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A VARIABLE EXPONENT BOUNDEDNESS OF THE STEKLOV OPERATOR

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Abstract. In this paper, a sufficiency condition for boundedness of the Steklov operator

$$S_h f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt, \quad h > 0$$

has been proved in variable exponent Lebesgue space $L^{p(\cdot)}(0,\infty)$. Here an infinite interval $(0,\infty)$ has been considered with a new decay condition on infinity. A finite interval $[0,2\pi]$ case with a local log- regularity condition has been studied previously in order to be applied on approximation problem.

1. Introduction

In this study, we derive a boundedness result for the classical Steklov operator

$$S_h f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt, \quad h > 0 \tag{1}$$

in variable exponent Lebesgue space $L^{p(\cdot)}(0,\infty)$ (For the notation and main properties of variable exponent Lebesgue space see, e.g. [1, 2]).

A boundedness problem for main integral operators in variable exponent Lebesgue spaces has been studied by many authors. A local log-regularity condition and a decay condition at infinity are used on the exponent functions in the study of boundedness and compactness problems for the main integral operators of harmonic analysis. For a survey of this topic see, e.g. in monographs [1, 2, 8, 14].

A function $p:(0,\infty)\to(1,\infty)$ satisfies local regularity condition if

$$|p(x)-p(y)|\ln\frac{1}{|x-y|} \leqslant C_1 \quad \text{where} \quad |x-y| < \frac{1}{2}. \tag{2}$$

A function $p:(0,\infty)\to(1,\infty)$ satisfies a decay condition if

$$|p(x)-p(\infty)|\ln(e+x) \leqslant C_2, \quad x > 0. \tag{3}$$

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The Steklov operator was considered in [5] by Edmunds and Nekvinda. It was given such example that Steklov operator is bounded and maximal operator is not bounded, it was shown that the local regularity condition is not enough for boundedness of Steklov operator see [5, Example 4.2].

Note that we have the estimates $S_h f \leqslant M^+ f$ and $M^+ f \leqslant Mf$, where M^+ is one-sided Hardy-Littlewood maximal operator. It is known that the one-side maximal operator M^+ is bounded on wider class of exponent functions then maximal function M, (see [13]).

In this paper, we continue a study on variable exponent boundedness of Steklov's operator, started by I. Sharapuddinov in case of bounded interval $[0,2\pi]$ in [15] (see also [16, 17]), where a local regularity condition has been assumed in order to apply it in the approximation problems in variable exponent spaces $L^{p(\cdot)}[0,2\pi]$ (see, also [6]). It is also was considered in the case of periodic functions f on $(0,\infty)$ not using a decay condition. In our study, we insert two type conditions for the exponent function $p(.)$, one is the same local regularity condition (2) and another is a new condition (5) below given near infinity, to govern the inequality

$$||S_h f||_{L^{p(\cdot)}(0,\infty)} \leqslant C||f||_{L^{p(\cdot)}(0,\infty)} \tag{4}$$

with a constant C independent on $0 < h < 1$.

In contrast to the mentioned works, we consider a case of infinite interval and not periodic functions f, therefore obligated some condition on infinity. Since it holds an inequality $S_h f(x) \leqslant Mf(x)$, via maximal operator, one can think, those results (see, e.g. a proper result in [3] or say, [12]) entail ours. It does not so by the followings.

A decay condition at infinity that we have used in this paper is the condition

$$|p(t)-p(x)|\ln(e+x) \leqslant C_3, \quad x < t \leqslant x+1,\, x > 0, \tag{5}$$

with a positive constant C_3 independent x.

In general, (5) is weaker than a decay condition (3). Indeed, since $t < y \leqslant t+1, \quad t > 0$ from (3) it follows

$$\begin{aligned} &|p(t)-p(y)|\ln(1+t) \\ &\leqslant |p(t)-p(\infty)|\ln(1+t)+|p(y)-p(\infty)|\ln(1+y) \leqslant 2C_2. \end{aligned}$$

