Mathematical
Inequalities
& Applications
Volume 24, Number 3 (2021), 635–643
doi:10.7153/mia-2021-24-44

A SHARPENED VERSION OF ACZÉL INEQUALITY BY ABSTRACT CONVEXITY

RAMAZAN TINAZTEPE* AND GULTEKIN TINAZTEPE

(*Communicated by J. Pečarić*)

Abstract. In this study, the Aczél inequality is considered and a new simple proof of the inequality is provided. An extension and a sharper version of this inequality are obtained by performing the results based on the optimality conditions of abstract convex functions.

Aczél inequality is one of the inequalities that have been studied and developed occasionally in the last fifty years. The statement of this inequality is as follows.

Let n be positive integer and let A,B,x_k,y_k $(1 \leqslant k \leqslant n)$ be real numbers such that

$$A^2 \geqslant \sum_{i=1}^{n} x_i^2 \ \text{ or } \ B^2 \geqslant \sum_{i=1}^{n} y_i^2. \tag{1}$$

Then

$$\left(A^2 - \sum_{i=1}^{n} x_i^2\right)\left(B^2 - \sum_{i=1}^{n} y_i^2\right) \leqslant \left(AB - \sum_{i=1}^{n} x_i y_i\right)^2$$

with the equality if only if the sequences $A,a_1,\ldots,a_n$ and $B,b_1,\ldots,b_n$ are proportional.

This inequality is originated from the study of Aczél [1]. Since then, many generalizations, refinements, extensions of it have been done by many researchers [5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 18, 19, 20, 21].

Most generalizations or refinements are based on the older refinement or generalization of the inequality and some useful inequalities given as lemmas naturally. In this study, after suggesting a new proof to the Aczél inequality, we give an extension and a sharper version of the inequality by a different approach. This approach uses the optimality condition of a function that is obtained in the context of abstract convexity. The notion of abstract convexity (concavity) uses the representation of a function as a supremum (infimum) of a certain class of minorant (majorant) functions. Some properties and examples of different abstract convex functions and the related studies involving inequalities can be seen in [2, 3, 4, 10, 17] and the references therein. In [11], it has been shown that a function with Lipschitz continuous gradient mapping is abstract concave with respect to a certain class of quadratic functions, which gives the necessary optimality condition for this kind of function. This property of abstract concave functions allows us to establish some new Aczél type inequalities.

Mathematics subject classification (2020): 26D07.
Keywords and phrases: Abstract concavity, refinement, Aczél inequality, global optimization.
* Corresponding author.

1. Preliminaries

Let us set up the notation and terminology used throughout the paper. $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^n$ is an Euclidean space; $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$ are nonnegative and positive orthants, respectively; X is a Hilbert space with the inner product $\langle .,. \rangle$ and the norm $\|x\| = \sqrt{\langle x,x \rangle}$; $B_*(x_0;r) = \{x \in X : \|x-x_0\|_* \leqslant r\}$ is a closed ball. Let $f : X \to \mathbb{R}$ and H be a set of real valued functions defined on X. f is said to be majorized by H if for all $h \in H$,

$$f(x) \leqslant h(x),\ x \in X$$

i.e., every element of H majorizes f.

Rubinov [10] defines abstract concave function as follows .

DEFINITION 1. Let H be a set of functions $h : \Omega \to \mathbb{R}$. A function $f : \Omega \to \mathbb{R}$ is called abstract concave with respect to H (or $H-concave$) if there exists a set $U \subset H$ such that

$$f(x) = \inf_{h \in U} h(x)$$

for all $x \in \Omega$.

Let H be the set of all quadratic functions h of the form

$$h(x) = a\|x\|^2 + \langle l,x \rangle + c,\ \ x \in X \tag{2}$$

where $a > 0,\ l \in X$ and $c \in \mathbb{R}$.

Let $\Omega \subset X$ and let H be the set of quadratic functions given in (2). Then a function $f : \Omega \to \mathbb{R}$ is abstract concave with respect to H if and only if f is majorized by H and f is upper semicontinuous (see [11]).

