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ON THE GENERALIZED VON NEUMANN–JORDAN TYPE CONSTANT FOR SOME CONCRETE BANACH SPACES

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Abstract. In this paper, we investigate the relations involving the generalized von Neumann-Jordan type constant of the absolute normalized norms $\|.\|_\psi$ and $\|.\|_\phi$, where the convex functions ψ and ϕ are comparable.These conclusions which not only contain some previous results, but also give the exact value of the generalized von Neumann-Jordan type constant for some practical examples in the application of geometric theory of Banach spaces.

1. Introduction

Let X be a Banach space with the unit ball B_X and the unit sphere S_X. Many geometric constants for a Banach space X have been investigated, such as the von Neumann-Jordan constant $C_{NJ}(X)$ [9] and the von Neumann-Jordan type constant $C_{-\infty}(X)$ [21]. On the one hand, it has been shown that these constants are very useful in geometric theory of Banach space, which enable us to classify several important concepts of Banach space such as uniformly non-squareness and normal structure[7, 16, 25, 28, 29, 30], on the other hand, the calculation of these geometric constants for some concrete spaces is also of some interest [6, 12, 13, 19]. It is well known that the exact values of the von Neumann-Jordan constants $C_{NJ}(X)$ have been calculated for many classical spaces, such as the Lebesgue space [3], the Cesàro space, the Lorentz sequence space [8] and the Bynum space [7] etc. Naturally, one hopes to know the exact values of the von Neumann-Jordan type constant $C_{-\infty}(X)$ for these spaces. Although the exact values of the von Neumann-Jordan type constant $C_{-\infty}(X)$ have been considered in some concrete Banach spaces [21, 25, 28, 29, 30]. However, the exact values for the von Neumann-Jordan type constant $C_{-\infty}(X)$ remain undiscovered for the absolute normalized norms of some concrete Banach spaces.

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2. Preliminaries

Firstly, let us recall the definition of the von Neumann-Jordan constant $C_{NJ}(X)$ and the von Neumann-Jordan type constant $C_{-\infty}(X)$,

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x \in S_X, y \in B_X \right\},$$

$$C_{-\infty}(X) = \sup\left\{ \frac{\min\{\|x+y\|^2, \|x-y\|^2\}}{\|x\|^2+\|y\|^2} : x, y \in X, (x,y) \neq (0,0) \right\}.$$

It is well known that $C_{-\infty}(X) \leqslant C_{\mathrm{NJ}}(X)$ and some properties among them have been indicated in [21, 25].

Recently, the von Neumann-Jordan type constant $C_{-\infty}(X)$ is generalized in the following form, for $1 \leqslant p < +\infty$,

$$C^{(p)}_{-\infty}(X) = \sup\left\{ \frac{\min\{\|x+y\|^p, \|x-y\|^p\}}{2^{p-2}(\|x\|^p+\|y\|^p)} : x, y \in X, (x,y) \neq (0,0) \right\}.$$

It is obvious that $C^{(2)}_{-\infty}(X) = C_{-\infty}(X)$, some geometric properties of Banach spaces X in terms of the new constant $C^{(p)}_{-\infty}(X)$ are investigated in [26, 27].

(i) Let X be a Banach space, then $\frac{1}{2^{p-2}} \leqslant C^{(p)}_{-\infty}(X) \leqslant 2$ for all $1 \leqslant p < +\infty$.

(ii) The Banach space X is uniformly nonsquare $\Leftrightarrow$ $C^{(p)}_{-\infty}(X) < 2$ for some $1 \leqslant p < +\infty$.

(iii) Let X be a Banach space, if there exists some $1 \leqslant p < +\infty$ such that $C^{(p)}_{-\infty}(X) < \frac{(1+\frac{1}{\mu(X)^p})^p}{2^{p-2}(1+\frac{1}{\mu(X)^{p(p-1)}})}$, then X has normal structure, where $\mu(X)$ is weak orthogonality coefficient.

(iv) Let X be a Banach space, if there exists some $1 \leqslant p < +\infty$ such that $C^{(p)}_{-\infty}(X) < \frac{(1+\frac{1}{R(1,X)^p})^p}{2^{p-2}(1+\frac{1}{R(1,X)^{p(p-1)}})}$, then X has normal structure, where $R(1,X)$ is Domínguez-Benavides coefficient.

Therefore, the calculation of the new constant $C^{(p)}_{-\infty}(X)$ is very important in geometric theory of Banach space, which not only enable us to classify several important concepts of Banach space, such as uniformly non-squareness and normal structure, but also give the exact values of the von Neumann-Jordan type constant $C_{-\infty}(X)$ for some concrete Banach spaces. In this paper, we are interested in determining the generalized von Neumann-Jordan type constant $C^{(p)}_{-\infty}(X)$ for the absolute normalized norms. As an application, we can compute the exact values of the generalized von Neumann-Jordan type $C^{(p)}_{-\infty}(X)$ for some concrete Banach spaces, such as the space ℓ_p, Cesàro space $ces^{(2)}_p$, Lorentz sequence spaces $d^{(2)}(\omega, q)$, Banach lattice X^p etc.

Firstly, let us recall that a norm on $\mathbb{R}^2$ is called absolute, if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $(z,w) \in \mathbb{R}^2$ and the norm is called normalized, if

$$\|(1,0)\| = \|(0,1)\| = 1.$$

Let N_α denote the family of all absolute normalized norms on $\mathbb{R}^2$, and Ψ denote the family of all continuous convex functions on $[0,1]$ such that

$$\psi(0) = \psi(1) = 1 \text{ and } \max\{1-t,t\} \leqslant \psi(t) \leqslant 1.$$

It has been shown that N_α and Ψ are a one-to-one correspondence in [1].

THEOREM 1. *If* $\|.\| \in N_\alpha$, *then* $\psi(t) = \|(1-t,t)\| \in \Psi$ *and conversely, if* $\psi(t) \in \Psi$, *then*

$$\|(z,\omega)\|_\psi := \begin{cases} (|z|+|\omega|)\psi\left(\frac{|\omega|}{|z|+|\omega|}\right), & (z,\omega) \neq (0,0), \\ 0, & (z,\omega) = (0,0). \end{cases}$$

is a norm and $\|.\|_\psi \in N_\alpha$.

In particular, for the ℓ_p norm the corresponding convex function $\psi_p(t)$ is given by

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{\frac{1}{p}}, & 1 \leqslant p < \infty, \\ \max\{1-t,t\}, & p = \infty. \end{cases}$$

By Theorem 1, we can also get some Banach spaces which have non-ℓ_p norms on $\mathbb{R}^2$, such as the X^p space, Cesàro sequence space and the following Examples 3, 4, 5, 6, 8, 9 in this paper.

For any $p \in (1,+\infty)$ and $X = \mathbb{R}^2$ with different absolute normalized norms, the norm of the space X^p is given by

$$\|x\| = \||x|^p\|_X^{\frac{1}{p}}.$$

It is proved that if X is a Banach lattice, then X^p space is a Banach lattice for $p \in (1,+\infty)$, some more results about X^p space can be found in [14, 15].

The Cesàro sequence space was defined by Shue in [20], it is very useful in the theory of matrix operators and others. For $1 < q < \infty$, let us restrict ourselves to the two-dimensional Cesàro sequence space $ces_q^{(2)}$, which is just $\mathbb{R}^2$ equipped with the norm defined by

$$\|(x,y)\| = \left(|x|^q + \left(\frac{|x|+|y|}{2}\right)^q\right)^{\frac{1}{q}}.$$

The geometry of Cesàro sequence spaces have been extensively studied in [4, 5, 10, 17, 18].