The reverse assertion is not true, as it follows from the example (see, [10])

$$p(x) = 4+\frac{1}{\sqrt{\ln(e+x)}} \quad x > 0. \tag{6}$$

It is not difficult to verify that, this function satisfies (5) but does not satisfy (3) the weaker decay condition

$$1 \in L^{s(\cdot)}, \quad \frac{1}{s(x)} = \left|\frac{1}{p(x)}-\frac{1}{p(\infty)}\right| \tag{7}$$

that usually is used in proving a boundedness result for maximal operators in $L^{p(\cdot)}$ (see, e.g. in [2], Remark 4.2.8). Since the condition (7) excludes the above example (6), our results does not follow from a boundedness result for maximal operator. To be sure it, note that (7) means

$$\int_0^\infty e^{-\frac{C}{\left|\frac{1}{p(x)}-\frac{1}{p(\infty)}\right|}}dx \leqslant 1$$

by some positive C. For the above example, the left hand side for any $C>0$ yields

$$\int_0^\infty e^{-2C\sqrt{\ln(e+x)}}dx = \int_1^\infty 2ue^{u^2-2Cu}du = \infty,$$

i.e. (7) is violated.

The boundedness results of the type of given in the present paper (e.g. as in (4)) is used in proving the density and embedding results for the variable exponent Sobolev spaces [2, 3, 4, 7]. There arises a need for a boundedness result for approximate operator (it can be e.g. the Steklov operator, a mollifying operator with smooth kernel etc.). Having such estimates and proving the approximation property in the smooth and compact support class of functions and further applying Banach-Steinhaus theorem is handed the density result. For example, let us show the convergence as $h \to +0$ for arbitrary function $f \in L^{p(\cdot)}(\mathbb{R}^+)$:

$$I_{p(\cdot)}(S_h f - f) = \int_0^\infty \left|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\right|^{p(x)} dx \to 0. \tag{8}$$

For a function $f \in L^{1,loc}(\mathbb{R}^+)$ the convergence $S_h f(x) \to f(x)$ a.e. $x \in \mathbb{R}$ is well known (see, e.g. [2]). Then the convergence (8) for a bounded compact support continues function $f(x)$ easily follows from this fact and the Lebesgue convergence theorem. For further establishing this convergence on functions $f \in L^{p(\cdot)}(\mathbb{R}^+)$ it suffices to apply the estimate (4) and the Banach-Steinhaus theorem and the fact on density of compact support continuous functions in $L^{p(\cdot)}$.

In this paper, we use following notation.

By C, C_i we denote a positive and greater then 1 constant depending on p^+ and C_1 from the conditions (2) . We use a notation $p^+ = \sup\{p(x) : x \in (0,\infty)\}$ and $p^- = \inf\{p(x) : x \in (0,\infty)\}$. Recall, the norm in variable exponent Lebesgue space $L^{p(\cdot)}(0,\infty)$ given as $\|f\| = \inf\left\{\lambda > 0 : I_{p(.)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}$ makes it a Banach space, with a modular $I_{p(.)}(f) = \int_0^\infty |f(t)|^{p(t)}dt$. We use the notation $\|f\|_{p(.)}$ or $\|f\|_{L^{p(\cdot)}(0,\infty)}$ for the $L^{p(\cdot)}(0,\infty)$ variable exponent Lebesgue norm of function f. For a function $p : [0,\infty) \to [1,\infty)$ denote $p'(x)$ the function satisfying $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p' = \infty$ if $p = 1$.

2. Main result

Main result of this paper is stated as following.

THEOREM 1. *Let $p : [0,\infty) \to (1,\infty)$ be a continuous function with $p^+ < \infty$ and the conditions (2), (5) be satisfied. Then for any measurable positive function f : $(0,\infty) \to (0,\infty)$ it holds the inequality (4) with a constant C_0 depending on C_1, C_3, p^-, p^+.*

We present two lemmas which we need in the proof.

LEMMA 1. *It holds an estimate*

$$\|\chi_{(x,x+h)}\|_{p'(.)} \leqslant C_5 h^{\frac{1}{(p^-_{x,h})'}}, \quad 0<h<1,\ x>0, \tag{9}$$

where C_5 is a constant greater then 1.

LEMMA 2. *It holds an inequality*

$$h^{\frac{p^-_{x,h}-p(x)}{p^-_{x,h}}} \leqslant e^{\frac{2C_1}{p^-}}, \quad 0<h<1, \quad x>0 \tag{10}$$

The proof of Lemma 1 easily follows from Remark 2.40 of [1] and proof of Lemma 2 follows from Lemma 4.1.6 of [2].