Assuming some differentiability conditions on f allows us to provide a certain way of building quadratic functions majorizing f. The following proposition elaborates this fact [11].

PROPOSITION 2. [11] *Let $\Omega \subset X$ be a convex set and let f be a differentiable mapping defined on an open set including Ω. Suppose that $\nabla f(x)$ is Lipschitz continuous on Ω, i.e.*

$$K = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x-y\|} < +\infty.$$

Let $a \geqslant K$ and for each $t \in \Omega$

$$f_t(x) = f(t) + \langle \nabla f(t), x-t \rangle + a\|x-t\|^2,\ \ x \in X.$$

Then $f(x) = \min_{t \in \Omega} f_t(x),\ x \in \Omega.$

In [11], different aspects of optimality conditions for the functions that can be expressed as the infimum of a family of convex functions over a convex set are studied in detail. One important result therein establishes a lower bound for the functions whose gradient is Lipschitz continuous, i.e., there exists $K>0$ satisfying that $\|\nabla f(x)-\nabla f(y)\| \leqslant K\|x-y\|$ for all $x,y\in X$.

It states that if f is a function with Lipschitz continuous gradient mapping and x^* is a global minimum point of f over X, then for some real number $a>K$,

$$f(x)-f(x^*) \geqslant \frac{1}{4a}\|\nabla f(x)\|^2 \tag{3}$$

for all $x\in X$.

The following theorem in [11] gives detailed information about such a real number "a" in (3).

THEOREM 3. *Let $\|.\|$ and $\|.\|_\circ$ be norms on $\mathbb{R}^n$. Let $\Omega\subset\mathbb{R}^n$ be a set with nonempty interior (denoted by $int(\Omega)$) and let $f\in C^1(\Omega)$. Suppose that the mapping $x\longmapsto\nabla f(x)$ is Lipschitz on Ω and*

$$K=\sup_{\substack{x,y\in X\\ x\neq y}}\frac{\|\nabla f(x)-\nabla f(y)\|}{\|x-y\|}<\infty.$$

Let f have global minimum at $x^*\in int(\Omega)$ over Ω. Define

$$M:=\max\{\|\nabla f(x)\|_\circ : x\in B_\circ(x^*;r)\}$$

where

$$B_\circ(x^*;r)=\{x:\|x-x^*\|_\circ\leqslant r\}\subset int(\Omega).$$

If q is a positive real number such that $B_\circ(x^*,r+q)\subset\Omega$ and $a\geqslant\max\left(K,\frac{M}{2q}\right)$, then

$$\frac{1}{4a}\|\nabla f(x)\|^2\leqslant f(x)-f(x^*),\ x\in B_\circ(x^*;r).$$

Theorem 3 enables us to obtain a sharper version of the Aczél inequality.

2. Main results

In this section, we first present a new simple proof of the Aczél inequality. This proof involves only straightforward algebraic calculations. Also it extends the sufficient condition (1) of the Aczél inequality. Next, by a similar argument in this proof, we give a lemma and obtain an Aczél type inequality. Then, a sharpened version for the Aczél inequality is presented.

THEOREM 4. *Let A and B any real numbers. If $x,y\in\mathbb{R}^n$ satisfies $AB\geqslant\frac{\|x\|\|y\|+\langle x,y\rangle}{2}$, then the following inequality holds:*

$$(A^2-\|y\|^2)(B^2-\|x\|^2)\leqslant(AB-\langle x,y\rangle)^2$$

Proof. Start with $0 \leqslant (A\|y\| - B\|x\|)^2$. Adding and subtracting $\|x\|^2\|y\|^2 - \langle x,y\rangle^2$ to the righthand side yields

$$0 \leqslant (A\|y\| - B\|x\|)^2 + \|x\|^2\|y\|^2 - \langle x,y\rangle^2 - \|x\|^2\|y\|^2 + \langle x,y\rangle^2 .$$

