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ON WEIGHTED HARDY INEQUALITY WITH TWO–DIMENSIONAL RECTANGULAR OPERATOR — EXTENSION OF THE E. SAWYER THEOREM

VLADIMIR D. STEPANOV* AND ELENA P. USHAKOVA

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Abstract. A characterization is obtained for those pairs of weights v and w on $\mathbb{R}^2_+$, for which the two–dimensional rectangular integration operator is bounded from a weighted Lebesgue space $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$ for $1 < p < q < \infty$, which is an essential complement to E. Sawyer's result [13] given for $1 < p \leqslant q < \infty$. Besides, we demonstrate that the E. Sawyer theorem is actual if $p = q$ only, for $p < q$ the criterion is the finiteness of the Muckenhoupt-type constant. The case $q < p$ is also discussed.

1. Introduction

Let $n \in \mathbb{N}$. For Lebesgue measurable functions $f(y_1,\dots,y_n)$ on $\mathbb{R}^n_+ := (0,\infty)^n$ the n–dimensional rectangular integration operator I_n is given by the formula

$$I_n f(x_1,\dots,x_n) := \int_0^{x_1}\dots\int_0^{x_n} f(y_1,\dots,y_n)\,dy_1\dots dy_n \qquad (x_1,\dots,x_n > 0).$$

The dual transformation I_n^* has the form

$$I_n^* f(x_1,\dots,x_n) := \int_{x_1}^{\infty}\dots\int_{x_n}^{\infty} f(y_1,\dots,y_n)\,dy_1\dots dy_n \qquad (x_1,\dots,x_n > 0).$$

Let $1 < p,q < \infty$ and $v,w \geqslant 0$ be weight functions on $\mathbb{R}^n_+$. Consider Hardy's inequality

$$\left(\int_{\mathbb{R}^n_+} (I_n f)^q w\right)^{\frac{1}{q}} \leqslant C_n \left(\int_{\mathbb{R}^n_+} f^p v\right)^{\frac{1}{p}} \qquad (f \geqslant 0) \tag{1}$$

on the cone of non–negative functions in weighted Lebesgue space $L^p_v(\mathbb{R}^n_+)$. The constant $C_n > 0$ in (1) is assumed to be the least possible and independent of f. For a fixed weight v and a parameter $p > 1$ the space $L^p_v(\mathbb{R}^n_+)$ consists of all measurable on $\mathbb{R}^n_+$ functions f such that $\int_{\mathbb{R}^n_+} |f|^p v < \infty$.

The problem of characterizing the inequality (1) is well known and has been considered by many authors (see [1, 3, 7, 11, 13, 15, 16] and references therein). The

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* Corresponding author.

one–dimensional case of this inequality has been completely studied (see [6, 4, 5, 12]). However, for $n > 1$ difficulties arise, preventing characterizing (1) without additional restrictions on v and w. Nevertheless, E. Sawyer's result is well known for arbitrary v, w in the case $1 < p \leqslant q < \infty$. To formulate it we denote $p' := p/(p-1)$ and $\sigma := v^{1-p'}$.

THEOREM 1. [13, Theorem 1A] *Let $n = 2$ and $1 < p \leqslant q < \infty$. The inequality* (1) *holds for all measurable non-negative functions f on $\mathbb{R}^2_+$ if and only if*

$$A_1 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left[I_2^* w(t_1,t_2)\right]^{\frac{1}{q}} \left[I_2\sigma(t_1,t_2)\right]^{\frac{1}{p'}} < \infty, \tag{2}$$

$$A_2 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_0^{t_1}\int_0^{t_2} (I_2\sigma)^q w\right)^{\frac{1}{q}} \left[I_2\sigma(t_1,t_2)\right]^{-\frac{1}{p}} < \infty, \tag{3}$$

$$A_3 := \sup_{(t_1,t_2)\in\mathbb{R}^2_+} \left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} (I_2^* w)^{p'} \sigma\right)^{\frac{1}{p'}} \left[I_2^* w(t_1,t_2)\right]^{-\frac{1}{q'}} < \infty, \tag{4}$$

and $C_2 \approx A_1 + A_2 + A_3$ with equivalence constants depending on parameters p and q only.

Note that in one–dimensional case the analogs of the conditions (2)–(4) are equivalent to each other [2]. For $n = 2$ this, generally speaking, is not true. Moreover, as shown in [13, § 4] for $p = q = 2$, no two of the conditions (2)–(4) guarantee (1). However, the construction of the second counterexample in [13, § 4] fails in the case $p < q$.

The purpose of this paper is to obtain new conditions for the fulfilment of Hardy's inequality (1) for $n = 2$ and $1 < p \neq q < \infty$. Relatively to the case $1 < p < q < \infty$, the solution to this problem is contained in Theorem 2 in a criterion form. In Theorem 3 we give separate necessary condition and sufficient condition on v and w, when (1) is true for $n = 2$ and $1 < q < p < \infty$. Recall that the criterion for (1) when $n = 2$ and $1 < p \leqslant q < \infty$, established in [13], is that the sum of three independent functionals is bounded (see Theorem 1). It is proven in Theorem 2 that for $1 < p < q < \infty$ the inequality (1) is characterized by only one Muckenhoupt-type functional.

Analogs of Theorems 2 and 3 are also valid for the dual operator I_2^* and mixed Hardy operators (see [13, Remark 1] for details).

In § 3, for completeness, we present known results about the operator I_n for arbitrary n, provided that at least one of the two weight functions in (1) is factorizable, that is, can be represented as a product of n one–dimensional functions.

Since $A_1 \leqslant C_2$, we may and shall assume that $I_2\sigma(x,y) < \infty$ and $I_2^* w(x,y) < \infty$ for any $(x,y) \in \mathbb{R}^2_+$. In particular, $\sigma, w \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$.

Throughout the work, the notation of the form $\Phi \lesssim \Psi$ means that the relation $\Phi \leqslant c\Psi$ holds with some constant $c > 0$, independent of Φ and Ψ. We write $\Phi \approx \Psi$ in the case of $\Phi \lesssim \Psi \lesssim \Phi$. The symbols $\mathbb{Z}$ and $\mathbb{N}$ are used for denoting the sets of integers and natural numbers, respectively. The characteristic function of the subset $E \subset \mathbb{R}^n_+$ is denoted by χ_E. Symbols $:=$ and $=:$ are used to define new values.

2. Main result

Denote $A := A_1$,

$$\alpha(p,q) := \frac{p^2(q-1)}{q-p}, \qquad p < q;$$

and

$$\begin{aligned} B :=& \left(\int_{\mathbb{R}^2_+} d_y \left[I_2\sigma(x,y) \right]^{\frac{r}{p'}} d_x \left(-\left[I_2^* w(x,y) \right]^{\frac{r}{q}} \right) \right)^{\frac{1}{r}} \\ =& \left(\int_{\mathbb{R}^2_+} \left[I_2\sigma(x,y) \right]^{\frac{r}{p'}} d_x d_y \left[I_2^* w(x,y) \right]^{\frac{r}{q}} \right)^{\frac{1}{r}} \\ =& \left(\int_{\mathbb{R}^2_+} \left[I_2^* w(x,y) \right]^{\frac{r}{q}} d_x d_y \left[I_2\sigma(x,y) \right]^{\frac{r}{p'}} \right)^{\frac{1}{r}}, \qquad q < p, \end{aligned}$$

where the last two equalities follow by integration by parts.

We start with some auxiliary technical statements.