3. Main results

LEMMA 1. *Let* $\|.\|_1$ *and* $\|.\|_2$ *be two equivalent norms on* X, *namely for* $b \geqslant a > 0$, $a\|.\|_2 \leqslant \|.\|_1 \leqslant b\|.\|_2$, *then*

$$\frac{a^p C_{-\infty}^{(p)}(\|.\|_2)}{b^p} \leqslant C_{-\infty}^{(p)}(\|.\|_1) \leqslant \frac{b^p C_{-\infty}^{(p)}(\|.\|_2)}{a^p}.$$

Moreover, if $\|.\|_1 = a\|.\|_2$, *then* $C_{-\infty}^{(p)}(\|.\|_1) = C_{-\infty}^{(p)}(\|.\|_2)$.

Proof. By the definition of $C_{-\infty}^{(p)}(\|.\|)$, we have that

$$\begin{aligned} C_{-\infty}^{(p)}(\|.\|_1) &= \sup\left\{\frac{\min\{\|x+y\|_1^p, \|x-y\|_1^p\}}{2^{p-2}(\|x\|_1^p + \|y\|_1^p)} : x,y \in X, (x,y) \neq (0,0)\right\} \\ &\leqslant \sup\left\{\frac{b^p \min\{\|x+y\|_2^p, \|x-y\|_2^p\}}{a^p 2^{p-2}(\|x\|_2^p + \|y\|_2^p)} : x,y \in X, (x,y) \neq (0,0)\right\} \\ &= \frac{b^p}{a^p} \sup\left\{\frac{\min\{\|x+y\|_2^p, \|x-y\|_2^p\}}{2^{p-2}(\|x\|_2^p + \|y\|_2^p)} : x,y \in X, (x,y) \neq (0,0)\right\} \\ &\leqslant \frac{b^p}{a^p} C_{-\infty}^{(p)}(\|.\|_2). \end{aligned}$$

Similarly, we can get the inequality

$$\frac{a^p C_{-\infty}^{(p)}(\|.\|_2)}{b^p} \leqslant C_{-\infty}^{(p)}(\|.\|_1). \quad \square$$

Let us put

$$M_1 = \max_{0 \leqslant t \leqslant 1} \frac{\phi(t)}{\psi(t)} \ \text{ and } \ M_2 = \max_{0 \leqslant t \leqslant 1} \frac{\psi(t)}{\phi(t)}.$$

THEOREM 2. *Let* $\psi(t), \phi(t) \in \Psi$ *and* $\psi(t) \leqslant \phi(t)$, *if the function* $\frac{\phi(t)}{\psi(t)}$ *attains its maximum at* $t = \frac{1}{2}$ *and* $C_{-\infty}^{(p)}(\|.\|_\phi) = \frac{1}{2^{p-1}\phi^p(\frac{1}{2})}$, *then*

$$C_{-\infty}^{(p)}(\|.\|_\psi) = \frac{1}{2^{p-1}\psi^p(\frac{1}{2})}.$$

Proof. By the condition of $\psi(t) \leqslant \phi(t)$ and the definition of M_1, we have that

$$\frac{1}{M_1}\|.\|_\phi \leqslant \|.\|_\psi \leqslant \|.\|_\phi.$$

Take $a = \frac{1}{M_1}$ and $b = 1$ in Lemma 1, which implies that

$$C_{-\infty}^{(p)}(\|.\|_\psi) \leqslant M_1^p C_{-\infty}^{(p)}(\|.\|_\phi).$$

It is noted that the function $\frac{\phi(t)}{\psi(t)}$ attains its maximum at $t=\frac{1}{2}$, i.e.,$M_1=\frac{\phi(\frac{1}{2})}{\psi(\frac{1}{2})}$ and $C_{-\infty}^{(p)}(\|.\|_\phi)=\frac{1}{2^{p-1}\phi^p(\frac{1}{2})}$, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) \leqslant M_1^p C_{-\infty}^{(p)}(\|.\|_\phi)=\frac{1}{2^{p-1}\psi^p(\frac{1}{2})}. \tag{1}$$

On the other hand, let us put $x_1=(1,1), y_1=(1,-1)$, it follows that

$$\|x_1\|_\psi=\|y_1\|_\psi=2\psi\Big(\frac{1}{2}\Big),$$

$$\|x_1+y_1\|_\psi=\|x_1-y_1\|_\psi=2,$$

$$\frac{\min\{\|x_1+y_1\|_\psi^p,\|x_1-y_1\|_\psi^p\}}{2^{p-2}(\|x_1\|_\psi^p+\|y_1\|_\psi^p)}=\frac{2^p}{2^{p-1}2^p\psi^p(\frac{1}{2})}=\frac{1}{2^{p-1}\psi^p(\frac{1}{2})} \leqslant C_{-\infty}^{(p)}(\|.\|_\psi). \tag{2}$$

By the inequality (1) and (2), we have that

$$C_{-\infty}^{(p)}(\|.\|_\psi)=M_1^p C_{-\infty}^{(p)}(\|.\|_\phi)=\frac{1}{2^{p-1}\psi^p(\frac{1}{2})}. \quad \square$$

THEOREM 3. *Let $\psi(t)\in\Psi$ and $\psi(t)\leqslant\phi(t)=\psi_p(t)$ ($2\leqslant p<\infty$), if the function $\frac{\psi_p(t)}{\psi(t)}$ attains its maximum at $t=\frac{1}{2}$, then*

$$C_{-\infty}^{(p)}(\|.\|_\psi)=M_1^p=\frac{1}{2^{p-1}\psi^p(\frac{1}{2})}.$$

Proof. By the condition of $\psi(t)\leqslant\psi_p(t)$ and Clarkson inequality,

$$\begin{aligned}\min\{\|x+y\|_\psi^p,\|x-y\|_\psi^p\} &\leqslant \frac{1}{2}(\|x+y\|_\psi^p+\|x-y\|_\psi^p)\\ &\leqslant \frac{1}{2}(\|x+y\|_p^p+\|x-y\|_p^p)\\ &\leqslant 2^{p-2}(\|x\|_p^p+\|y\|_p^p)\\ &\leqslant 2^{p-2}M_1^p(\|x\|_\psi^p+\|y\|_\psi^p).\end{aligned}$$

The definition of $C_{-\infty}^{(p)}(\|.\|_\psi)$ implies that

$$C_{-\infty}^{(p)}(\|.\|_\psi)\leqslant M_1^p. \tag{3}$$

On the other hand, note that the function $\frac{\psi_p(t)}{\psi(t)}$ attains its maximum at $t=\frac{1}{2}$, i.e. $M_1=\frac{\psi_p(\frac{1}{2})}{\psi(\frac{1}{2})}$. Let us put $x_2=(\frac{1}{2},\frac{1}{2}), y_2=(\frac{1}{2},-\frac{1}{2})$, then

$$\min\{\|x_2+y_2\|_\psi^p,\|x_2-y_2\|_\psi^p\}=1=2^{p-1}\psi_p^p\Big(\frac{1}{2}\Big),$$

$$\|x_2\|_\psi^p = \|y_2\|_\psi^p = \psi^p\left(\frac{1}{2}\right).$$

The definition of $C_{-\infty}^{(p)}(\|.\|_\psi)$ implies that

$$C_{-\infty}^{(p)}(\|.\|_\psi) \geqslant \frac{\min\{\|x_2+y_2\|_\psi^p, \|x_2-y_2\|_\psi^p\}}{2^{p-2}(\|x_2\|_\psi^p + \|y_2\|_\psi^p)} = \frac{\psi_p^p(\frac{1}{2})}{\psi^p(\frac{1}{2})} = M_1^p = \frac{1}{2^{p-1}\psi^p(\frac{1}{2})}. \tag{4}$$