On the other hand $\frac{\|x\|\|y\|+\langle x,y\rangle}{2} \leqslant AB$ implies $\|x\|^2\|y\|^2 - \langle x,y\rangle^2 \leqslant 2AB(\|x\|\|y\| - \langle x,y\rangle)$. Using this above yields

$$0 \leqslant (A\|y\| - B\|x\|)^2 + 2AB(\|x\|\|y\| - \langle x,y\rangle) - \|x\|^2\|y\|^2 + \langle x,y\rangle^2 .$$

Expanding the squared expression and simplifications, we get

$$0 \leqslant A^2\|y\|^2 + B^2\|x\|^2 - 2AB\langle x,y\rangle - \|x\|^2\|y\|^2 + \langle x,y\rangle^2$$

Substracting $A^2\|y\|^2 + B^2\|x\|^2 - \|x\|^2\|y\|^2$ from and adding A^2B^2 to both sides yields

$$\left(A^2 - \|x\|^2\right)\left(B^2 - \|y\|^2\right) \leqslant (AB - \langle x,y\rangle)^2 . \quad \square$$

COROLLARY 5. (Aczél Inequality) *If* $A \geqslant \|x\|$ *or* $B \geqslant \|y\|$, *then*

$$\left(A^2 - \|x\|^2\right)\left(B^2 - \|y\|^2\right) \leqslant (AB - \langle x,y\rangle)^2 .$$

The following extension of the Aczél inequality does not involve any conditions. It will allow us to set up a function to use abstract convexity approach to get a refinement.

LEMMA 6. *Let* A, B *real numbers and* $x, y \in \mathbb{R}^n$. *Then*

$$(A^2 - \|y\|^2)(B^2 - \|x\|^2) + \langle x,y\rangle^2 - (\|x\|\|y\|)^2 \leqslant (AB - \langle x,y\rangle)^2 .$$

Proof. From $0 \leqslant \|Ax - By\|^2$, it is clear that

$$-A^2\|x\|^2 - B^2\|y\|^2 \leqslant -2AB\langle x,y\rangle$$

Adding $A^2B^2 + \langle x,y\rangle^2 + \|x\|^2\|y\|^2$ to both sides, we have

$$\begin{aligned} & -A^2\|x\|^2 - B^2\|y\|^2 + A^2B^2 + \langle x,y\rangle^2 + \|x\|^2\|y\|^2 \\ & \quad \leqslant -2AB\langle x,y\rangle + A^2B^2 + \langle x,y\rangle^2 + \|x\|^2\|y\|^2 \\ & (A^2 - \|y\|^2)(B^2 - \|x\|^2) + \langle x,y\rangle^2 - (\|x\|\|y\|)^2 \leqslant (AB - \langle x,y\rangle)^2 \quad \square \end{aligned}$$

The following theorem establishes an Aczél type inequality and it employs the abstract convexity approach via Theorem 3.

THEOREM 7. *Let $A, B, x_k, y_k, (1 \leqslant k \leqslant n)$ be real numbers. Then*

$$\left(A^2-\sum_{i=1}^{n} x_i^2\right)\left(B^2-\sum_{i=1}^{n} y_i^2\right)+\frac{1}{2\sqrt{n}}\sum_{k=1}^{n}\left[Bx_k-Ay_k\right]^2+\left(\sum_{i=1}^{n} x_i y_i\right)^2-\sum_{i=1}^{n} x_i^2\sum_{i=1}^{n} y_i^2$$
$$\leqslant\left(AB-\sum_{i=1}^{n} x_i y_i\right)^2.$$

Proof. Let $y=(y_1,y_2,\ldots,y_n)$ be arbitrary point in $\mathbb{R}^n$ and let $f_y(x)$ be given by

$$f_y(x)=\left(AB-\sum_{i=1}^{n} x_i y_i\right)^2-\left(A^2-\sum_{i=1}^{n} x_i^2\right)\left(B^2-\sum_{i=1}^{n} y_i^2\right)-\left(\sum_{i=1}^{n} x_i y_i\right)^2+\sum_{i=1}^{n} x_i^2\sum_{i=1}^{n} y_i^2$$

for $\mathbb{R}^n_{++}$. One can express the Aczél type inequality in Lemma 6 as $f_y(x)\geqslant 0$ under required conditions. Now accepting the function in Theorem 3 as $f_y(x)$, we can have the sharper version of the inequality.