LEMMA 1. *Let $0 \leqslant a < b < \infty$ and $0 \leqslant c < d < \infty$. If $1 < p < q < \infty$ then*

$$\mathbf{V}_{(a,b)\times(c,d)}(w,\sigma) := \int_a^b \int_c^d w(x,y) \left(\int_a^x \int_c^y \sigma \right)^q dy\,dx \leqslant \alpha(p,q) \left(\int_a^b \int_c^d \sigma \right)^{\frac{q}{p}} A^q.$$

Proof. Assume $1 < p < q < \infty$ and write

$$\begin{aligned} \mathbf{V}_{(a,b)\times(c,d)}(w,\sigma) =& \int_a^b \int_c^d \left(\int_a^x \int_c^y \sigma \right)^q d_y \left[-\int_y^d w(x,t)\,dt \right] dx \\ =& q \int_a^b \int_c^d \left(\int_a^x \int_c^y \sigma \right)^{q-1} \left(\int_a^x \sigma(s,y)\,ds \right) \left(\int_y^d w(x,t)\,dt \right) dy\,dx \\ =& q \int_c^d \int_a^b \left(\int_a^x \int_c^y \sigma \right)^{q-1} \left(\int_a^x \sigma(s,y)\,ds \right) d_x \left[-\int_x^b \int_y^d w \right] dy \\ =& q \int_a^b \int_c^d \Bigg\{ (q-1) \left(\int_a^x \int_c^y \sigma \right)^{q-2} \left(\int_a^x \sigma(s,y)\,ds \right) \left(\int_c^y \sigma(x,t)\,dt \right) \\ &+ \left(\int_a^x \int_c^y \sigma \right)^{q-1} \sigma(x,y) \Bigg\} \left(\int_x^b \int_y^d w \right) dx\,dy. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{V}_{(a,b)\times(c,d)}(w,\sigma) \leqslant qA^q \int_a^b \int_c^d \Bigg\{ &(q-1) \left(\int_a^x \int_c^y \sigma \right)^{\frac{q}{p}-2} \left(\int_a^x \sigma(s,y)\,ds \right) \left(\int_c^y \sigma(x,t)\,dt \right) \\ &+ \left(\int_a^x \int_c^y \sigma \right)^{\frac{q}{p}-1} \sigma(x,y) \Bigg\} dx\,dy. \end{aligned}$$

The assertion of the lemma follows from the chain of estimates:

$$\begin{aligned}
&q\int_a^b\int_c^d\Big\{(q-1)\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-2}\Big(\int_a^x\sigma(s,y)\,ds\Big)\Big(\int_c^y\sigma(x,t)\,dt\Big)\\
&\quad+\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-1}\sigma(x,y)\Big\}\,dx\,dy\\
=&p\int_a^b\int_c^d\Big\{\frac{q}{p}\Big(\frac{q}{p}-1+\frac{q}{p'}\Big)\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-2}\Big(\int_a^x\sigma(s,y)\,ds\Big)\Big(\int_c^y\sigma(x,t)\,dt\Big)\\
&\quad+\frac{q}{p}\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-1}\sigma(x,y)\Big\}\,dx\,dy\\
\leqslant&p\int_a^b\int_c^d\Big\{\frac{q}{p}\Big(\frac{q}{p}-1\Big)\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-2}\Big(\int_a^x\sigma(s,y)\,ds\Big)\Big(\int_c^y\sigma(x,t)\,dt\Big)\\
&\quad+\frac{q}{p}\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-1}\sigma(x,y)\Big\}\,dx\,dy\\
&\quad+\frac{p^2q^2}{p'q(q-p)}\int_a^b\int_c^d\Big\{\frac{q}{p}\Big(\frac{q}{p}-1\Big)\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-2}\Big(\int_a^x\sigma(s,y)\,ds\Big)\Big(\int_c^y\sigma(x,t)\,dt\Big)\\
&\quad+\frac{q}{p}\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-1}\sigma(x,y)\Big\}\,dx\,dy\\
=&\Big[p+\frac{pq(p-1)}{q-p}\Big]\int_a^b\int_c^d\Big\{\frac{q}{p}\Big(\frac{q}{p}-1\Big)\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-2}\Big(\int_a^x\sigma(s,y)\,ds\Big)\Big(\int_c^y\sigma(x,t)\,dt\Big)\\
&\quad+\frac{q}{p}\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-1}\sigma(x,y)\Big\}\,dx\,dy\\
=&\alpha(p,q)\int_a^b\int_c^d\Big\{\frac{q}{p}\Big(\frac{q}{p}-1\Big)\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-2}\Big(\int_a^x\sigma(s,y)\,ds\Big)\Big(\int_c^y\sigma(x,t)\,dt\Big)\\
&\quad+\frac{q}{p}\Big(\int_a^x\int_c^y\sigma\Big)^{\frac{q}{p}-1}\sigma(x,y)\Big\}\,dx\,dy=\alpha(p,q)\Big(\int_a^b\int_c^d\sigma\Big)^{\frac{q}{p}}.\quad\square
\end{aligned}$$

A similar statement holds with the opposite integration of w and the proof follows by the same arguments.

LEMMA 2. *Let* $0<a<b\leqslant\infty$ *and* $0<c<d\leqslant\infty$. *If* $1<p<q<\infty$ *then*

$$\mathbf{W}_{(a,b)\times(c,d)}(\sigma,w):=\int_a^b\int_c^d\sigma(x,y)\Big(\int_x^b\int_y^d w\Big)^{p'}dy\,dx\leqslant\alpha(q',p')\Big(\int_a^b\int_c^d w\Big)^{\frac{p'}{q'}}A^{p'}.$$

Introduce notations: $\alpha:=\alpha(p,q)$, $\alpha':=\alpha(q',p')$,

$$\mathbb{C}_{\alpha,\alpha'}:=3^{3q}\Big[\Big(\frac{2^4}{3}\Big)^q\max\Big\{\alpha,2q(q')^{\frac{q}{p'}}\Big\}\Big(\frac{2^{p-1}}{2^{p-1}-1}\Big)^{\frac{q}{p}}+3^{\frac{1}{p}+\frac{1}{q'}}(\alpha')^{\frac{1}{p'}}\Big].$$

The main result of this paper for $1<p<q<\infty$ is the following statement.

THEOREM 2. *Let $1 < p < q < \infty$. Then the inequality*

$$\left(\int_{\mathbb{R}_+^2} (I_2 f)^q w\right)^{\frac{1}{q}} \leqslant C_2 \left(\int_{\mathbb{R}_+^2} f^p v\right)^{\frac{1}{p}} \qquad (f \geqslant 0) \tag{5}$$

holds if and only if $A < \infty$. Besides,

$$A \leqslant C_2 \leqslant \mathbb{C}_{\alpha,\alpha'} A.$$

Proof. The necessity part of the statement follows from Theorem 1 (by substituting $f = \chi_{(0,s)\times(0,t)}$ into the initial inequality (5)). To establish the sufficiency, similarly to how it was done in E. Sawyer's paper [13] for the case $1 < p \leqslant q < \infty$, we show that the conditions of the theorem are sufficient, limiting ourselves to proving the inequality (5) on the subclass $M \subset L_v^p(\mathbb{R}_+^2)$ of all functions $f \geqslant 0$ bounded on $\mathbb{R}_+^2$ with compact supports contained in the set $\{I_2\sigma > 0\}$. Then the inequality (5) for arbitrary $0 \leqslant f \in L_v^p(\mathbb{R}_+^2)$ follows by the standard arguments.

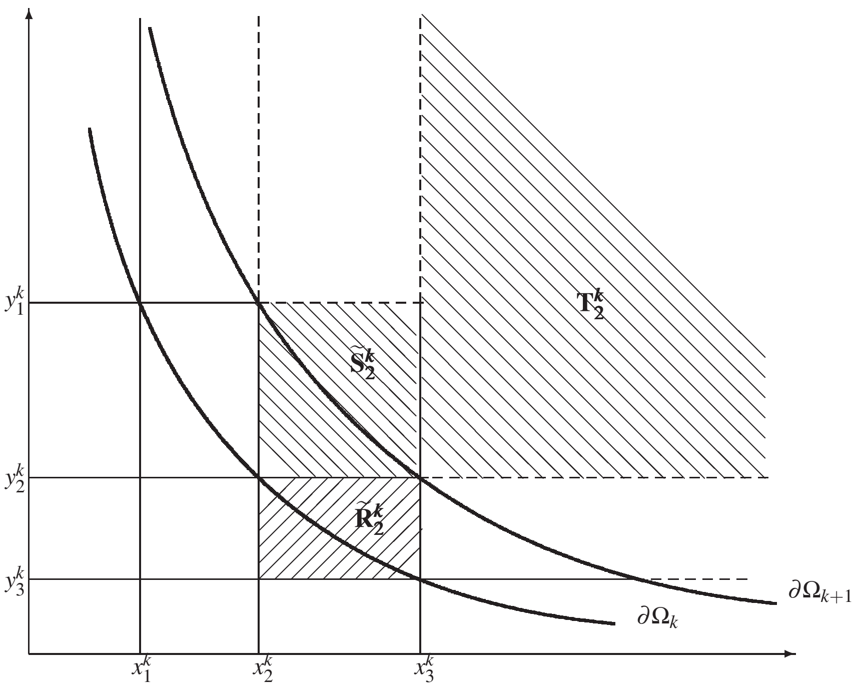

Fig. 1

Suppose $A < \infty$ and fix $f \in M$. In analogy with the proof of [13, Theorem 1A], we define the domains

$$\Omega_k := \left\{ I_2 f > 3^k \right\}, \qquad k \in \mathbb{Z}.$$

Then, by our assumptions on f, there exists $K \in \mathbb{Z}$ such that $\Omega_k \neq \varnothing$ for $k \leqslant K$,