By the inequality (3) and (4), we can get that

$$C_{-\infty}^{(p)}(\|.\|_\psi) = M_1^p = \frac{1}{2^{p-1}\psi^p(\frac{1}{2})}. \quad \square$$

COROLLARY 1. *Let X^p be a two-dimensional Banach spaces, if the corresponding function ψ_X attains its minimum at the point $t = \frac{1}{2}$. For $2 \leqslant p < \infty$, then*

$$C_{-\infty}^{(p)}(\|.\|_{X^p}) = \frac{1}{2^{p-1}\psi_{X^p}^p(\frac{1}{2})}.$$

Proof. It is clear that $\|x\| = \||x|^p\|_X^{\frac{1}{p}} \in \mathbb{N}_\alpha$ from the norm of the space X^p, and its corresponding convex function is

$$\psi_{X^p}(t) = \|(1-t,t)\|_{X^p} = [(1-t)^p + t^p]^{\frac{1}{p}}\,\psi_X^{\frac{1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right).$$

Since $\psi_X \leqslant 1$, then $\psi_{X^p}(t) \leqslant \psi_p(t)$, it is easy to check that the function

$$\frac{\psi_p(t)}{\psi_{X^p}(t)} = \psi_X^{\frac{-1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right).$$

For arbitrary $t \in [0,1]$, the variable $s = \frac{t^p}{(1-t)^p+t^p}$ is also belongs to $[0,1]$. Since the function $\psi_X(t)$ attains its minimum at the point $t = \frac{1}{2}$, then $\psi_X\left(\frac{t^p}{(1-t)^p+t^p}\right)$ attains its minimum at $t = \frac{1}{2}$, this implies that the function $\psi_X^{\frac{-1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right)$ attains its maximum at $\frac{1}{2}$. By Theorem 3, we have that

$$C_{-\infty}^{(p)}(\|.\|_{X^p}) = \frac{1}{2^{p-1}\psi_{X^p}^p(\frac{1}{2})}. \quad \square$$

THEOREM 4. *Let $\psi(t), \phi(t) \in \Psi$ and $\psi(t) \geqslant \phi(t)$, if the function $\frac{\psi(t)}{\phi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $C_{-\infty}^{(p)}(\|.\|_\phi) = 2\phi^p(\frac{1}{2})$, then*

$$C_{-\infty}^{(p)}(\|.\|_\psi) = 2\psi^p\left(\frac{1}{2}\right).$$

Proof. By the condition of $\psi(t) \geqslant \phi(t)$ and the definition of M_2, we have that

$$\|.\|_\phi \leqslant \|.\|_\psi \leqslant M_2\|.\|_\phi.$$

Take $a=1$ and $b=M_2$ in Lemma 1, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) \leqslant M_2^p C_{-\infty}^{(p)}(\|.\|_\phi).$$

It is noted that the function $\frac{\psi(t)}{\phi(t)}$ attains its maximum at $t=\frac{1}{2}$, i.e.,$M_2 = \frac{\psi(\frac{1}{2})}{\phi(\frac{1}{2})}$ and $C_{-\infty}^{(p)}(\|.\|_\phi) = 2\phi^p(\frac{1}{2})$, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) \leqslant M_2^p C_{-\infty}^{(p)}(\|.\|_\phi) = 2\psi^p\left(\frac{1}{2}\right). \tag{5}$$

On the other hand, let us put $x_3=(1,0), y_3=(0,1)$, then

$$\|x_3\| = \|y_3\| = 1,$$

$$\|x_3+y_3\|_\psi = \|x_3-y_3\|_\psi = 2\psi\left(\frac{1}{2}\right),$$

$$\frac{\min\{\|x_3+y_3\|_\psi^p, \|x_3-y_3\|_\psi^p\}}{2^{p-2}(\|x_3\|_\psi^p + \|y_3\|_\psi^p)} = \frac{2^p\psi^p(\frac{1}{2})}{2^{p-1}} = 2\psi^p\left(\frac{1}{2}\right) \leqslant C_{-\infty}^{(p)}(\|.\|_\psi). \tag{6}$$

By the inequality (5) and (6), we have that

$$C_{-\infty}^{(p)}(\|.\|_\psi) = M_2^p C_{-\infty}^{(p)}(\|.\|_\phi) = 2\psi^p\left(\frac{1}{2}\right). \quad \square$$

THEOREM 5. *Let* $\psi(t) \in \Psi$ *and* $\psi(t) \geqslant \phi(t) = \psi_p(t)$ *(*$1 \leqslant p \leqslant 2$*), then*

$$C_{-\infty}^{(p)}(\|.\|_\psi) = 2^{2-p}M_2^p.$$

Proof. By the condition of $\psi(t) \geqslant \psi_p(t)$ and the Clarkson inequality, we can get

$$\begin{aligned}
\min\{\|x+y\|_\psi^p, \|x-y\|_\psi^p\} &\leqslant \frac{1}{2}(\|x+y\|_\psi^p + \|x-y\|_\psi^p) \\
&\leqslant \frac{1}{2}M_2^p(\|x+y\|_p^p + \|x-y\|_p^p) \\
&\leqslant M_2^p(\|x\|_p^p + \|y\|_p^p) \\
&\leqslant M_2^p(\|x\|_\psi^p + \|y\|_\psi^p).
\end{aligned}$$

The definition of $C_{-\infty}^{(p)}(\|.\|_\psi)$ implies that

$$C_{-\infty}^{(p)}(\|.\|_\psi) \leqslant 2^{2-p}M_2^p. \tag{7}$$

On the other hand, if the function $\frac{\psi(t)}{\psi_p(t)}$ attains its maximum at $t = t_0 \in [0,1]$, i.e. $M_2 = \frac{\psi(t_0)}{\psi_p(t_0)}$. Let us put $x_0 = (1-t_0, 0), y_0 = (0, t_0)$, then

$$\|x_0\|_\psi^p = (1-t_0)^p, \quad \|y_0\|_\psi^p = t_0^p,$$

$$\begin{aligned} \min\{\|x_0+y_0\|_\psi^p, \|x_0-y_0\|_\psi^p\} &= \psi^p(t_0) \\ &= M_2^p[(1-t_0)^p + t_0^p] \\ &= M_2^p(\|x_0\|_p^p + \|y_0\|_p^p). \end{aligned}$$

$$C_{-\infty}^{(p)}(\|.\|_\psi) \geqslant \frac{\min\{\|x_0+y_0\|_\psi^p, \|x_0-y_0\|_\psi^p\}}{2^{p-2}(\|x_0\|_\psi^p + \|y_0\|_\psi^p)} = 2^{2-p}M_2^p. \tag{8}$$

By the inequality (7) and (8), we have that

$$C_{-\infty}^{(p)}(\|.\|_\psi) = 2^{2-p}M_2^p. \quad \Box$$

In fact, we can also get some results related to the general mean from Theorem 2 and Theorem 4. Firstly, we give the definition of general weighted mean of order s,

$$m^{[s]}(a,b;\omega,1-\omega) = \begin{cases} (\omega a^s + (1-\omega)b^s)^{\frac{1}{s}}, & s \neq 0, +\infty, -\infty \\ a^\omega b^{1-\omega}, & s = 0 \\ \max\{a,b\}, & s = \infty \\ \min\{a,b\}, & s = -\infty \end{cases}$$

where a,b are positive real numbers, $\omega \in (0,1)$. In the following, let us state a conclusion related to the general mean and then applied it to the weighted mean of order s.