The function $f_y(x)$ is nonnegative for all $x\in\mathbb{R}^n$ and it is clear $f_y(x)=0$ if and only if $x^*=\lambda y$, $\lambda=\frac{A}{B}k$, $k\in\mathbb{R}$. It yields that x^* is global minimum point over $\mathbb{R}^n$. A trivial calculation shows that

$$\nabla f_y(x)=2\left[x_1B^2-ABy_1, x_2B^2-ABy_2,\ldots,x_nB^2-ABy_n\right]$$

Thus

$$\|\nabla f(x)\|^2=4\sum_{k=1}^{n}\left[x_kB^2-ABy_k\right]^2$$

Accepting $\|.\|=\|.\|_2$ and $\|.\|_\circ=\|.\|_\infty$ we can define

$$\begin{aligned}\Omega=B_\infty(\lambda y;d)&=\{x\in\mathbb{R}^n:\|x-\lambda y\|_\infty\leqslant d\}\\&=\{x\in\mathbb{R}^n:\lambda y_i-d\leqslant x_i\leqslant\lambda y_i+d, i=1,\ldots,n\}\end{aligned}$$

where $\lambda':=\min_i\{\lambda y_i\}>d>0$. Since $d<\lambda y_i$, it follows that $\Omega\subset\mathbb{R}^n_{++}$. To show the fulfillment of Lipschitz condition, let us define the following function

$$\rho_i(x)=x_iB^2-ABy_i$$

and estimate $\|\nabla\rho_i(x)\|$ for $x\in\Omega$. For $i\in\{1,\ldots,n\}$,

$$\frac{\partial\rho_i}{\partial x_i}(x)=B^2$$
$$\frac{\partial\rho_i}{\partial x_j}(x)=0\quad(j\neq i)$$

so

$$\|\nabla\rho_i(x)\|^2=B^4$$

Let $x,z\in\Omega$. The mean value theorem implies that there exist numbers $\theta_i\in(0,1)$, $i=1,\ldots,n$ such that

$$\begin{aligned}\left\|\nabla f_y(x)-\nabla f_y(z)\right\| &= 2\left\|[\rho_1(x)-\rho_1(z)],[\rho_2(x)-\rho_2(z)],\ldots,[\rho_n(z)-\rho_n(z)]\right\|\\ &= 2\left(\sum_{k=1}^{n}[\rho_k(x)-\rho_k(z)]^2\right)^{\frac{1}{2}}\\ &= 2\left(\sum_{k=1}^{n}[\nabla\rho_k(x+\theta_k(z-x))(x-z)]^2\right)^{\frac{1}{2}}\end{aligned}$$

Taking into account that $x+\theta_i(z-x)\in\Omega$ for all i and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\sum_{k=1}^{n}[\nabla\rho_k(x+\theta_k(z-x))(x-z)]^2\right)^{\frac{1}{2}} &\leqslant \left(\sum_{k=1}^{n}\|\nabla\rho_k(x+\theta_k(z-x))\|^2\right)^{\frac{1}{2}}\|x-z\|\\ &\leqslant \left(\sum_{i=1}^{n}B^4\right)^{\frac{1}{2}}\|z-x\|\\ &= \left(nB^4\right)^{\frac{1}{2}}\|z-x\|\end{aligned}$$