$\Omega_k = \varnothing$ for $k > K$, $\bigcup_{k\in\mathbb{Z}} \Omega_k = \mathbb{R}^2_+$ and

$$3^k < I_2 f(x,y) \leqslant 3^{k+1}, \qquad k \leqslant K, \qquad (x,y) \in (\Omega_k \setminus \Omega_{k+1}).$$

We can write down that

$$\int_{\mathbb{R}^2_+} (I_2 f)^q w = \sum_{k\leqslant K-2} \int_{\Omega_{k+2}\setminus\Omega_{k+3}} (I_2 f)^q w \leqslant 3^{3q} \sum_{k\leqslant K-2} 3^{kq} |\Omega_{k+2}\setminus\Omega_{k+3}|_w,$$

where $|\Omega_{k+2}\setminus\Omega_{k+3}|_w := \int_{\Omega_{k+2}\setminus\Omega_{k+3}} w$ and $\Omega_K \setminus \Omega_{K+1} = \Omega_K$, since Ω_{K+1} is empty.

Next, as in the proof of [13, Theorem 1A], we introduce rectangles. For this, we fix k such that $\Omega_{k+1} \neq \varnothing$, and choose points (x_j^k, y_j^k), $1 \leqslant j \leqslant N = N_k$, lying on the boundary $\partial\Omega_k$ in such a way to have (x_j^k, y_{j-1}^k) belonging to $\partial\Omega_{k+1}$ for $2 \leqslant j \leqslant N$ and $\Omega_{k+1} \subset \bigcup_{j=1}^N S_j^k$, where S_j^k is a rectangle of the form $(x_j^k,\infty)\times(y_j^k,\infty)$. We also define rectangles $\widetilde{S}_j^k = (x_j^k, x_{j+1}^k)\times(y_j^k, y_{j-1}^k)$ for $1 \leqslant j \leqslant N$ and $R_j^k = (0, x_{j+1}^k)\times(0, y_j^k)$, $\widetilde{R}_j^k = (x_j^k, x_{j+1}^k)\times(y_{j+1}^k, y_j^k)$ and $T_j^k = (x_{j+1}^k,\infty)\times(y_j^k,\infty)$ for $1 \leqslant j \leqslant N-1$. Put $y_0^k = x_{N+1}^k = \infty$ (see Figure 1).

Now we choose the sets $E_j^k \subset T_j^k$ so that $E_j^k \cap E_i^k = \varnothing$ for $j \neq i$ and $\bigcup_j E_j^k = (\Omega_{k+2}\setminus\Omega_{k+3}) \cap \left(\bigcup_j T_j^k\right)$. Since $\Omega_{k+2}\setminus\Omega_{k+3} \subset \Omega_{k+1} \subset \left(\bigcup_j T_j^k\right) \cup \left(\bigcup_j \widetilde{S}_j^k\right)$, then

$$3^{-3q} \int_{\mathbb{R}^2_+} (I_2 f)^q w \leqslant \sum_{k,j} 3^{kq} \big|E_j^k\big|_w + \sum_{k,j} 3^{kq} \big|\widetilde{S}_j^k \cap (\Omega_{k+2} - \Omega_{k+3})\big|_w =: I + II. \tag{6}$$

To estimate II we denote $D_j^k := \widetilde{S}_j^k \setminus \Omega_{k+3}$ and turn to the reasoning by E. Sawyer on page 6 in [13], from which it follows that

$$I_2(\chi_{D_j^k} f)(x,y) > 3^k \quad \text{if} \quad (x,y) \in \widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3}).$$

Further, according to [13, p. 6],

$$\begin{aligned}
\big|\widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3})\big|_w &\leqslant 3^{-k} \int_{\widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3})} I_2(\chi_{D_j^k} f)(x,y) w(x,y)\, dxdy \\
&\leqslant 3^{-k} \int_{D_j^k} \left(\int_{x_j^k}^x \int_{y_j^k}^y f \right) w(x,y)\, dxdy \\
&= 3^{-k} \int_{D_j^k} f(s,t) \left(\int_s^\infty \int_t^\infty w\chi_{D_j^k} \right) dsdt \\
&\leqslant 3^{-k} \left(\int_{D_j^k} f^p v \right)^{\frac{1}{p}} \left(\int_{D_j^k} \sigma(s,t) \left(\int_s^\infty \int_t^\infty w\chi_{D_j^k} \right)^{p'} dsdt \right)^{\frac{1}{p'}}.
\end{aligned}$$

By applying Lemma 2 to $(a,b)\times(c,d) = \widetilde{S}_j^k$, we obtain for $p < q$ that

$$\mathbf{W}_{\widetilde{S}_j^k}(\sigma\chi_{D_j^k}, w\chi_{D_j^k}) = \int_{D_j^k} \sigma(s,t) \left(\int_s^\infty \int_t^\infty w\chi_{D_j^k} \right)^{p'} dsdt \leqslant \alpha' A^{p'} \big|D_j^k\big|_w^{\frac{p'}{q'}}.$$

From this and Hölder's inequality with q and q'

$$(\alpha')^{-\frac{1}{p'}}\cdot II\leqslant A\sum_{k,j}3^{k(q-1)}\Big(\int_{D_j^k}f^pv\Big)^{\frac{1}{p}}|D_j^k|_w^{\frac{1}{q'}}\leqslant A\Big(\sum_{k,j}3^{kq}|D_j^k|_w\Big)^{\frac{1}{q'}}\Big[\sum_{k,j}\Big(\int_{D_j^k}f^pv\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}}.$$

Thus, Jensen's inequality with p/q and the estimate $\sum_{k,j}\chi_{D_j^k}\leqslant\sum_k\chi_{\Omega_k\setminus\Omega_{k+3}}\leqslant 3$ entail

$$II\leqslant 3^{\frac{1}{p}}(\alpha')^{\frac{1}{p'}+\frac{1}{q'}}A\Big(\int_{\mathbb{R}^2_+}f^pv\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^2_+}(I_2f)^qw\Big)^{\frac{1}{q'}}. \tag{7}$$

To evaluate I in (6), in accordance with the proof of [13, Theorem 1A, pp. 8–9], we put $g\sigma:=f$ and write:

$$3^qI=\sum_{k,j}3^{(k+1)q}|E_j^k|_w=\sum_{k,j}|E_j^k|_w\Big(\int_{R_j^k}f\Big)^q=\sum_{k,j}|E_j^k|_w|R_j^k|_\sigma^q\Big(\frac{1}{|R_j^k|_\sigma}\int_{R_j^k}g\sigma\Big)^q. \tag{8}$$

For an integer l, by Γ_l we denote the set of pairs (k,j) such that $\big|E_j^k\big|_w>0$ and

$$2^l<\frac{1}{|R_j^k|_\sigma}\int_{R_j^k}g\sigma\leqslant 2^{l+1},\qquad (k,j)\in\Gamma_l.$$

For fixed l the family $\{U_i^l\}_{i=1}^{i(l)}$ consists of maximal rectangles from the collection $\{R_j^k\}_{(k,j)\in\Gamma_l}$, that is, each R_j^k with $(k,j)\in\Gamma_l$ is contained in some U_i^l (or coincides with it). In [13, p. 8] it is shown that $\widetilde{U}_i^l$ are disjoint for fixed l, where we denote $\widetilde{U}_i^l=\widetilde{R}_i^l$ if $U_i^l=R_i^l$.