COROLLARY 2. *Let $\psi(t), \phi(t) \in \Psi$ and $\psi(t) \leqslant \phi(t)$, $m(t) := m(\psi(t), \phi(t))$ be a mean of functions $\psi(t), \phi(t)$, if the function $m(t)$ be a convex function, then*

(i) $\frac{m(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $C_{-\infty}^{(p)}(\|.\|_\psi) = 2\psi^p(\frac{1}{2})$, then

$$C_{-\infty}^{(p)}(\|.\|_m) = 2m^p\left(\frac{1}{2}\right).$$

(ii) $\frac{\phi(t)}{m(t)}$ attains its maximum at $t = \frac{1}{2}$ and $C_{-\infty}^{(p)}(\|.\|_\phi) = \frac{1}{2^{p-1}\phi^p(\frac{1}{2})}$, then

$$C_{-\infty}^{(p)}(\|.\|_m) = \frac{1}{2^{p-1}m^p(\frac{1}{2})}.$$

Proof. The general mean $m(t)$ has the property

$$\psi(t) \leqslant m(t) \leqslant \phi(t).$$

Since $\psi(t), \varphi(t) \in \Psi$ and the assumption of the function $m(t)$ is convex, it is easy to check that $m(t) \in \Psi$. Now, statements of the results follows by the Theorem 2 and Theorem 4. $\square$

For the general case $\psi(t) \in \Psi$, we can only estimate the lower bound or upper bound of the generalized von Neumann-Jordan type constant $C_{-\infty}^{(p)}(\|.\|_\psi)$.

COROLLARY 3. *Let $\psi(t) \in \Psi$, then*

(i) if $1 \leqslant p \leqslant 2$, then

$$2^{2-p} M_2^p \leqslant C_{-\infty}^{(p)}\left(\|\cdot\|_\psi\right) \leqslant 2^{2-p} M_1^p M_2^p,$$

(ii) if $2 \leqslant p \leqslant \infty$, then

$$C_{-\infty}^{(p)}\left(\|\cdot\|_\psi\right) \leqslant M_1^p M_2^p.$$

Proof. (i) It is well known that from Theorem 5

$$C_{-\infty}^{(p)}\left(\|\cdot\|_\psi\right) \geqslant 2^{2-p} M_2^p.$$

Note that the inequality

$$\frac{1}{M_1}\|.\|_p \leqslant \|.\|_\psi \leqslant M_2\|.\|_p.$$

Take $a = \frac{1}{M_1}$ and $b = M_2$ in Lemma 1, we have that

$$C_{-\infty}^{(p)}\left(\|\cdot\|_\psi\right) \leqslant C_{-\infty}^{(p)}(\|.\|_p) M_1^p M_2^p. \tag{9}$$

If $1 \leqslant p \leqslant 2$, it is known that $C_{-\infty}^{(p)}(\|.\|_p) = 2^{2-p}$ from Example 1, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) \leqslant 2^{2-p} M_1^p M_2^p.$$

(ii)If $2 \leqslant p \leqslant \infty$, it is known that $C_{-\infty}^{(p)}(\|.\|_p) = 1$ from Example 1, we can similarly get the estimate (ii)

$$C_{-\infty}^{(p)}\left(\|\cdot\|_\psi\right) \leqslant M_1^p M_2^p$$

from the inequality (9). $\square$

However, we can get some conditions of $\psi(t)$ that the von Neumann-Jordan type constant $C_{-\infty}(\|.\|_\psi)$ coincides with the upper bound $M_1^2 M_2^2$.

THEOREM 6. *Let* $\psi(t)\in\Psi$ *and* $\psi(t)=\psi(1-t)$ *for all* $t\in[0,1]$. *If* $M_1=\frac{\psi_2(\frac{1}{2})}{\psi(\frac{1}{2})}$ *and* $M_2=\max_{0\leqslant t\leqslant 1}\frac{\psi(t)}{\psi_2(t)}$, *then*

$$C_{-\infty}(\|.\|_\psi)=M_1^2M_2^2.$$

Proof. By the definition of M_1, M_2 and inequality (9), we can have that

$$C_{-\infty}(\|.\|_\psi)\leqslant C_{-\infty}(\|.\|_2)M_1^2M_2^2.$$

Since $C_{-\infty}(\|.\|_2)=1$, which implies that

$$C_{-\infty}(\|.\|_\psi)\leqslant M_1^2M_2^2. \tag{10}$$

Take an arbitrary $t\in[0,1]$ and put $x=(t,1-t)$, $y=(1-t,t)$, then

$$\|x\|_\psi=\|y\|_\psi=\psi(t).$$

$$\|x+y\|_\psi=\|(1,1)\|_\psi=2\psi\Big(\frac{1}{2}\Big),\quad \|x-y\|_\psi=\|(2t-1,1-2t)\|_\psi=2|2t-1|\psi\Big(\frac{1}{2}\Big).$$

$$\begin{aligned}\frac{4\min\{\|x\|_\psi^2,\|y\|_\psi^2\}}{\|x+y\|_\psi^2+\|x-y\|_\psi^2}&=\frac{\psi^2(t)}{(1+(2t-1)^2)\psi^2(\frac{1}{2})}\\&=\frac{\psi^2(t)}{2\psi_2^2(t)\psi^2(\frac{1}{2})}\\&=\frac{\psi^2(t)}{\psi_2^2(t)}\frac{\psi_2^2(\frac{1}{2})}{\psi^2(\frac{1}{2})}\\&=M_1^2M_2^2.\end{aligned}$$

Since t is arbitrary, from the equivalent definition of $C_{-\infty}(\|.\|_\psi)$, then

$$C_{-\infty}(\|.\|_\psi)\geqslant M_1^2M_2^2. \tag{11}$$

The inequalities (10) and (11) show that $C_{-\infty}(\|.\|_\psi)=M_1^2M_2^2$. □

THEOREM 7. *Let* $\psi(t)\in\Psi$ *and* $\psi(t)=\psi(1-t)$ *for all* $t\in[0,1]$. *If there exist unique points* $t_1,t_2\in[0,\frac{1}{2}]$ *such that*

$$M_1=\frac{\psi_2(t_1)}{\psi(t_1)},\ \ M_2=\frac{\psi(t_2)}{\psi_2(t_2)}\ \ \textit{and}\ \ (1-t_1)(1-t_2)=\frac{1}{2},$$

then

$$C_{-\infty}(\|.\|_\psi)=M_1^2M_2^2.$$

Proof. On the one hand, by Theorem 6, we have that

$$C_{-\infty}(\|.\|_\psi) \leqslant M_1^2 M_2^2. \tag{12}$$

On the other hand, note that $(1-t_1)(1-t_2)=\frac{1}{2}$, put $x=(1-t_1,t_1)$, $y=(t_1,t_1-1)$, then $x+y=(1,2t_1-1)$ $x-y=(1-2t_1,1)$ and

$$\|x\|_\psi = \psi(t_1) = \frac{\psi_2(t_1)}{M_1}, \ \ \|y\|_\psi = \psi(1-t_1) = \frac{\psi_2(t_1)}{M_1},$$

$$\|x+y\|_\psi = (2-2t_1)\psi\left(\frac{1-2t_1}{2-2t_1}\right) = \frac{\psi(t_2)}{(1-t_2)} = \frac{M_2\psi_2(t_2)}{(1-t_2)},$$

$$\|x-y\|_\psi = (2-2t_1)\psi\left(\frac{1}{2-2t_1}\right) = \frac{\psi(1-t_2)}{(1-t_2)} = \frac{M_2\psi_2(t_2)}{(1-t_2)}.$$

Since

$$\sqrt{2}(1-t)\psi_2\left(\frac{1}{2-2t}\right) = \psi_2(t).$$

Consequently

$$C_{-\infty}(\|.\|_\psi) \geqslant \frac{\min\{\|x+y\|_\psi^2, \|x-y\|_\psi^2\}}{(\|x\|_\psi^2 + \|y\|_\psi^2)} = M_1^2 M_2^2. \tag{13}$$

By the inequalities (12) and (13), we have that $C_{-\infty}(\|.\|_\psi) = M_1^2 M_2^2$. $\square$

4. Some Examples

In this section, we will calculate the exactly values of $C_{-\infty}^{(p)}(X)$ for some concrete Banach spaces. These results which not only give the exact value of the generalized von Neumann-Jordan type constant $C_{-\infty}^{(p)}(X)$, but also give some new supplement results about the von Neumann-Jordan type constant $C_{-\infty}(X)$ for some concrete Banach spaces.