Proof of Theorem 1. Let f be a positive measurable function on $(0,\infty)$ such that $||f||_{p(.)} \leqslant 1$. To prove Theorem 1, it suffices to show $\|S_h f(x)\|_{p(.)} \leqslant C_4^{1/p^-}$, or more explicitly, $I_{p(.)}(S_h f) \leqslant C_4$ by some constant C_3 depending on C_1, C_3, p^-, p^+. Let $p : (0,\infty) \to (1,\infty)$ be a continuous function satisfying the conditions (2) and (5). Let $x \in (0,\infty)$ be fixed. Denote $p^-_{x,h} = \inf\{p(t) : x<t<x+h\}$ and $p^+_{x,h} = \sup\{p(t) : x<t<x+h\}$. Using Minkowski's inequality for $p(.)$ -norms we have

$$\|S_h f\|_{L^{p(.)}(0,\infty)} \leqslant \|S_h f\|_{L^{p(.)}(0,1)} + \|S_h f\|_{L^{p(.)}(1,\infty)}.$$

To get an estimation for the left hand side, we get estimation of proper modulars:

$$i_1 = \int_0^1 (S_h f(x))^{p(x)} dx \quad \text{and} \quad i_2 = \int_1^\infty (S_h f(x))^{p(x)} dx.$$

By definition,

$$i_1 = \int_0^1 \Big(S_h f(x)\Big)^{p(x)} dx = \int_0^1 \Big(\frac{1}{h}\int_x^{x+h} f(t)dt\Big)^{p(x)} dx.$$

Using Hölder's inequality for $p(x)$-norms it follows that

$$\int_x^{x+h} f(t)dt \leqslant 2\|f(.)\chi_{(x,x+h)}(.)\|_{p(.)}\|\chi_{(x,x+h)}(.)\|_{p'(.)}.$$

From this and the assumption $\|f\|_{p(.)} \leqslant 1$, for $x>0$ we get

$$\int_x^{x+h} f(t)dt \leqslant 2\|\chi_{(x,x+h)}\|_{p'(.)}. \tag{11}$$

Now, we pass to the estimation of i_1.

$$\begin{aligned} i_1 &= \int_0^1 \Big(\frac{1}{h}\int_x^{x+h} f(t)dt\Big)^{p(x)} dx \\ &= \int_0^1 \left(\frac{\frac{1}{h}\int_x^{x+h} f(t)dt}{C_5 h^{-\frac{1}{p^-_{x,h}}}}\right)^{p(x)} C_5^{p(x)} h^{-\frac{p(x)}{p^-_{x,h}}} dx, \quad h>0. \end{aligned} \tag{12}$$

By using (11) and estimate (9) of Lemma 1, the parentheses term does not exceed 1. To increase its value, we may decrease the exponent $p(x)$ to $p^-_{x,h}$. Then

$$i_1 \leqslant \int_0^1 \frac{\left(\frac{1}{h}\int_x^{x+h} f(t)dt\right)^{p^-_{x,h}}}{C_5^{p^-}} C_5^{p^+} h^{\frac{p^-_{x,h}-p(x)}{p^-_{x,h}}} dx. \tag{13}$$

From this applying Holder's inequality

$$\frac{1}{h}\int_x^{x+h} f(t)dt \leqslant \left(\frac{1}{h}\int_x^{x+h} f(t)^{p^-_{x,h}} dt\right)^{\frac{1}{p^-_{x,h}}}$$

the right hand side of (13) is exceeded by

$$C_5^{p^+-p^-}\int_0^1 \left(\frac{1}{h}\int_x^{x+h} f(t)^{p^-_{x,h}} dt\right) h^{\frac{p^-_{x,h}-p(x)}{p^-_{x,h}}} dx.$$

Whence,

$$i_1 \leqslant \frac{C_5^{p^+-p^-}}{h}\int_0^1 \left(\int_x^{x+h} f(t)^{p^-_{x,h}} dt\right) h^{\frac{p^-_{x,h}-p(x)}{p^-_{x,h}}} dx. \tag{14}$$

Using Lemma 2, from (14) it follows

$$i_1 \leqslant C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}} \int_0^1 \left(\frac{1}{h}\int_x^{x+h} f(t)^{p^-_{x,h}} dt\right) dx. \tag{15}$$