Let χ_i^l be the characteristic function of the union of the sets E_j^k over all $(k,j)\in\Gamma_l$ such that $R_j^k\subset U_i^l$. Further, following [13, (2.13)], we arrive to

$$\begin{aligned}\sum_{(k,j)\in\Gamma_l}|E_j^k|_w|R_j^k|_\sigma^q&=\sum_{i=1}^{i(l)}\sum_{(k,j):\,R_j^k\subset U_i^l}\int_{E_j^k}w\big[I_2(\chi_{U_i^l}\sigma)(x_{j+1}^k,y_j^k)\big]^q\\&\leqslant\sum_{i=1}^{i(l)}\int_{\mathbb{R}^2_+}\chi_i^lw\big[I_2(\chi_{U_i^l}\sigma)\big]^q.\end{aligned} \tag{9}$$

In analogy with [13, (2.8)], let us first show the validity of the estimate

$$\int_{\mathbb{R}^2_+}\chi_i^lw\big[I_2(\chi_{U_i^l}\sigma)\big]^q\leqslant\max\Big\{\alpha,2q(q')^{\frac{q}{p'}}\Big\}A^q|U_i^l|_\sigma^{\frac{q}{p}} \tag{10}$$

for $U_i^l=(0,a)\times(0,b)$. In view of Lemma 1,

$$\mathbf{V}_{U_i^l}=\int_{U_i^l}\chi_i^lw(I_2\sigma)^q\leqslant\alpha A^q|U_i^l|_\sigma^{\frac{q}{p}}.$$

On the rectangles $(a,\infty)\times(b,\infty)$, $(0,a)\times(b,\infty)$ and $(a,\infty)\times(0,b)$ analogous estimates were established in [13, (2.8)] (see also [6, §1.3.2]). Therefore, (10) is true.

Continuing (9), we obtain, using [13, (2.11)] and Jensen's inequality with p/q:

$$\sum_{(k,j)\in\Gamma_l}|E_j^k|_w|R_j^k|_\sigma^q \leqslant \max\Big\{\alpha, 2q(q')^{\frac{q}{p'}}\Big\}A^q\sum_i|U_i^l|_\sigma^{\frac{q}{p}}$$
$$\leqslant \max\Big\{\alpha, 2q(q')^{\frac{q}{p'}}\Big\}A^q\sum_i\Big(2^{-l+3}\int_{\widetilde{U}_i^l\cap\{g>2^{l-3}\}}g\sigma\Big)^{\frac{q}{p}}$$
$$\leqslant 2^{\frac{3q}{p}}\max\Big\{\alpha, 2q(q')^{\frac{q}{p'}}\Big\}A^q 2^{-\frac{lq}{p}}\left(\int_{\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}}.$$

The last estimate is valid due to the fact that for fixed l the rectangles $\widetilde{U}_i^l$ do not intersect (see [13, p. 8]). Combining this with (8) and taking into account the relation

$$\sum_l 2^{l(p-1)}\chi_{\{g>2^{l-3}\}} \leqslant \frac{3^{p-1}2^{p-1}}{2^{p-1}-1}g^{p-1} \qquad \text{for} \quad p>1,$$

we obtain since $q>p$:

$$I \leqslant \Big(\frac{2}{3}\Big)^q\sum_l 2^{lq}\sum_{(k,j)\in\Gamma_l}|E_j^k|_w|R_j^k|_\sigma^q$$
$$\leqslant 2^{\frac{3q}{p}}\Big(\frac{2}{3}\Big)^q\max\Big\{\alpha, 2q(q')^{\frac{q}{p'}}\Big\}A^q\sum_l 2^{lq}\left(2^{-l}\int_{\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}}$$
$$\leqslant 2^{\frac{3q}{p}}\Big(\frac{2}{3}\Big)^q\max\Big\{\alpha, 2q(q')^{\frac{q}{p'}}\Big\}A^q\left(\sum_l 2^{l(p-1)}\int_{\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}}$$
$$\leqslant \Big(\frac{2^4}{3}\Big)^q\max\Big\{\alpha, 2q(q')^{\frac{q}{p'}}\Big\}\Big(\frac{2^{p-1}}{2^{p-1}-1}\Big)^{\frac{q}{p}}A^q\left(\int_{\mathbb{R}_+^2}f^p v\right)^{\frac{q}{p}}. \qquad (11)$$

The (11) and (7) lead to the required upper bound, where the final upper estimate

$$\int_{\mathbb{R}_+^2}(I_2 f)^q w \leqslant C\left(\int_{\mathbb{R}_+^2}f^p v\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}_+^2}(I_2 f)^q w\right)^{\frac{1}{q'}} + C^q\left(\int_{\mathbb{R}_+^2}f^p v\right)^{\frac{q}{p}}$$

follows from (6) combined with (7) and (11), where $C = A\cdot\mathbb{C}_{\alpha,\alpha'}$. □

REMARK 1. Recall that in the case $p\leqslant q$ the best constant C_2 of the two–dimensional inequality (5) is equivalent to $\sum_{i=1}^3 A_i$ (see Theorem 1). However, by virtue of the statements of Lemmas 1 and 2, for $p<q$ the following inequalities take place:

$$A_1 \leqslant C_2 \leqslant \mathbb{C}_{1,1}\big[A_1+A_2+A_3\big] \leqslant \mathbb{C}_{1,1}\big[1+\alpha(p,q)^{\frac{1}{q}}+\alpha(q',p')^{\frac{1}{p'}}\big]A_1. \qquad (12)$$

Moreover,

$$\lim_{p\uparrow q}\big[\alpha(p,q)+\alpha(q',p')\big] = \infty.$$

Thus, the last estimate in (12) and the upper bound in Theorem 2 have blow-up for $p\uparrow q$.

The one–dimensional analog of the condition (2) is the boundedness of the Muckenhoupt constant [8], of the condition (3) – the boundedness of the Tomaselli functional [14, definition (11)], and the analog of the constant B is the Maz'ya–Rozin functional [6, § 1.3.2]. The constants have been generalized to the scales of equivalent conditions in [10] (see also [2] for the case $p \leqslant q$). In the following theorem we find a sufficient condition for the inequality (5) to hold in the case $q < p$, having the form (13), where B_v is a two–dimensional analog of the constant $\mathscr{B}^{(1)}_{MR}(\frac{1}{r})$ from [10] in the one–dimensional case. A necessary condition is given with the functional B.

THEOREM 3. *Let* $1 < q < p < \infty$. *Then the inequality* (5) *holds if*

$$B_v := \left(\int_{\mathbb{R}^2_+} \sigma(u,z)\left(\int_u^\infty \int_z^\infty (I_2\sigma)^{q-1} w\right)^{\frac{r}{q}} du\,dz\right)^{\frac{1}{r}} < \infty, \tag{13}$$

where

$$C_2 \leqslant 3^{3q}2^{2q}B_v.$$

If (5) *is valid, then* $B < \infty$*, moreover,*

$$2^{-\frac{1}{p'}}\left(\frac{q}{r}\right)^{\frac{1}{q}}\left(\frac{p'}{r}\right)^{\frac{1}{p'}} B \leqslant C_2.$$

Proof. (*Sufficiency*) We apply Sawyer's scheme of partitioning $\mathbb{R}^2_+$ into rectangles from the proof of Theorem 2. Compared to Figure 1, Figure 2 below has a rectangle $Q^k_j = (0,x^k_j) \times (0,y^k_j)$ added.