EXAMPLE 1. If X is the ℓ_p $(1 \leqslant p \leqslant \infty)$ space, then

$$C_{-\infty}^{(p)}(X) = \begin{cases} 2^{2-p}, & 1 \leqslant p \leqslant 2, \\ 1, & 2 < p < \infty. \end{cases}$$

In particular, $C_{-\infty}^{(p)}(\|.\|_1) = C_{-\infty}^{(p)}(\|.\|_\infty) = 2$.

Proof. Let $1 \leqslant p \leqslant 2$, then $\psi_p(t) \geqslant \psi_2(t)$ and

$$\psi_p(t) \leqslant 2^{\frac{1}{p}-\frac{1}{2}}\psi_2(t),$$

where the constant $2^{\frac{1}{p}-\frac{1}{2}}$ is the best possible. On the other hand, the function $\frac{\psi_p(t)}{\psi_2(t)}$ attains maximum at $t=\frac{1}{2}$.

$$\frac{\psi_p(\frac{1}{2})}{\psi_2(\frac{1}{2})}=\frac{((1-\frac{1}{2})^p+(\frac{1}{2})^p)^{\frac{1}{p}}}{((1-\frac{1}{2})^2+(\frac{1}{2})^2)^{\frac{1}{2}}}=2^{\frac{1}{p}-\frac{1}{2}}.$$

Therefore, by Theorem 4, we have

$$C_{-\infty}^{(p)}(\|.\|_p)=2\psi_p^p\left(\frac{1}{2}\right)=2^{2-p}. \tag{14}$$

Similarly, for $2<p<\infty$, then $\psi_p(t)\leqslant\psi_2(t)$. By Theorem 3, then

$$C_{-\infty}^{(p)}(\|.\|_p)=\frac{1}{2^{p-1}\psi_p^p(\frac{1}{2})}=1. \tag{15}$$

Let $p=\infty$, since

$$\psi_\infty(t)=\begin{cases}1-t, & 0\leqslant t\leqslant\frac{1}{2},\\ t, & \frac{1}{2}<t<1.\end{cases}$$

(i) Let $0\leqslant t\leqslant\frac{1}{2}$, $\frac{\psi_p(t)}{\psi_\infty(t)}=\frac{((1-t)^p+t^p)^{\frac{1}{p}}}{1-t}=g(t)$, then $g'(t)>0$ and $M_1=g(\frac{1}{2})=2^{\frac{1}{p}}$.

(ii) Let $\frac{1}{2}\leqslant t\leqslant 1$, $\frac{\psi_p(t)}{\psi_\infty(t)}=\frac{((1-t)^p+t^p)^{\frac{1}{p}}}{t}=h(t)$, then $h'(t)<0$ and $M_1=h(\frac{1}{2})=2^{\frac{1}{p}}$.

Therefore, $C_{-\infty}^{(p)}(\|.\|_\infty)=M_1^p=2$ by Theorem 3. □

EXAMPLE 2. Let $X=\mathbb{R}^2$, the convex function $\psi(t)$ is defined on $[0,1]$ as

$$\psi_X(t)=(1-t+t^2)^{\frac{1}{2}}.$$

The corresponding norm is

$$\|(x,y)\|=(|x|^2+|x||y|+|y|^2)^{\frac{1}{2}}.$$

It is obvious that $\|(x,y)\|$ is an absolute normalized norm on $\mathbb{R}^2$. By a standard discussion, it is easy to check that the corresponding function $\psi_X(t)=\sqrt{1-t+t^2}$ attains its minimum at the point $\frac{1}{2}$. For $p\geqslant 2$, then the corresponding space X^p has the norm

$$\|(x,y)\|=((|x|^{2p}+|x|^p|y|^p+|y|^{2p})^{\frac{1}{2p}}.$$

And the corresponding convex function is

$$\psi_{X^p}(t)=\|(1-t,t)\|_{X^p}=[(1-t)^p+t^p]^{\frac{1}{p}}\psi_X^{\frac{1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right).$$

By Corollary 1, we have that

$$C_{-\infty}^{(p)}(\|.\|_{\psi_{X^p}})=\frac{1}{2^{p-1}\psi_{X^p}^p(\frac{1}{2})}=\frac{2\sqrt{3}}{3}.$$

EXAMPLE 3. Let $0<\omega<1$ and $2 \leqslant q<\infty$. The two-dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ is $\mathbb{R}^2$ with the norm

$$\|(x,y)\|_{\omega,q}=((x^*)^q+\omega(y^*)^q)^{\frac{1}{q}},$$

where (x^*,y^*) is the rearrangement of $(|x|,|y|)$ satisfying $x^* \geqslant y^*$, then

$$C_{-\infty}^{(p)}(\|.\|_{\omega,q})=2\Big(\frac{1}{1+\omega}\Big)^{\frac{p}{q}}.$$

Proof. It is well known that $\|(x,y)\|_{\omega,q}$ is an absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is

$$\psi_{\omega,q}(t)=\begin{cases}((1-t)^q+\omega t^q)^{\frac{1}{q}}, & 0 \leqslant t \leqslant \frac{1}{2},\\ (t^q+\omega(1-t)^q)^{\frac{1}{q}}, & \frac{1}{2} \leqslant t \leqslant 1.\end{cases}$$

It is easy to check that $\psi_{\omega,q}(t) \leqslant \psi_q(t)$. Since $0<\omega<1$, $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ is symmetric with respect to $t=\frac{1}{2}$, it suffices to consider $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ for $t \in [0,\frac{1}{2}]$. For any $t \in [0,\frac{1}{2}]$, put $f(t)=\frac{\psi_q(t)^q}{\psi_{\omega,q}(t)^q}$. Taking derivative of the function $f(t)$, then

$$f'(t)=\frac{q(1-\omega)[t(1-t)]^{q-1}}{[(1-t)^q+\omega t^q]^2}.$$

We always have $f'(t) \geqslant 0$ for $0 \leqslant t \leqslant \frac{1}{2}$, this implies that the function $f(t)$ is increased for $0 \leqslant t \leqslant \frac{1}{2}$. Therefore, the function $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ attains its maximum at $t=\frac{1}{2}$. By Theorem 3, then

$$C_{-\infty}^{(p)}(\|.\|_{\omega,q})=2\Big(\frac{1}{1+\omega}\Big)^{\frac{p}{q}}. \quad \Box$$