Since $p(t) \geqslant p^-_{x,h}$ for $t \in (x, x+h)$, it is clear that

$$\begin{aligned}\int_x^{x+h} f(t)^{p^-_{x,h}} dt &= \int_x^{x+h} \chi_{\{f(t)\geqslant 1\}}(t) f(t)^{p^-_{x,h}} dt + \int_x^{x+h} \chi_{\{f(t)<1\}}(t) f(t)^{p^-_{x,h}} dt \\ &\leqslant \int_x^{x+h} f(t)^{p(t)} dt + \int_x^{x+h} dt.\end{aligned} \tag{16}$$

From (15) using (16) and Fubini's theorem it follows that

$$i_1 \leqslant C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}} \frac{1}{h}\int_0^1 \left(\int_x^{x+h} f(t)^{p(t)} dt\right) dx + C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}}$$

or

$$\begin{aligned} i_1 &\leqslant C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}} \frac{1}{h}\int_0^1 \left(\int_x^{x+h} f(t)^{p(t)} dt\right) dx + C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}} \\ &\leqslant C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}} \left(\int_0^{1+h} f(t)^{p(t)} dt + 1\right) \leqslant 2C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}}\end{aligned}$$

by assumption $I_{p(.)}(f) \leqslant 1$ due to [9, (2.9)].

Therefore, it has been proved that

$$i_1 \leqslant 2C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}}. \tag{17}$$

Derive an estimation for i_2,

$$i_2 = \int_1^{\infty} \left(\frac{1}{h} \int_x^{x+h} f(t)dt \right)^{p(x)} dx, \quad h > 0.$$

Using the estimate (9) of Lemma 1 for $x > 1$ it follows that

$$\int_x^{x+h} f(t)dt \leqslant C_5 h^{\frac{1}{(p_{x,h}^-)'}}, \quad 1 < x < \infty, \quad h > 0. \tag{18}$$

We shall use this estimation in our further argues. Since

$$\frac{\frac{1}{h}\int_x^{x+h} f(t)dt}{C_5 h^{-\frac{1}{p_{x,h}^-}}} \leqslant 1, \quad x > 1, \quad h > 0,$$

it follows that

$$\begin{aligned} i_2 &\leqslant \int_1^{\infty} \left(\frac{\frac{1}{h}\int_x^{x+h} f(t)dt}{C_5 h^{-\frac{1}{p_{x,h}^-}}} \right)^{p(x)} C_5^{p(x)} h^{\frac{-p(x)}{p_{x,h}^-}} dx \\ &\leqslant C_5^{p^+-p^-} \int_1^{\infty} \left(\frac{1}{h} \int_x^{x+h} f(t)dt \right)^{p_{x,h}^-} h^{\frac{p_{x,h}^- - p(x)}{p_{x,h}^-}} dx, \quad h > 0 \end{aligned}$$

since $p(x) \geqslant p_{x,h}^-$.

Now, using Holder's inequality

$$\frac{1}{h} \int_x^{x+h} f(t)dt \leqslant \left(\frac{1}{h} \int_x^{x+h} f(t)^{p_{x,h}^-} dt \right)^{\frac{1}{p_{x,h}^-}}$$

it follows that

$$i_2 \leqslant C_5^{p^+-p^-} \frac{1}{h} \int_1^{\infty} h^{\frac{p_{x,h}^- - p(x)}{p_{x,h}^-}} \left(\int_x^{x+h} f(t)^{p_{x,h}^-} dt \right) dx, \quad h > 0. \tag{19}$$

Use the estimate (10) of Lemma 2 for $x > 1$:

$$h^{\frac{p_{x,h}^- - p(x)}{p_{x,h}^-}} = e^{\frac{p(x) - p_{x,h}^-}{p_{x,h}^-} \ln \frac{1}{h}} \leqslant e^{\frac{2C_1}{p^-}}, \quad h > 0, \ x > 1.$$

Therefore, from (19) it follows

$$i_2 \leqslant C_5^{p^+-p^-} e^{\frac{2C_1}{p^-}} \frac{1}{h} \int_1^\infty \left[\int_x^{x+h} f(t)^{p_{x,h}^-} dt \right] dx. \tag{20}$$