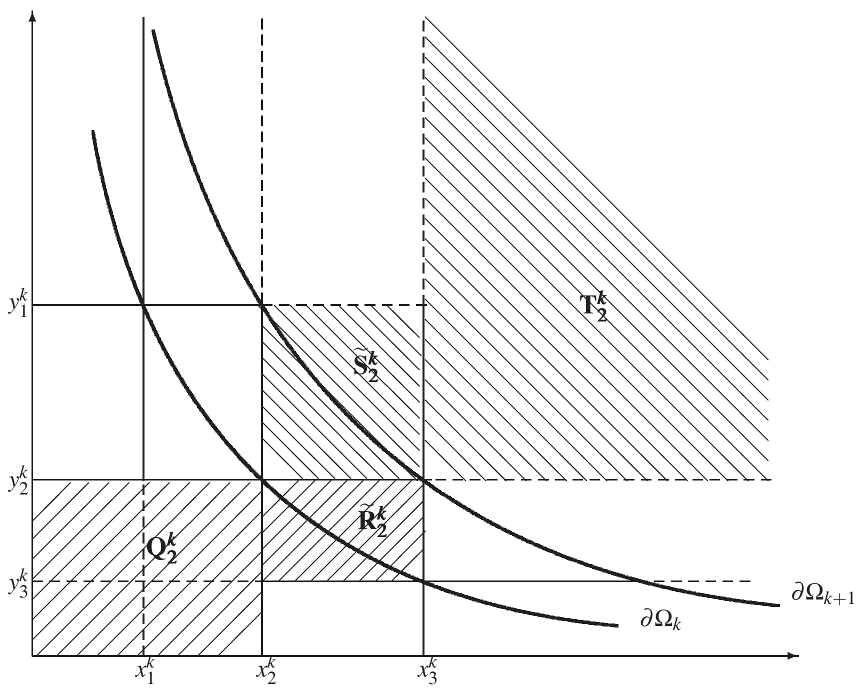

Fig. 2

Denote $\widetilde{E}_j^k := E_j^k \cup \left(\widetilde{S}_j^k \cap (\Omega_{k+2} - \Omega_{k+3})\right)$. Then (see (6))

$$\int_{\mathbb{R}_+^2} (I_2 f)^q w \leqslant 3^{3q} \sum_{k,j} 3^{kq} |\widetilde{E}_j^k|_w. \tag{14}$$

Put $g\sigma := f$ and write

$$\sum_{k,j} 3^{kq} |\widetilde{E}_j^k|_w = \sum_{k,j} |\widetilde{E}_j^k|_w \left(\int_{Q_j^k} f\right)^q = \sum_{k,j} |\widetilde{E}_j^k|_w |Q_j^k|_\sigma^q \left(\frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k} g\sigma\right)^q. \tag{15}$$

For an integer l, by Γ_l we denote the set of pairs (k,j) such that $|\widetilde{E}_j^k|_w > 0$ and

$$2^l < \frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k} g\sigma \leqslant 2^{l+1}, \qquad (k,j) \in \Gamma_l.$$

In analogy with the proof of [13, Theorem 1A], we show that

$$2^{l-1} < \frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k} g\sigma\chi_{\{g>2^{l-1}\}}, \qquad \text{for all } \ j,k. \tag{16}$$

Indeed, this follows from the fact that

$$\begin{aligned} 2^l &< \frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k} g\sigma = \frac{1}{|Q_j^k|_\sigma} \left[\int_{Q_j^k \cap \{g>2^{l-1}\}} g\sigma + \int_{Q_j^k \cap \{g \leqslant 2^{l-1}\}} g\sigma\right] \\ &\leqslant \frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k \cap \{g>2^{l-1}\}} g\sigma + 2^{l-1}. \end{aligned}$$

Further, we write for fixed l:

$$\begin{aligned} \sum_{(k,j)\in\Gamma_l} |\widetilde{E}_j^k|_w |Q_j^k|_\sigma^q &\overset{(16)}{\leqslant} 2^{-l+1} \sum_{(k,j)\in\Gamma_l} |\widetilde{E}_j^k|_w |Q_j^k|_\sigma^{q-1} \int_{Q_j^k} g\sigma\chi_{\{g>2^{l-1}\}} \\ &\leqslant 2^{-l+1} \sum_{(k,j)\in\Gamma_l} \int_{\widetilde{E}_j^k} w(x,y) \left[I_2\sigma(x,y)\right]^{q-1} \left(\int_0^x \int_0^y g\sigma\chi_{\{g>2^{l-1}\}}\right) dx\,dy. \end{aligned}$$

Combining the last estimate and (15), we obtain

$$\begin{aligned} \sum_{k,j} 3^{kq} |\widetilde{E}_j^k|_w &\leqslant 2^q \sum_l 2^{lq} \sum_{(k,j)\in\Gamma_l} |\widetilde{E}_j^k|_w |Q_j^k|_\sigma^q \\ &\leqslant 2^{q+1} \sum_l 2^{l(q-1)} \sum_{(k,j)\in\Gamma_l} \int_{\widetilde{E}_j^k} w(x,y) \left[I_2\sigma(x,y)\right]^{q-1} \left(\int_0^x \int_0^y g\sigma\chi_{\{g>2^{l-1}\}}\right) dx\,dy \\ &\leqslant 2^{2q} \sum_l \sum_{(k,j)\in\Gamma_l} \int_{\widetilde{E}_j^k} w(x,y) \left[I_2\sigma(x,y)\right]^{q-1} \left(\int_0^x \int_0^y g^q\sigma\chi_{\{g>2^{l-1}\}}\right) dx\,dy \\ &\leqslant 2^{2q} \sum_{k,j} \int_{\widetilde{E}_j^k} w(x,y) \left[I_2\sigma(x,y)\right]^{q-1} \left(\int_0^x \int_0^y g^q\sigma\right) dx\,dy. \end{aligned}$$

From this and Hölder's inequality with exponents p/q and r/q, we find that

$$\begin{aligned}
2^{-2q}\sum_{k,j}3^{kq}\big|\widetilde{E}_j^k\big|_w &\leqslant \sum_{k,j}\int_{\widetilde{E}_j^k} w(x,y)\big[I_2\sigma(x,y)\big]^{q-1}\Big(\int_0^x\int_0^y g^q(s,t)\sigma(s,t)\,ds\,dt\Big)dx\,dy \\
&= \int_{\mathbb{R}^2_+} w(x,y)\big[I_2\sigma(x,y)\big]^{q-1}\Big(\int_0^x\int_0^y g^q(s,t)\sigma(s,t)\,ds\,dt\Big)dx\,dy \\
&= \int_{\mathbb{R}^2_+} g^q(s,t)\sigma(s,t)\Big(\int_s^\infty\int_t^\infty w(x,y)\big[I_2\sigma(x,y)\big]^{q-1}dx\,dy\Big)ds\,dt \\
&\leqslant \Big(\int_{\mathbb{R}^2_+} g^p\sigma\Big)^{\frac{q}{p}}\Big(\int_{\mathbb{R}^2_+}\sigma(s,t)\Big(\int_s^\infty\int_t^\infty (I_2\sigma)^{q-1}w\Big)^{\frac{r}{q}}ds\,dt\Big)^{\frac{q}{r}} \\
&= B_v^q\Big(\int_{\mathbb{R}^2_+} g^p\sigma\Big)^{\frac{q}{p}}, \qquad (17)
\end{aligned}$$

since the sets $\widetilde{E}_j^k$ are disjoint and $g^p\sigma = f^p v$. The estimates (14) and (17) imply the validity of (5) for all f from the subclass M.

(*Necessity*) We apply the test function

$$f(s,y)=\sigma(s,y)\Big[\int_s^\infty\big[I_2\sigma(x,y)\big]^{\frac{r}{q'}}\big[I_2^*w(x,y)\big]^{\frac{r}{p}}\Big(\int_y^\infty w(x,t)\,dt\Big)dx\Big]^{\frac{1}{p}} =: \sigma(s,y)J(s,y)$$

into (5). Then

$$\begin{aligned}
\int_{\mathbb{R}^2_+} f^p v &= \int_{\mathbb{R}^2_+}\sigma(s,y)\big[J(s,y)\big]^p ds\,dy \\
&= \int_{\mathbb{R}^2_+}\big[I_2\sigma(x,y)\big]^{\frac{r}{q'}}\big[I_2^*w(x,y)\big]^{\frac{r}{p}}\Big(\int_y^\infty w(x,t)\,dt\Big)\Big(\int_0^x\sigma(s,y)ds\Big)dx\,dy \\
&= \frac{p'q}{r^2}\int_{\mathbb{R}^2_+} d_y\big[I_2\sigma(x,y)\big]^{\frac{r}{p'}}\,d_x\Big[-\big[I_2^*w(x,y)\big]^{\frac{r}{q}}\Big] = \frac{p'q}{r^2}B^r. \qquad (18)
\end{aligned}$$

To estimate the left–hand side of the inequality (5), we write

$$\begin{aligned}
\big[J(s,y)\big]^p &= \frac{q}{r}\big[I_2\sigma(s,y)\big]^{\frac{r}{q'}}\big[I_2^*w(s,y)\big]^{\frac{r}{q}} \\
&\quad+\frac{q}{q'}\int_s^\infty\big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1}\big[I_2^*w(x,y)\big]^{\frac{r}{q}}\Big(\int_0^y\sigma(x,t)\,dt\Big)dx \\
&=: \frac{q}{r}\big[J_1(s,y)\big]^p+\frac{q}{q'}\big[J_2(s,y)\big]^p. \qquad (19)
\end{aligned}$$