EXAMPLE 4. Let $X=\mathbb{R}^2$ with the norm $\|.\|_{p,q,\lambda}=\max\{\|.\|_p,\lambda\|.\|_q\}$, where $1 \leqslant q \leqslant p \leqslant \infty$ and $\lambda \in [2^{\frac{1}{p}-\frac{1}{q}},1]$, then

$$C_{-\infty}^{(p)}(\|.\|_{p,q,\lambda})=\begin{cases}2\lambda^p 2^{\frac{p}{q}-p}, & \text{if } 1 \leqslant q<p \leqslant 2,\\ \frac{2^{1-\frac{p}{q}}}{\lambda^p}, & \text{if } 2 \leqslant q<p \leqslant \infty.\end{cases}$$

Proof. It is very easy to check that $\|.\|_{p,q,\lambda}=\max\{\|.\|_p,\lambda\|.\|_q\} \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(t)=\|(1-t,t)\|_{p,q,\lambda}=\max\{\psi_p(t),\lambda\psi_q(t)\}.$$

Let $t_0 \in [0,\frac{1}{2}]$ be a point such that $\psi_p(t_0)=\lambda\psi_q(t_0)$, then

$$\psi(t)=\begin{cases}\psi_p(t), & t \in [0,t_0],\\ \lambda\psi_q(t), & t \in [t_0,\frac{1}{2}].\end{cases}$$

In fact, $\psi(t)$ is symmetric with respect to $t=\frac{1}{2}$, which is expanded to the whole interval $[0,1]$.

(i) Suppose that $1\leqslant q<p\leqslant 2$, from the definition of $\psi(t)$, it is obvious that $\psi(t)\geqslant \psi_p(t)$ and the function

$$\frac{\psi(t)}{\psi_p(t)}=\begin{cases} 1, & t\in[0,t_0]\cup[1-t_0,1],\\ \frac{\lambda\psi_q(t)}{\psi_p(t)}, & t\in[t_0,1-t_0].\end{cases}$$

attains its maximum at $t=\frac{1}{2}$. Hence, by Theorem 5, we can have that

$$C_{-\infty}^{(p)}(\|.\|_{p,q,\lambda})=2\lambda^p 2^{\frac{p}{q}-p}.$$

(ii) Suppose that $2\leqslant q<p\leqslant\infty$, since $\psi_p(t)\leqslant\psi_q(t)$ and $\lambda\psi_q(t)\leqslant\psi_q(t)$, then $\psi(t)\leqslant\psi_q(t)$, it is easy to check that the function

$$\frac{\psi_q(t)}{\psi(t)}=\begin{cases} \frac{\psi_q(t)}{\psi_p(t)}, & t\in[0,t_0]\cup[1-t_0,1],\\ \frac{1}{\lambda}, & t\in[t_0,1-t_0].\end{cases}$$

attains its maximum at $t=\frac{1}{2}$. By Theorem 3, then

$$C_{-\infty}^{(p)}(\|.\|_{p,q,\lambda})=\frac{2^{1-\frac{p}{q}}}{\lambda^p}. \quad \square$$

In the following, we will consider a wide class of absolute normalized norms which involve weighted means of p-norms after normalization.

EXAMPLE 5. Let $1\leqslant p<q\leqslant\infty$, $1\leqslant s<\infty$ and $\lambda>0$, the convex function $\psi_{\lambda,p,q,s}(t)$ is defined on $[0,1]$ as

$$\psi_{\lambda,p,q,s}(t)=(1+\lambda)^{-\frac{1}{s}}(\psi_p^s(t)+\lambda\psi_q^s(t))^{\frac{1}{s}}.$$

i.e. $\psi_{\lambda,p,q,s}(t)$ is a weighted mean of order s of functions ψ_p and ψ_q with weights $\frac{1}{1+\lambda}$ and $\frac{\lambda}{1+\lambda}$. The corresponding norm is

$$\|.\|_{\lambda,p,q,s}=(1+\lambda)^{-\frac{1}{s}}(\|.\|_p^s+\lambda\|.\|_q^s)^{\frac{1}{s}}.$$

Then

(i) If $1\leqslant p<q\leqslant 2$, then $C_{-\infty}^{(p)}(\|.\|_{\lambda,p,q,s})=2(1+\lambda)^{\frac{-p}{s}}(2^{\frac{s}{p}}+\lambda 2^{\frac{s}{q}})^{\frac{p}{s}}$.

(ii) If $2\leqslant p<q\leqslant\infty$, then $C_{-\infty}^{(p)}(\|.\|_{\lambda,p,q,s})=2(1+\lambda)^{\frac{p}{s}}(2^{\frac{s}{q}}+\lambda 2^{\frac{s}{q}})^{\frac{-p}{s}}$.

Proof. Since $\psi_{\lambda,p,q,s}(t)$ is a weighted mean of order s of functions $\psi_p(t)$ and $\psi_q(t)$, then

$$\psi_q(t) \leqslant \psi_{\lambda,p,q,s}(t) \leqslant \psi_p(t).$$

(i) Let $1 \leqslant p < q \leqslant 2$, since $\psi_{\lambda,p,q,s}(t) \geqslant \psi_q(t)$ and $\frac{\psi^s_{\lambda,p,q,s}(t)}{\psi^s_q(t)}$ attains its maximum at the same point as $\frac{\psi_p(t)}{\psi_q(t)}$ attains its maximum at $t = \frac{1}{2}$ by the simple calculation. Take $\psi = \psi_q(t)$ and $\phi = \psi_p(t)$ in Corollary 2 (i), we have

$$C^{(p)}_{-\infty}(\|.\|_{\lambda,p,q,s}) = 2\psi^p_{\lambda,p,q,s}(\frac{1}{2}) = 2(1+\lambda)^{\frac{-p}{s}}(2^{\frac{s}{p}} + \lambda 2^{\frac{s}{q}})^{\frac{p}{s}}.$$

(ii) Suppose that $2 \leqslant p < q \leqslant \infty$, since $\psi_{\lambda,p,q,s}(t) \leqslant \psi_p(t)$ and $\frac{\psi_p(t)}{\psi_{\lambda,p,q,s}(t)}$ attains its maximum at $t = \frac{1}{2}$. Similarly, take $\psi = \psi_q(t)$ and $\phi = \psi_p(t)$ in Corollary 2 (ii), then

$$C^{(p)}_{-\infty}(\|.\|_{\lambda,p,q,s}) = \frac{1}{2^{p-1}\psi^p_{\lambda,p,q,s}(\frac{1}{2})} = 2(1+\lambda)^{\frac{p}{s}}(2^{\frac{s}{q}} + \lambda 2^{\frac{s}{q}})^{\frac{-p}{s}}. \quad \square$$

REMARK 1.

(i) In fact, take $q = 2$ in Example 3 and take $p = 2$, $q = 1$ or $p = \infty$, $q = 2$ in Example 4, these concrete Banach spaces which have been studied in the paper [8, 9], some classical constants such as von Neumann-Jordan constant $C_{NJ}(X)$ have been calculated for these spaces. Now, we get the exact values of $C^{(p)}_{-\infty}(\|.\|_{p,q,\lambda})$ for the general Banach space. However, there are some problems which remain unsolved: the exact values of $C^{(p)}_{-\infty}(\|.\|_{\omega,q})$ for the case $1 \leqslant q < 2$ and $C^{(p)}_{-\infty}(\|.\|_{p,q,\lambda})$ for the case $1 \leqslant q < 2 < p \leqslant \infty$, $\lambda \in (2^{\frac{1}{p}-\frac{1}{q}}, 2^{\frac{1}{2}-\frac{1}{q}})$.

