \leqslant \frac{L(\Gamma K)}{L(K)} \leqslant \frac{1}{2}.$$

Proof. Consider a fixed direction $u \in \mathbb{S}^1$ and suppose the x-axis is the line orthogonal to u and the y-axis is in the direction of u. Let K^+ and K^- be the parts of K over and below the x-axis, respectively. Now we apply to K the Steiner symmetrization (see for instance [7]) with respect to the y-axis and name the symmetrized body as K_{sim}. Set

$$K^+_{\text{sim}} = \{(x,y) \in K_{\text{sim}} : y \geqslant 0\}, \text{ and } K^-_{\text{sim}} = \overline{K_{\text{sim}} \setminus K^+_{\text{sim}}}.$$

Let us denote by T^+ the trapezium contained in the upper half-space of the plane that coincides with K^+_{sim} on the x-axis, and has the same area and height as K^+_{sim}. Define T as the trapezium having bases parallel to the x-axis, tangent to K_{sim} on both bases, and which coincides with T^+ in the half-space above the x-axis. Let T^- be the trapezium resulting from the restriction of T to the half-space below the x axis. We clearly have that $A(T^-) \geqslant A(K^-_{\text{sim}})$.

The Steiner symmetrization with respect to the y-axis preserve the y-coordinates of the centroids, so we have that y^+_K is the y-coordinate of the centroids of K^+ and K^+_{sim}. Analogously, we have that y^-_K is the y-coordinate of the centroids of K^- and K^-_{sim}. Denote the y-coordinates of the centroids of T^+ and T^- by y^+_T and y^-_T, respectively. From Lemma 3 we know that

$$y^+_K \leqslant y^+_T \text{ and } y^-_K \geqslant y^-_T,$$

and using (4) it follows that

$$w_{\Gamma K}(u) \leqslant y^+_K - y^-_K \leqslant y^+_T - y^-_T.$$

Now, there are two possible cases.

(a) The height of T^+ is greater than or equal to $\frac{w_K(u)}{2}$. By a known result in Convexity (see for instance [2]) we also have that the distance from the centroid of a convex body to a support line is at least one third of the width in the direction orthogonal to such line. This implies that the height of T^+ is at most $\frac{2}{3}(w_K(u))$. It follows from Lemma 2 that $y^+_T - y^-_T \leqslant \frac{w_K(u)}{2}$ holds. Hence we have that $w_{\Gamma K}(u) \leqslant \frac{w_K(u)}{2}$.

(b) The height of T^+ is less than $\frac{w_K(u)}{2}$. Suppose the length of the bases of T^+ are λ_1 and λ_2, where λ_1 is the base of T^+ on the x-axis. By the choice of T^+ we have that $\lambda_1 > \lambda_2$. We have two subcases. First subcase arises when the area of K^+_{sim} is more than half the area of K_{sim}. Let Q be the trapezium with the same area and height as K^-_{sim}, contained in the halfplane below the x-axis and coinciding with K_{sim} on the x-axis. Let λ_3 be the length of the other base of Q. Since the area of Q is smaller than the area of T^+ and its height is greater than the height of T^+, we have that $\lambda_3 < \lambda_2 < \lambda_1$. It follows that $-y_Q \leqslant \frac{h_K(-u)}{2}$, where y_Q is y-coordinate of the centroid of Q. By Lemma 3 we have that $-y_K^- \leqslant -y_Q \leqslant \frac{h_K(-u)}{2}$ and since $y_K^+ \leqslant \frac{h_K(u)}{2}$, it follows that $y_K^+ - y_K^+ \leqslant \frac{h_K(u)}{2} + \frac{h_K(-u)}{2} = \frac{w_K(u)}{2}$.

Now consider the subcase when the area of K^+_{sim} is less than half the area of K_{sim}. Then

$$-y_K^- < y_K^+ < \frac{1}{2}\left(\frac{w_K(u)}{2}\right),$$

which implies that $y_K^+ - y_K^- < \frac{w_K(u)}{2}$.

By Remark 1 we have that

$$w_{\Gamma K}(u) \leqslant \frac{w_K(u)}{2}.$$

Since u is an arbitrary direction, by Cauchy's formula for the perimeter of K (see [7]) we have that

$$L(\Gamma K) = \int_0^\pi w_{\Gamma K}(u)d\theta \leqslant \frac{1}{2}\int_0^\pi w_K(u)d\theta = \frac{1}{2}L(K).$$

Therefore,

$$\frac{L(\Gamma K)}{L(K)} \leqslant \frac{1}{2}.$$

Now for the lower bound we proceed as follows: if K is considered to be a centrally symmetric set then every centroid c_u^+ is in the boundary of ΓK and the exterior normal vector at c_u^+ is precisely the unit vector u (see [6] or [1]). This means that the width of ΓK in direction u is exactly $y_K^+ - y_K^-$ and by the result of Convexity mentioned at the beginning of case (a) we have that $y_K^+ - y_K^- \geqslant \frac{1}{3}w_K(u)$, for every $u \in \mathbb{S}^1$. Using again Cauchy's formula, it follows that

$$\frac{L(\Gamma K)}{L(K)} \geqslant \frac{1}{3}.$$

This concludes the proof. □

REMARK 2. For the proof of the upper bound in the inequality, it is not necessary to assume that K is centrally symmetric. Furthermore, using a result proved by M. Fradelizi in [3] we can prove that if K is not a centrally symmetric convex body then

$$\frac{L(\Gamma K)}{L(K)} \geqslant \frac{1}{4}.$$

However, we believe that it must be true that $\frac{L(\Gamma K)}{L(K)} \geqslant \frac{1}{3}$ in this case as well.

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