Then, for our chosen f,

$$\begin{aligned}
F(u,z) &:= \int_0^u\int_0^z f = \int_0^u\int_0^z\sigma(s,y)J(s,y)\,dy\,ds \\
&\geqslant 2^{-\frac{1}{p'}}\Big(\Big(\frac{q}{r}\Big)^{\frac{1}{p}}\int_0^u\int_0^z\sigma(s,y)J_1(s,y)\,dy\,ds+\Big(\frac{q}{q'}\Big)^{\frac{1}{p}}\int_0^u\int_0^z\sigma(s,y)J_2(s,y)\,dy\,ds\Big) \\
&=: 2^{-\frac{1}{p'}}\big(F_1+F_2\big).
\end{aligned}$$

To estimate F_2, we observe that

$$\Big(\frac{q'}{q}\Big)^{\frac{1}{p}} F_2 = \int_0^u \int_0^z \sigma(s,y) J_2(s,y)\,dyds$$
$$\geqslant \big[I_2^* w(u,z)\big]^{\frac{r}{qp}} \int_0^u \int_0^z \sigma(s,y) \Big[\int_s^u \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^y \sigma(x,t)\,dt\Big) dx\Big]^{\frac{1}{p}} dyds.$$

Since

$$\int_s^u \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^y \sigma(x,t)\,dt\Big) dx \leqslant \frac{q'}{r} \big[I_2\sigma(u,y)\big]^{\frac{r}{q'}}, \tag{20}$$

then

$$\int_0^u \int_0^z \sigma(s,y) \Big[\int_s^u \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^y \sigma(x,t)\,dt\Big) dx\Big]^{1-\frac{1}{p'}} dyds$$
$$\geqslant \Big(\frac{q'}{r}\Big)^{-\frac{1}{p'}} \int_0^u \int_0^z \sigma(s,y) \big[I_2\sigma(u,y)\big]^{-\frac{r}{q'p'}} \Big[\int_s^u \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^y \sigma(x,t)\,dt\Big) dx\Big] dyds$$
$$\geqslant \Big(\frac{q'}{r}\Big)^{-\frac{1}{p'}} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \int_0^u \int_0^z \sigma(s,y) \Big[\int_s^u \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^y \sigma(x,t)\,dt\Big) dx\Big] dyds$$
$$= \Big(\frac{q'}{r}\Big)^{-\frac{1}{p'}} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \int_0^u \int_0^z \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^x \sigma(s,y)\,ds\Big) \Big(\int_0^y \sigma(x,t)\,dt\Big) dydx$$

and, therefore,

$$F_2 \geqslant \Big(\frac{q}{q'}\Big)^{\frac{1}{p}} \Big(\frac{r}{q'}\Big)^{\frac{1}{p'}} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \big[I_2^* w(u,z)\big]^{\frac{r}{qp}}$$
$$\times \int_0^u \int_0^z \big[I_2\sigma(x,y)\big]^{\frac{r}{q'}-1} \Big(\int_0^x \sigma(s,y)\,ds\Big) \Big(\int_0^y \sigma(x,t)\,dt\Big) dxdy$$
$$=: \Big(\frac{q}{r}\Big)^{\frac{1}{p}} \frac{r}{q'} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \big[I_2^* w(u,z)\big]^{\frac{r}{qp}} \mathbf{J}_2(u,z).$$

For F_1 we obtain:

$$F_1 = \Big(\frac{q}{r}\Big)^{\frac{1}{p}} \int_0^u \int_0^z \sigma(s,y) \big[I_2\sigma(s,y)\big]^{\frac{r}{q'p}} \big[I_2^* w(s,y)\big]^{\frac{r}{qp}} dyds$$
$$\geqslant \Big(\frac{q}{r}\Big)^{\frac{1}{p}} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \big[I_2^* w(u,z)\big]^{\frac{r}{qp}} \int_0^u \int_0^z \sigma(s,y) \big[I_2\sigma(s,y)\big]^{\frac{r}{q'}} dyds$$
$$=: \Big(\frac{q}{r}\Big)^{\frac{1}{p}} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \big[I_2^* w(u,z)\big]^{\frac{r}{qp}} \mathbf{J}_1(u,z).$$

It holds that

$$F(u,z) \geqslant 2^{-\frac{1}{p'}} \Big(\frac{q}{r}\Big)^{\frac{1}{p}} \big[I_2\sigma(u,z)\big]^{-\frac{r}{q'p'}} \big[I_2^* w(u,z)\big]^{\frac{r}{qp}} \Big(\mathbf{J}_1(u,z) + \frac{r}{q'} \mathbf{J}_2(u,z)\Big).$$

Integrating by parts we find:

$$\begin{aligned}\mathbf{J}_2(u,z) &= \frac{q'}{r}\int_0^u dx\int_0^z\Big(\int_0^y \sigma(x,t)\,dt\Big)d_y\left[I_2\sigma(x,y)\right]^{\frac{r}{q'}}\\ &= \frac{q'}{r}\int_0^u\Big(\int_0^z \sigma(x,t)\,dt\Big)\left[I_2\sigma(x,z)\right]^{\frac{r}{q'}}dx-\frac{q'}{r}\mathbf{J}_1(u,z)\\ &= \frac{q'p'}{r^2}\left[I_2\sigma(u,z)\right]^{\frac{r}{p'}}-\frac{q'}{r}\mathbf{J}_1(u,z).\end{aligned}$$

Hence,

$$F(u,z)\geqslant 2^{-\frac{1}{p'}}\left(\frac{q}{r}\right)^{\frac{1}{p}}\frac{p'}{r}\left[I_2\sigma(u,z)\right]^{\frac{r}{qp'}}\left[I_2^*w(u,z)\right]^{\frac{r}{qp}}. \tag{21}$$

We write making use of (19):

$$\begin{aligned}\int_{\mathbb{R}_+^2}(I_2f)^q w &= \int_{\mathbb{R}_+^2} f(x,y)\left(\int_x^\infty\int_y^\infty w(u,z)\left[F(u,z)\right]^{q-1}dzdu\right)dxdy\\ &\geqslant 2^{-\frac{1}{p'}}\int_{\mathbb{R}_+^2}\sigma(x,y)\left(\int_x^\infty\int_y^\infty wF^{q-1}\right)\left\{\left(\frac{q}{r}\right)^{\frac{1}{p}}\left[I_2\sigma(x,y)\right]^{\frac{r}{q'p}}\left[I_2^*w(x,y)\right]^{\frac{r}{qp}}\right.\\ &\quad\left.+\left(\frac{q}{q'}\right)^{\frac{1}{p}}\left[\int_x^\infty\left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1}\left[I_2^*w(s,y)\right]^{\frac{r}{q}}\Big(\int_0^y\sigma(s,t)\,dt\Big)ds\right]^{\frac{1}{p}}\right\}dxdy\\ &=:2^{-\frac{1}{p'}}\left(G_1+G_2\right).\end{aligned} \tag{22}$$

G_1 is evaluated with (21) as follows:

$$\begin{aligned}G_1 &= \left(\frac{q}{r}\right)^{\frac{1}{p}}\int_{\mathbb{R}_+^2}\sigma(x,y)\left[I_2\sigma(x,y)\right]^{\frac{r}{q'p}}\left[I_2^*w(x,y)\right]^{\frac{r}{qp}}\left(\int_x^\infty\int_y^\infty w\right)\left[F(x,y)\right]^{q-1}dxdy\\ &\geqslant 2^{-\frac{q-1}{p'}}\left(\frac{q}{r}\right)^{\frac{q}{p}}\left(\frac{p'}{r}\right)^{q-1}\int_{\mathbb{R}_+^2}\sigma(x,y)\left[I_2\sigma(x,y)\right]^{\frac{r}{q'}}\left[I_2^*w(x,y)\right]^{\frac{r}{q}}dxdy.\end{aligned} \tag{23}$$