(ii) In particular, take $p = 2$, $q = \infty$, $s = 2$ in Example 5, the concrete Banach space which has been studied in some papers [12, 22, 23, 24]. The generalized von Neumann-Jordan type constant $C^{(p)}_{-\infty}(\|.\|_{\lambda,p,q,s})$ is calculated for the general case in the paper. However, the exact value of $C^{(p)}_{-\infty}(\|.\|_{\lambda,p,q,s})$ for the case $1 \leqslant p < 2 < q \leqslant \infty$ remain undiscovered.

In the above Examples, the maximum value M_1 and M_2 always attains at $t = \frac{1}{2}$. However, there are some examples that maximum value M_2 attains not at $t = \frac{1}{2}$.

EXAMPLE 6. If the corresponding convex function is given by

$$\psi(t) = \begin{cases} \psi_2(t) \ \ (0 \leqslant t \leqslant \frac{1}{2}), \\ (2-\sqrt{2})t + \sqrt{2} - 1 \quad (\frac{1}{2} \leqslant t \leqslant 1), \end{cases}$$

then

$$C^{(p)}_{-\infty}(\|.\|_{\psi}) = 2^{2-p}M_2^p = 2^{2-p}(4-2\sqrt{2})^{\frac{p}{2}}.$$

Proof. Let $\psi(t) \in \Psi$ and the norm of $\|\|_\psi$ is

$$\|(a,b)\|_\psi = \begin{cases} \sqrt{|a|^2+|b|^2} \ (|a| \geqslant |b|), \\ (\sqrt{2}-1)|a|+|b| \quad ((|a| \leqslant |b|). \end{cases}$$

Since $\psi(t) \geqslant \psi_2(t)$, from Theorem 5, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) = 2^{2-p}M_2^p = 2^{2-p}\frac{\psi^p(\frac{\sqrt{2}}{2})}{\psi_2^p(\frac{\sqrt{2}}{2})} = 2^{2-p}(4-2\sqrt{2})^{\frac{p}{2}}. \quad \square$$

EXAMPLE 7. Let X be two-dimensional Cesàro space $ces_q^{(2)}$, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) = 2^{2-p}M_2^p = 2^{2-p}\max_{0 \leqslant t \leqslant 1}\frac{\psi^p(t)}{\psi_2^p(t)}.$$

Where

$$\psi(t) = \left[\frac{2^q(1-t)^q}{1+2^q} + \left(\frac{1-t}{(1+2^q)^{1/q}}+t\right)^q\right]^{\frac{1}{q}}$$

and

$$\psi_2(t) = ((1-t)^2+t^2)^{\frac{1}{2}}.$$

Proof. Let us first define

$$|x,y| = \left\|\left(\frac{2x}{(1+2^q)^{\frac{1}{q}}}, 2y\right)\right\|_{ces_q^{(2)}}$$

for $(x,y) \in \mathbb{R}^2$. $ces_q^{(2)}$ is isometrically isomorphic to $(\mathbb{R}^2, |.|)$ and $|.|$ is an absolute and normalized norm ([17]), and the corresponding convex function is given by

$$\psi(t) = \left[\frac{2^q(1-t)^q}{1+2^q} + \left(\frac{1-t}{(1+2^q)^{1/q}}+t\right)^q\right]^{\frac{1}{q}}.$$

Note that $\psi(t) \geqslant \psi_2(t)$ and Theorem 5, then

$$C_{-\infty}^{(p)}(\|.\|_\psi) = 2^{2-p}M_2^p = 2^{2-p}\max_{0 \leqslant t \leqslant 1}\frac{\psi^p(t)}{\psi_2^p(t)}. \quad \square$$

REMARK 2. In fact, the function $\frac{\psi(t)}{\psi_2(t)}$ attains the maximum at $t = \frac{1}{2}$ if and only if $q = 2$ for the two-dimensional Cesàro space $ces_q^{(2)}$.

As the application, we will present two practical examples [6, 11] which satisfy the conditions of Theorem 6 and Theorem 7, thus the exact value of the von Neumann-Jordan type constant $C_{-\infty}(X)$ coincides with their upper bound in some concrete Banach spaces.

EXAMPLE 8. Let $\frac{1}{2} \leqslant \beta \leqslant 1$, X_β^* is the Banach space and its corresponding function is

$$\psi_\beta^*(t) = \begin{cases} 1 - \frac{2\beta-1}{\beta}s, & if\ \ 0 \leqslant s \leqslant \frac{1}{2}, \\ \frac{1-\beta}{\beta} + \frac{2\beta-1}{\beta}s, & if\ \ \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

Then

(i) If $\frac{1}{2} \leqslant \beta \leqslant \frac{1}{\sqrt{2}}$, then $C_{-\infty}(\|.\|_{\psi_\beta^*}) = \frac{\beta^2+(1-\beta)^2}{\beta^2}$.

(ii) If $\frac{1}{\sqrt{2}} < \beta \leqslant 1$, then $C_{-\infty}(\|.\|_{\psi_\beta^*}) = 2(\beta^2+(1-\beta)^2)$.

Proof. Note that $\psi_\beta^*(t)$ is symmetric, therefore we discuss the function $g_1(s) = \frac{\psi_2(s)}{\psi_\beta^*(s)}$ on $[0,\frac{1}{2}]$, then

$$M_1 = \begin{cases} 1 \ \ (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]), \\ \frac{\psi_2(\frac{1}{2})}{\psi_\beta^*(\frac{1}{2})} = \sqrt{2}\beta \quad (\beta \in (\frac{1}{\sqrt{2}}, 1]). \end{cases}$$

Similarly, we discuss the function $g_2(s) = \frac{\psi_\beta^*(s)}{\psi_2(s)}$ on $[0,\frac{1}{2}]$, then

$$M_2 = \frac{\sqrt{(1-\beta)^2+\beta^2}}{\beta}.$$

(ii) If $\beta \in (\frac{1}{\sqrt{2}}, 1]$, since $\psi_\beta^*(1-\beta) = \psi_\beta^*(\beta)$ and $\frac{\psi_2(\frac{1}{2})}{\psi_\beta^*(\frac{1}{2})} = M_1 = \sqrt{2}\beta$. By Theorem 6, we have

$$C_{-\infty}(\|.\|_{\psi_\beta^*}) = M_1^2 M_2^2 = 2(\beta^2+(1-\beta)^2), \ \ \beta \in \left(\frac{1}{\sqrt{2}}, 1\right]$$

(i) For each $\frac{1}{2} \leqslant \beta \leqslant \frac{1}{\sqrt{2}}$, it is easy to check that X_β^* is isometrically isomorphic to $X_{\frac{1}{2\beta}}^*$ under the identification

$$X_\beta^* \ni (x_1,x_2) \leftrightarrow \frac{1}{2\beta}(x_1+x_2, x_1-x_2) \in X_{\frac{1}{2\beta}}^*,$$

since $\max\{|x_1+x_2|,|x_1-x_2|\} = |x_1|+|x_2|$ for all $x_1,x_2 \in \mathbb{R}$. If $\beta \in (\frac{1}{2}, \frac{1}{\sqrt{2}}]$, then $\frac{1}{2\beta} \in (\frac{1}{\sqrt{2}}, 1]$ and

$$\begin{aligned} C_{-\infty}(X_\beta^*) &= C_{-\infty}(X_{\frac{1}{2\beta}}^*) \\ &= 2\Big(\Big(\frac{1}{2\beta}\Big)^2 + \Big(1-\Big(\frac{1}{2\beta}\Big)\Big)^2\Big) \\ &= \frac{\beta^2+(1-\beta)^2}{\beta^2}. \quad \square \end{aligned}$$

EXAMPLE 9. Let $0 \leqslant c \leqslant 1$, the corresponding convex function is given by

$$\psi_c(t) = \max\left\{1 - ct, 1 - c + ct, 1 - \frac{c^2}{2}\right\} \text{ for } 0 \leqslant t \leqslant 1.$$

Then

(i) If $0 \leqslant c \leqslant -1+\sqrt{3}$, then $C_{-\infty}(\|.\|_{\psi_c}) = \frac{(2-c^2)^2}{2}$.