It follows that

$$\left\|\nabla f_y(x)-\nabla f_y(z)\right\|\leqslant a_1(\lambda,d)\,\|x-z\|,\ \ x,z\in\Omega$$

where

$$a_1(\lambda,d)=2\sqrt{n}B^2.$$

Thus it is deduced that the mapping $x\rightarrow\nabla f(x)$ is Lipschitz continuous on Ω with the Lipschitz constant $K\leqslant a_1(\lambda,d)$. Let us choose a positive number $r\in(0,d)$. Clearly $B_\infty(x^*,r)\subset\Omega$ and we can take $q=d-r$. Let us estimate $M=\max\{\|\nabla f_y(x)\|_\infty : x\in B_\infty(x^*,r)\}$ as follows:

$$\begin{aligned}M &= \max_{x\in B_\infty(x^*,r)}\{\|\nabla f(x)\|_\infty\} = 2\max_{x\in B_\infty(x^*,r)}\left\{\max_{1\leqslant i\leqslant n}\left|x_iB^2-ABy_i\right|\right\}\\ &\leqslant 2\max_{B_\infty(x^*,r)}\left\{\max\{|x_i|\}B^2+\max\{|y_i|\}|AB|\right\}\\ &\leqslant 2\left\{(\lambda\max|y_i|+r)B^2+\max\{|y_i|\}|AB|\right\}\end{aligned}$$

Let

$$a_2(\lambda,d,r)=\frac{M}{2(d-r)}$$

and

$$a(d)=\max\left\{a_1(\lambda,d),a_2(\lambda,d,r)\right\}.$$

Since $a_1(\lambda,d)$ is constant, $\lim\limits_{d\to r^+} a_2(\lambda,d,r)=+\infty$ and decreasing on (r,λ') so the function $d\longmapsto a(d)$ takes its minimum on the interval (r,λ') and it equals to $a_1(\lambda,d)$. Let

$$a=\min_{r<d<\lambda'} a(d)=2\sqrt{n}B^2.$$

Hence

$$\left(A^2-\sum_{i=1}^n x_i^2\right)\left(B^2-\sum_{i=1}^n y_i^2\right)+\frac{1}{2\sqrt{n}}\sum_{k=1}^n [x_kB-Ay_k]^2+\left(\sum_{i=1}^n x_iy_i\right)^2-\sum_{i=1}^n x_i^2\sum_{i=1}^n y_i^2$$
$$\leqslant\left(AB-\sum_{i=1}^n x_iy_i\right)^2$$

for all $x\in\mathbb{R}^n$ such that $\|x-\lambda y\|_\infty\leqslant r$. Since we can choose Ω as large as required with extending d and $a(d)=a_1(\lambda,d)$ is independent of the choice of d and λ, the inequality holds for all $x,y\in\mathbb{R}^n$. $\square$

We can express the inequality above in the norm notation as follows

$$(A^2-\|y\|^2)(B^2-\|x\|^2)+\frac{1}{2\sqrt{n}}\|Ax-By\|^2+\langle x,y\rangle^2-(\|x\|\,\|y\|)^2\leqslant(AB-\langle x,y\rangle)^2$$

By applying Theorem 3 in a similar way to the Aczél inequality, we have the following sharper version of it.

THEOREM 8. *Let $A,B,x_k,y_k,(1\leqslant k\leqslant n),\lambda,r$ be real numbers such that $A^2\geqslant\sum_{k=1}^n x_k^2$ and $B^2\geqslant\sum_{k=1}^n y_k^2$. Then*

$$\left(A^2-\sum_{i=1}^n x_i^2\right)\left(B^2-\sum_{i=1}^n y_i^2\right)+\max\{C,D\}\leqslant\left(AB-\sum_{i=1}^n x_iy_i\right)^2 \tag{4}$$

where

$$C=\frac{\sum\limits_{k=1}^n\left[x_k\left(B^2-\sum\limits_{i=1}^n y_i^2\right)-y_k\left(AB-\sum\limits_{i=1}^n x_iy_i\right)\right]^2}{2n^2\left[nB^4+(n-1)\left(\sum\limits_{i=1}^n y_i^2-2B^2\right)\sum\limits_{k=1}^n y_k^2\right]^{\frac{1}{2}}},$$

$$D=\frac{\sum\limits_{k=1}^n\left[y_k\left(A^2-\sum\limits_{i=1}^n x_i^2\right)-x_k\left(AB-\sum\limits_{i=1}^n x_iy_i\right)\right]^2}{2n^2\left[nA^4+(n-1)\left(\sum\limits_{i=1}^n x_i^2-2A^2\right)\sum\limits_{k=1}^n x_k^2\right]^{\frac{1}{2}}}$$