It is true for G_2:

$$\begin{aligned}\left(\frac{q'}{q}\right)^{\frac{1}{p}}G_2 &= \int_{\mathbb{R}_+^2}\sigma(x,y)\left[\int_x^\infty\left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1}\left[I_2^*w(s,y)\right]^{\frac{r}{q}}\Big(\int_0^y\sigma(s,t)\,dt\Big)ds\right]^{\frac{1}{p}}\\ &\quad\times\left(\int_x^\infty\int_y^\infty w(u,z)\left[F(u,z)\right]^{q-1}dzdu\right)dxdy\\ &= \int_{\mathbb{R}_+^2}\int_0^u\sigma(x,y)\left[\int_x^\infty\left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1}\left[I_2^*w(s,y)\right]^{\frac{r}{q}}\Big(\int_0^y\sigma(s,t)\,dt\Big)ds\right]^{\frac{1}{p}}dx\\ &\quad\times\left(\int_y^\infty w(u,z)\left[F(u,z)\right]^{q-1}dz\right)dudy\\ &\geqslant \int_{\mathbb{R}_+^2}\int_0^u\sigma(x,y)\left[\int_x^u\left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1}\left[I_2^*w(s,y)\right]^{\frac{r}{q}}\Big(\int_0^y\sigma(s,t)\,dt\Big)ds\right]^{\frac{1}{p}}dx\\ &\quad\times\left(\int_y^\infty w(u,z)\left[F(u,z)\right]^{q-1}dz\right)dudy\end{aligned}$$

$$\geqslant \int_{\mathbb{R}^2_+} \left[I_2^* w(u,y)\right]^{\frac{r}{pq}} \int_0^u \sigma(x,y) \left[\int_x^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1} \left(\int_0^y \sigma(s,t)\,dt\right) ds\right]^{1-\frac{1}{p'}} dx$$
$$\times \left(\int_y^\infty w(u,z)\left[F(u,z)\right]^{q-1} dz\right) dudy$$
$$\overset{(20)}{\geqslant} \left(\frac{r}{q'}\right)^{\frac{1}{p'}} \int_{\mathbb{R}^2_+} \left[I_2\sigma(u,y)\right]^{-\frac{r}{q'p'}} \left[I_2^* w(u,y)\right]^{\frac{r}{qp}}$$
$$\times \int_0^u \sigma(x,y) \left[\int_x^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1} \left(\int_0^y \sigma(s,t)\,dt\right) ds\right] dx$$
$$\times \left(\int_y^\infty w(u,z)\left[F(u,z)\right]^{q-1} dz\right) dudy$$
$$\geqslant \left(\frac{r}{q'}\right)^{\frac{1}{p'}} \int_{\mathbb{R}^2_+} \left[I_2\sigma(u,y)\right]^{-\frac{r}{q'p'}} \left[I_2^* w(u,y)\right]^{\frac{r}{qp}} \left(\int_y^\infty w(u,z)\,dz\right) \left[F(u,y)\right]^{q-1}$$
$$\times \left[\int_0^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1} \left(\int_0^s \sigma(x,y)\,dx\right) \left(\int_0^y \sigma(s,t)\,dt\right) ds\right] dudy.$$

Integrating by parts we find

$$\int_0^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}-1} \left(\int_0^s \sigma(x,y)\,dx\right) \left(\int_0^y \sigma(s,t)\,dt\right) ds$$
$$= \frac{q'}{r} \left(\int_0^u \sigma(x,y)\,dx\right) \left[I_2\sigma(u,y)\right]^{\frac{r}{q'}} dx - \frac{q'}{r} \int_0^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}} \sigma(s,y)\,ds.$$

Hence, continuing the reasoning, we obtain for G_2 using (21):

$$\left(\frac{q'}{q}\right)^{\frac{1}{p}} G_2 \geqslant 2^{-\frac{q-1}{p'}} \left(\frac{q'}{r}\right)^{\frac{1}{p}} \left(\frac{q}{r}\right)^{\frac{q-1}{p}} \left(\frac{p'}{r}\right)^{q-1} \int_{\mathbb{R}^2_+} \left[I_2^* w(u,y)\right]^{\frac{r}{p}} \left(\int_y^\infty w(u,z)\,dz\right)$$
$$\times \left[\left[I_2\sigma(u,y)\right]^{\frac{r}{q'}} \int_0^u \sigma(x,y)\,dx - \int_0^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}} \sigma(s,y)ds\right] dudy. \quad (24)$$

Since

$$\int_{\mathbb{R}^2_+} \left[I_2^* w(u,y)\right]^{\frac{r}{p}} \left(\int_y^\infty w(u,z)\,dz\right) \left[\int_0^u \left[I_2\sigma(s,y)\right]^{\frac{r}{q'}} \sigma(s,y)\,ds\right] dudy$$
$$= \frac{q}{r} \int_{\mathbb{R}^2_+} \left[I_2^* w(u,y)\right]^{\frac{r}{q}} \left[I_2\sigma(u,y)\right]^{\frac{r}{q'}} \sigma(u,y)\,dudy$$

then from (22) we obtain, applying (23) and (24),

$$2^{\frac{q}{p'}} \int_{\mathbb{R}^2_+} (I_2 f)^q w \geqslant \left(\frac{q}{r}\right)^{\frac{q}{p}} \left(\frac{p'}{r}\right)^{q-1} \int_{\mathbb{R}^2_+} \sigma(x,y) \left[I_2\sigma(x,y)\right]^{\frac{r}{q'}} \left[I_2^* w(x,y)\right]^{\frac{r}{q}} dxdy$$
$$+ \left(\frac{q}{r}\right)^{\frac{q}{p}} \left(\frac{p'}{r}\right)^{\frac{q}{q'}} \int_{\mathbb{R}^2_+} \left[I_2^* w(u,y)\right]^{\frac{r}{p}} \left(\int_y^\infty w(u,z)\,dz\right)$$
$$\times \left[I_2\sigma(u,y)\right]^{\frac{r}{q'}} \left(\int_0^u \sigma(x,y)\,dx\right) dudy$$

$$
\begin{aligned}
&-\left(\frac{q}{r}\right)^{\frac{q}{p}}\left(\frac{p'}{r}\right)^{q-1}\frac{q}{r}\int_{\mathbb{R}^2_+}\sigma(x,y)\left[I_2\sigma(x,y)\right]^{\frac{r}{q'}}\left[I_2^*w(x,y)\right]^{\frac{r}{q}}dxdy\\
&=\left(\frac{q}{r}\right)^{\frac{q}{p}}\left(\frac{p'}{r}\right)^{q-1}\frac{q}{p}\int_{\mathbb{R}^2_+}\sigma(x,y)\left[I_2\sigma(x,y)\right]^{\frac{r}{q'}}\left[I_2^*w(x,y)\right]^{\frac{r}{q}}dxdy\\
&\quad+\left(\frac{q}{r}\right)^{\frac{q}{p}+1}\left(\frac{p'}{r}\right)^{q}\int_{\mathbb{R}^2_+}d_u\left(-\left[I_2^*w(u,y)\right]^{\frac{r}{q}}\right)d_y\left[I_2\sigma(u,y)\right]^{\frac{r}{p'}}\\
&\geqslant\left(\frac{q}{r}\right)^{\frac{q}{p}+1}\left(\frac{p'}{r}\right)^{q}B^r.
\end{aligned}
$$

In view of (18), the required lower bound for C_2 is proven. □

There is also a dual statement of the first part of Theorem 3 with the functional

$$
B_w := \left(\int_{\mathbb{R}^2_+} w(u,z)\left(\int_0^u\int_0^z (I_2^*w)^{p'-1}\sigma\right)^{\frac{r}{p'}} du\,dz\right)^{\frac{1}{r}}
$$

instead of B_v. The proof is similar and can be carried out through the operator I_2^*f. If the weights v and w are factorizable, then the condition $B_v < \infty$ (or $B_w < \infty$) is necessary and sufficient for the (5) to hold if $1 < q < p < \infty$, moreover $C_2 \approx B_v \approx B_w$.

3. Multidimensional case with factorizable weights

It was established by A. Wedestig in [15] (see also [16]) for the case $n = 2$ that if the weight function v in (1) is factorizable, that is, $v(x_1,x_2) = v_1(x_1)v_2(x_2)$, then it is possible to characterize the inequality (1) by only one functional for all $1 < p \leqslant q < \infty$.