(ii) If $-1+\sqrt{3} < c \leqslant 1$, then $C_{-\infty}(\|.\|_{\psi_c}) = \frac{2(c^2-2c+2)^2}{(2-c^2)^2}$.

Proof. As the discussion in [6], if $0 \leqslant c \leqslant -1+\sqrt{3}$, then $\psi_c(t) \geqslant \psi_2(t)$. From Theorem 5, then

$$C_{-\infty}(\|.\|_{\psi_c}) = M_2^2 = \frac{(2-c^2)^2}{2}.$$

If $-1+\sqrt{3} < c \leqslant 1$, then

$$M_1^2 = \frac{\psi_2^2(t_1)}{\psi_c^2(t_1)} = \frac{2(c^2-2c+2)}{(2-c^2)}, \quad M_2^2 = \frac{\psi_c^2(t_2)}{\psi_2^2(t_2)} = c^2 - 2c + 2,$$

where $t_1 = \frac{c}{2}, t_2 = \frac{1-c}{2-c}$, which satisfy the condition $(1-t_1)(1-t_2) = \frac{1}{2}$ in Theorem 7, then

$$C_{-\infty}(\|.\|_{\psi_c}) = M_1^2 M_2^2 = \frac{2(c^2-2c+2)^2}{(2-c^2)^2}. \quad \square$$

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REFERENCES

[1] F. BONSALL AND J. DUNCAN, *Numerical Ranges II*, London Mathematical Society Lecture Notes Series, **10**, Cambridge Univ. Press, New York, 1973.

[2] J. CLARKSON, *Uniform convex spaces*, Trans. Amer. Math. Soc., **40** (1936), 396–414.

[3] J. CLARKSON, *The von Neumann-Jordan constant for the Lebesgue space*, Ann. of Math., **38** (1937), 114–115.

[4] Y. CUI AND H. HUDZIK, *Some geometric properties related to fixed point theory in Cesàro spaces*, Collect. Math., **50** (1999), 277–288.

[5] Y. CUI, L. JIE AND R. PLUCIENNIK, *Local uniform nonsquareness in Cesàro sequence spaces*, Comment. Math., **27** (1997), 47–58.

[6] T. IKEDA AND M. KATO, *Notes on von Neumann-Jordan and James constants for absolute norms on* $\mathbb{R}^2$, Mediterr. J. Math., **11** (2014), 633–642.

[7] A. JIMÉNEZ-MELADO, E. LLORENS-FUSTER AND S. SAEJUNG, *The von Newman-Jordan constant, weak orthogonality and normal structure in Banach spaces*, Proc. Am. Math. Soc., **134** (2006), 355–364.

[8] M. KATO AND L. MALIGRANDA, *On James and von Neumann-Jordan constants of Lorentz sequence spaces*, J. Math. Anal. Appl., **258** (2001), 457–465.

[9] M. KATO, L. MALIGRANDA AND Y. TAKAHASHI, *On James and Jordan-von Neumann constants and normal structure coefficient of Banach spaces*, Studia Math., **144** (2001), 275–295.

[10] L. MALIGRANDA, N. PETROT AND S. SUANTAI, *On the James constant and B-convexity of Cesàro and Cesàro-Orlicz sequence spaces*, J. Math. Anal. Appl., **326** (2007), 312–331.

[11] H. MIZUGUCHI, *The Dunkl-Williams constant of symmetric octagonal norms on R^2*, Nihonkai Math. J., **23** (2012), 93–113.

[12] H. MIZUGUCHI AND K. SAITO, *Some geometric constants of absolute normalized norms on $\mathbb{R}^2$*, Ann. Funct. Anal., **2** (2011), 22–33.

[13] H. MIZUGUCHI AND K. SAITO, *On the upper bound of some geometric constants of absolute normalized norms on $\mathbb{R}^2$*, Mediterr. J. Math., **13** (2016), 309–322.

[14] H. L. NIKOLOVA AND L. E. PERSSON, *Some properties of X^p spaces*, in: J. Musielak, H. Hudzik, R. Urbanski (eds.), Function spaces, Teubner-Texte zur Matematik **120** (1991), 174–185.

[15] L. PERSSON (Eds), *Some elementary inequalities in connection with X^p spaces*, Constructive theory of functions, Publishing House of the Bulgarian Academy of Sciences, (1988), 367–376.

[16] S. SAEJUNG, *On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property*, J. Math. Anal. Appl., **323** (2006), 1018–1024.

[17] S. SAEJUNG, *Another look at Cesàro sequence spaces*, J. Math. Anal. Appl., **366** (2010), 530–537.

[18] W. SANHAN AND S. SUANTAI, *Some geometric properties of Cesàro sequence space*, Kyungpook Math. J., **43** (2003), 191–197.

[19] K. SAITO, M. KATO AND Y. TAKAHASHI, *Von Neumann-Jordan constant of absolute normalized norms on C^2*, J. Math. Anal. Appl., **244** (2000), 515–532.

[20] J. SHUE, *On the Cesàro sequence spaces*, Tamkang J. Math., **1** (1970), 143–150.

[21] Y. TAKAHASHI, *Some geometric constants of Banach spaces – a unified approach*, in: Banach and Function Spaces II, Proc. International Symposium (ISBFS 2006) held in Kitakyushu (Sept. 14–17, 2006), Edited by Mikio Kato and Lech Maligranda, Yokohama Publ., Yokohama, (2008), 191–220.

[22] C. YANG, *A note on Jordan-Von Neumann constant for $Z_{p,q}$ space*, Journal of Mathematical Inequalities, **2** (2015), 499–504.

[23] C. YANG AND H. LI, *An inequality between Jordan-von Neumann constant and James constant*, Appl. Math. Lett., **23** (2010), 277–281.

[24] C. YANG AND H. LI, *An inequality on Jordan-von Neumann constant and James constant on $Z_{p,q}$ space*, Journal of Mathematical Inequalities, **7** (2013), 97–102.

[25] C. YANG AND Y. WANG, *Some properties on von Neumann-Jordan type constant of Banach spaces*, Acta Mathematica Scientia, **32A (1)** (2012), 212–221.

[26] Z. ZUO, *The generalized von Neumann-Jordan type constant and fixed points for multivalued nonexpansive mappings*, Science Asia, **45** (2019), 292–297.

[27] Z. ZUO, *The normal structure and parametrized von Neumann-Jordan type constant*, Acta Mathematica Scientia, **63** (2020), 655–660.

[28] Z. ZUO AND C. TANG, *Jordan-von Neumann type constant and fixed points for multivalued nonexpansive mappings*, Journal of Mathematical Inequalities, **10** (2016), 649–657.

[29] Z. ZUO AND C. TANG, *On James and Jordan von-Neumann type constants and the normal structure in Banach spaces*, Topological Methods in Nonlinear Analysis, **49** (2017), 615–623.

[30] Z. ZUO AND C. TANG, *On Jordan-von Neumann type constants and normal structure in Banach spaces*, Acta Mathematica Sinica, **60** (2017), 383–388.

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