Proof. First, by choosing $y \in \mathbb{R}^n$ such that $B^2 \geqslant \sum_{k=1}^n y_k^2$ and applying the similar arguments in the proof of Theorem 7 to the function

$$f_y(x) = \left(AB - \sum_{i=1}^n x_i y_i\right)^2 - \left(A^2 - \sum_{i=1}^n x_i^2\right)\left(B^2 - \sum_{i=1}^n y_i^2\right) \quad (x \in \mathbb{R}^n)$$

we have

$$\left(A^2 - \sum_{i=1}^n x_i^2\right)\left(B^2 - \sum_{i=1}^n y_i^2\right) + \frac{\sum\limits_{k=1}^n \left[x_k\left(B^2 - \sum\limits_{i=1}^n y_i^2\right) - y_k\left(AB - \sum\limits_{i=1}^n x_i y_i\right)\right]^2}{2n^2\left[nB^4 + (n-1)\left(\sum\limits_{i=1}^n y_i^2 - 2B^2\right)\sum\limits_{k=1}^n y_k^2\right]^{\frac{1}{2}}}$$
$$\leqslant \left(AB - \sum_{i=1}^n x_i y_i\right)^2 \tag{5}$$

for all $x \in \mathbb{R}^n$.

Second, choosing $x \in \mathbb{R}^n$ such that $A^2 \geqslant \sum_{k=1}^n x_k^2$ and applying the similar arguments in the proof of Theorem 7 to the function

$$f_x(y) = \left(AB - \sum_{i=1}^n x_i y_i\right)^2 - \left(A^2 - \sum_{i=1}^n x_i^2\right)\left(B^2 - \sum_{i=1}^n y_i^2\right) \quad (y \in \mathbb{R}^n)$$

we have

$$\left(A^2 - \sum_{i=1}^n x_i^2\right)\left(B^2 - \sum_{i=1}^n y_i^2\right) + \frac{\sum\limits_{k=1}^n \left[x_k\left(B^2 - \sum\limits_{i=1}^n y_i^2\right) - y_k\left(AB - \sum\limits_{i=1}^n x_i y_i\right)\right]^2}{2n^2\left[nB^4 + (n-1)\left(\sum\limits_{i=1}^n y_i^2 - 2B^2\right)\sum\limits_{k=1}^n y_k^2\right]^{\frac{1}{2}}}$$
$$\leqslant \left(AB - \sum_{i=1}^n x_i y_i\right)^2 \tag{6}$$

for all $x \in \mathbb{R}^n$. Combining 5 and 6 with assumptions, one can have 4. □

REFERENCES

[1] J. ACZÉL, *Some general methods in the theory of functional equations in one variable. New applications of functional equations*, Uspekhi Mat. Nauk, **11** (3) (1956), 3–68.

[2] G. ADILOV, A. M. RUBINOV, $\mathbb{B}$*-convex sets and functions*, Numer. Funct. Anal. Optim., **27** (3–4) (2006), 237–257.

[3] G. ADILOV, G. TINAZTEPE, *The sharpening of some inequalities via abstract convexity*, Math. Inequal. Appl., **12** (1) (2009), 33–51.

[4] G. ADILOV, I. YESILCE, $\mathbb{B}^{-1}$*-convex functions*, J. Convex Anal., **24** (2) (2017), 505–517.

[5] K. CHEN, F. YAN, H. Y. YUE, *Extensions and applications of power-type Aczél-Vasić-Pečarić's inequalities*, Math. Probl. Eng., (2019), `https://doi.org/10.1155/2019/1323474`.