THEOREM 4. [16, Theorem 1.1] *Let* $n = 2$, $1 < p \leqslant q < \infty$, $s_1, s_2 \in (1,p)$ *and* $v(x_1,x_2) = v_1(x_1)v_2(x_2)$. *Then the inequality* (1) *holds for all* $f \geqslant 0$ *if and only if*

$$
\begin{aligned}
A_W(s_1,s_2) :&= \sup_{(t_1,t_2)\in\mathbb{R}^2_+}\left[I_1\sigma_1(t_1)\right]^{\frac{s_1-1}{p}}\left[I_1\sigma_2(t_2)\right]^{\frac{s_2-1}{p}}\\
&\qquad\times\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty}(I_1\sigma_1)^{\frac{q(p-s_1)}{p}}(I_1\sigma_2)^{\frac{q(p-s_2)}{p}}w\right)^{\frac{1}{q}} < \infty,
\end{aligned}
$$

where $\sigma_i := v_i^{1-p'}$, $i = 1,2$. *Moreover,* $C_2 \approx A_W(s_1,s_2)$ *with equivalence constants dependent on parameters* p, q *and* s_1, s_2 *only.*

The result of this theorem can be generalized to $n > 2$.

A number of statements similar to [16, Theorem 1.1] were obtained in [11] under the condition that weight functions v or w satisfy

$$
v(y_1,\dots,y_n) = v_1(y_1)\dots v_n(y_n) \tag{25}
$$

or

$$
w(x_1,\dots,x_n) = w_1(x_1)\dots w_n(x_n). \tag{26}
$$

THEOREM 5. [11, Theorems 2.1, 2.2] *Let* $1 < p \leqslant q < \infty$ *and the weight function* v *satisfy the condition* (25). *Then the inequality* (1) *holds for all* $f \geqslant 0$
(i) *if and only if* $A_{M_n} < \infty$, *where*

$$A_{M_n} : = \sup_{(t_1,\dots,t_n)\in\mathbb{R}^n_+} \big[I_n^* w(t_1,\dots,t_n)\big]^{\frac{1}{q}} \big[I_1\sigma_1(t_1)\big]^{\frac{1}{p'}} \dots \big[I_1\sigma_n(t_n)\big]^{\frac{1}{p'}};$$

(ii) *if and only if* $A_{T_n} < \infty$, *where*

$$A_{T_n} = \sup_{(t_1,\dots,t_n)\in\mathbb{R}^n_+} \big[I_1\sigma_1(t_1)\big]^{-\frac{1}{p}} \dots \big[I_1\sigma_n(t_n)\big]^{-\frac{1}{p}} \left(\int_0^{t_1} \dots \int_0^{t_n} (I_1\sigma_1)^q \dots (I_1\sigma_n)^q w\right)^{\frac{1}{q}}.$$

Besides, $C_n \approx A_{M_n} \approx A_{T_n}$ *with equivalence constants depending on* p, q *and* n.

THEOREM 6. [11, Theorems 2.4, 2.5] *Let* $1 < p \leqslant q < \infty$ *and the weight* w *satisfy the condition* (26). *Then the inequality* (1) *is true*
(i) *if and only if* $A^*_{M_n} < \infty$, *where with* $\sigma := v^{1-p'}$

$$A^*_{M_n} : = \sup_{(t_1,\dots,t_n)\in\mathbb{R}^n_+} \big[I_n\sigma(t_1,\dots,t_n)\big]^{\frac{1}{p'}} \big[I_1^* w_1(t_1)\big]^{\frac{1}{q}} \dots \big[I_1^* w_n(t_n)\big]^{\frac{1}{q}};$$

(ii) *if and only if* $A^*_{T_n} < \infty$, *where*

$$A^*_{T_n} = \sup_{(t_1,\dots,t_n)\in\mathbb{R}^n_+} \big[I_1^* w_1(t_1)\big]^{-\frac{1}{q'}} \dots \big[I_1^* w_n(t_n)\big]^{-\frac{1}{q'}} \left(\int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} (I_1^* w_1)^{p'} \dots (I_1^* w_n)^{p'} \sigma\right)^{\frac{1}{p'}}.$$

Besides, $C_n \approx A^*_{M_n} \approx A^*_{T_n}$ *with equivalence constants depending on* p, q *and* n.

Next assertions are devoted to the case $1 < q < p < \infty$ and we use multidimensional analogs of Maz'ya-Rozin [6, § 1.3.2] and Persson-Stepanov [9, § Theorem 3] functionals.

THEOREM 7. [11, Theorems 3.1, 3.2] *Let* $1 < q < p < \infty$. *Suppose that the weight function* v *in* (1) *satisfies the condition* (25) *and* $I_1\sigma_1(\infty) = \ldots = I_1\sigma_n(\infty) = \infty$. *Then* (1) *is valid for all* $f \geqslant 0$ *on* $\mathbb{R}^n_+$ *with* $C_n < \infty$ *independent of functions* f
(i) *if and only if* $B_{MR_n} < \infty$, *where*

$$B_{MR_n} := \left(\int_{\mathbb{R}^n_+} \big[I_n^* w(t_1,\dots,t_n)\big]^{\frac{r}{q}} \big[I_1\sigma_1(t_1)\big]^{\frac{r}{q'}} \sigma_1(t_1) \dots \big[I_1\sigma_n(t_n)\big]^{\frac{r}{q'}} \sigma_n(t_n)\, dt_1 \dots dt_n\right)^{\frac{1}{r}};$$

(ii) *if and only if* $B_{PS_n} < \infty$, *where*

$$B_{PS_n} := \left(\int_{\mathbb{R}^n_+} \left(\int_0^{t_1} \dots \int_0^{t_n} \big[I_1\sigma_1(t_1)\big]^q \dots \big[I_1\sigma_n(t_n)\big]^q w(x_1,\dots,x_n)\, dx_1 \dots dx_n\right)^{\frac{r}{q}} \right.$$
$$\left. \times \big[I_1\sigma_1(t_1)\big]^{-\frac{r}{q}} \sigma_1(t_1) \dots \big[I_1\sigma_n(t_n)\big]^{-\frac{r}{q}} \sigma_n(t_n)\, dt_1 \dots dt_n\right)^{\frac{1}{r}}.$$

Moreover, $C_n \approx B_{MR_n} \approx B_{PS_n}$ *with equivalence constants dependent on* p, q *and* n.

THEOREM 8. [11, Theorems 3.3, 3.4] *Let* $1 < q < p < \infty$*. Assume that* w *in* (1) *satisfies* (26) *and* $I_1^* w_1(0) = \ldots = I_1^* w_n(0) = \infty$*. Then* (1) *is valid for all* $f \geqslant 0$ *on* $\mathbb{R}_+^n$ *with* $C_n < \infty$ *independent of functions* f
(i) *if and only if* $B_{MR_n}^* < \infty$*, where*

$$B_{MR_n}^* := \left(\int_{\mathbb{R}_+^n} \left[I_n \sigma(t_1, \ldots, t_n) \right]^{\frac{r}{p'}} \left[I_1^* w_1(t_1) \right]^{\frac{r}{p}} w_1(t_1) \ldots \left[I_1^* w_n(t_n) \right]^{\frac{r}{p}} w_n(t_n)\, dt_1 \ldots dt_n \right)^{\frac{1}{r}};$$

(ii) *if and only if* $B_{PS_n}^* < \infty$*, where*

$$B_{PS_n}^* := \left(\int_{\mathbb{R}_+^n} \left(\int_{t_1}^{\infty} \ldots \int_{t_n}^{\infty} (I_1^* w_1)^{p'} \ldots (I_1^* w_n)^{p'} \sigma \right)^{\frac{r}{p'}} \right.$$
$$\left. \times \left[I_1^* w_1(t_1) \right]^{-\frac{r}{p'}} w_1(t_1) \ldots \left[I_1^* w_n(t_n) \right]^{-\frac{r}{p'}} w_n(t_n)\, dt_1 \ldots dt_n \right)^{\frac{1}{r}}.$$

Moreover, $C_n \approx B_{MR_n}^* \approx B_{PS_n}^*$ *with equivalence constants dependent on* p*,* q *and* n*.*

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Vladimir D. Stepanov
Computing Center of FEB RAS
Kim Yu Chen 65, Khabarovsk 680000, Russia
e-mail: stepanov@mi-ras.ru

Elena P. Ushakova
V. A. Trapeznikov Institute of Control Sciences of RAS
Profsoyuznaya 65, Moscow 117997, Russia
e-mail: elenau@inbox.ru