The interior integral is estimated as

$$\begin{aligned}
\int_x^{x+h} f(t)^{p_{x,h}^-} dt &= \int_x^{x+h} \Big(f(t)(1+t^2) \Big)^{p_{x,h}^-} \Big(\frac{1}{1+t^2} \Big)^{p_{x,h}^-} dt \\
&= \int_x^{x+h} \Big(f(t)(1+t^2) \Big)^{p_{x,h}^-} \chi_{\{s: f(s)(1+s^2) \geqslant 1\}}(t) \Big(\frac{1}{1+t^2} \Big)^{p_{x,h}^-} dt \\
&\quad + \int_x^{x+h} \Big(f(t)(1+t^2) \Big)^{p_{x,h}^-} \chi_{\{s: f(s)(1+s^2) < 1\}}(t) \Big(\frac{1}{1+t^2} \Big)^{p_{x,h}^-} dt \\
&\leqslant \int_x^{x+h} f(t)^{p(t)} (1+t^2)^{p(t)-p_{x,h}^-} dt + \int_x^{x+h} \frac{dt}{1+t^2}.
\end{aligned} \tag{21}$$

By using the condition (5),

$$(1+t^2)^{p(t)-p_{x,h}^-} = e^{2(p(t)-p_{x,h}^-)\ln(1+t)} \leqslant e^{4C_3}, \quad 1 < x < t \leqslant x+h. \tag{22}$$

Indeed, as in the proof of (10) for $t \in (x, x+h)$ it holds

$$\Big[p(t) - p_{x,h}^- \Big] \ln(1+t) \leqslant 2|p(t) - p(\xi)| \ln(1+t) \leqslant 2C_3,$$

where $\xi \in (x, x+h)$.

Using (22) and (21), it follows that

$$\int_x^{x+h} f(t)^{p_{x,h}^-} dt \leqslant \Bigg(e^{2C_3} \int_x^{x+h} f(t)^{p(t)} dt + \int_x^{x+h} \frac{dt}{1+t^2} \Bigg). \tag{23}$$

Insert (23) into the estimate (20). Then

$$\begin{aligned}
i_2 &\leqslant C_5^{p^+-p^-} e^{2C_3+2C_1} \frac{1}{h} \int_1^\infty \Bigg(\int_x^{x+h} f(t)^{p(t)} dt \Bigg) dx \\
&\quad + C_5^{p^+-p^-} e^{2C_1} \frac{1}{h} \int_1^\infty \Bigg(\int_x^{x+h} \frac{dt}{1+t^2} \Bigg) dx.
\end{aligned} \tag{24}$$

Apply here Fubini's theorem; then

$$\begin{aligned}
i_2 &\leqslant C_5^{p^+-p^-} e^{2C_3+2C_1} \frac{1}{h} \int_1^\infty f(t)^{p(t)} \Bigg(\int_{t-h}^t dx \Bigg) dt \\
&\quad + C_5^{p^+-p^-} e^{2C_1} \frac{1}{h} \int_1^\infty \Bigg(\int_{t-h}^t dx \Bigg) \frac{dt}{1+t^2} \\
&\leqslant C_5^{p^+-p^-} e^{2C_3+2C_1} \int_1^\infty f(t)^{p(t)} dt + C_5^{p^+-p^-} e^{2C_1} \int_1^\infty \frac{dt}{1+t^2}
\end{aligned} \tag{25}$$

since $I_{p(.)}(f) \leqslant 1$ and the last integral is equal $\frac{\pi}{2}$, we have proved that

$$i_2 \leqslant C_6 \tag{26}$$

with

$$C_6 = C_5^{p^+-p^-} e^{2C_3+2C_1} + C_5^{p^+-p^-} e^{2C_1} \frac{\pi}{2}.$$

Combining the estimates (17) and (26) we get

$$I_{p(.)}(S_h f(x)) \leqslant C_7 := C_6 + 2C_5^{p^+-p^-} e^{2C_1}.$$

Using an inequality between modular and $p(.)$ norm (see e.g. [9] or [7]), the last inequality gives an estimate

$$\|S_h f(x)\|_{p(\cdot)} \leqslant C_7^{\frac{1}{p^-}}.$$

Theorem 1 has been proved. $\square$

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