[6] J. L. DÍAZ-BARRERO, M. GRAU-SÁNCHEZ, P. G. POPESCU, *Refinements of Aczél, Popoviciu and Bellman's inequalities*, Comput. Math. Appl., **56** (9) (2008), 2356–2359, https://doi.org/10.1016/j.camwa.2008.05.013.

[7] G. FARID, J. P. PEČARIĆ, A. U. REHMAN, *On refinements of Aczél, Popoviciu, Bellman's inequalities and related results*, J. Inequal. Appl., **2010** (2010), Article 579567, pages 17, https://doi.org/10.1155/2010/579567.

[8] M. H. HA, *Properties and refinements of Aczél type inequalities*, J. Inequal. Appl., **12** (1) (2018), 175–189, https://doi.org/10.7153/jmi-2018-12-14.

[9] Z. HU, A. XU, *Refinements of Aczél and Bellman's inequalities*, Comput. Math. Appl., **59** (9) (2010), 3078–3083, https://doi.org/10.1016/j.camwa.2010.02.027.

[10] A. M. RUBINOV, *Abstract Convexity and Global Optimization*, Springer Science Business Media, New York, 2013.

[11] A. M. RUBINOV, Z. Y. WU, *Optimality conditions in global optimization and their applications*, Math. Program., **120** (1) (2009), 101–123.

[12] J. F. TIAN, W. L. WANG, *Reversed versions of Aczél-type inequality and Bellman-type inequality*, J. Math. Inequal., **9** (2) (2015), 417–424, https://doi.org/10.7153/jmi-09-35.

[13] J. TIAN, Y. SUN, *New refinements of generalized Aczél inequality*, J. Inequal. Appl., **2014** (2014), Article 239, 14 pages.

[14] J. F. TIAN, S. Y. WANG, *Refinements of Generalized Aczél's inequality and Bellman's inequality and their applications*, J. Appl. Math., **2013** (2013), https://doi.org/10.1155/2013/645263.

[15] J. TIAN, W.-L. WANG, *Refinements of Aczél-type inequality and their applications*, J. Appl. Math., **2014** (2014), https://doi.org/10.1155/2014/581354.

[16] J. TIAN, Y. ZHOU, *Note on Aczél-type inequality and Bellman-type inequality*, J. Nonlinear Sci. Appl., **9** (3) (2016), 1316–1322.

[17] G. TINAZTEPE, S. KEMALI, S. SEZER, Z. EKEN, *The Sharper Form of Brunn-Minkowski Type Inequality for Boxes*, Hacet. J. Math. Stat., **50** (2), (2021), 377–386.

[18] S. VONG, *On a generalization of Aczél's inequality*, Appl. Math. Lett., **24** (8) (2011), 1301–1307, https://doi.org/10.1016/j.aml.2011.02.020.

[19] S. WU, *Improvement of Aczél's inequality and Popoviciu's inequality*, J. Math. Inequal., **9** (2007), https://doi.org/10.1155/2007/72173.

[20] W. YANG, *Refinements of generalized Aczél-Popoviciu's inequality and Bellman's inequality*, Comput. Math. Appl., **59** (11) (2010), 3570–3577, https://doi.org/10.1016/j.camwa.2010.03.050.

[21] X. ZHOU, *Some generalizations of Aczél, Bellman's inequalities and related power sums*, J. Inequal. Appl., **2012** (2012), Article 130, 11 pages, https://doi.org/10.1186/1029-242X-2012-130.

(Received September 20, 2020)

Ramazan Tinaztepe
Imam Abdulrahman Bin Faisal University
Deanship of Preparatory Year and Supporting Studies
Dammam, 31441, P.O. Box 1982, KSA
e-mail: ttinaztepe@iau.edu.sa

Gultekin Tinaztepe
Akdeniz University
Vocational School of Technical Sciences
Dumlupinar Bulvari, Antalya 07058, Turkey
e-mail: gtinaztepe@akdeniz.edu.tr