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# ON COMPLEX $L_p$ AFFINE ISOPERIMETRIC INEQUALITIES

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*Abstract.* Recently, Haberl [18] established the complex version of the Petty projection inequality and the Busemann-Petty centroid inequality. In this paper, we define the complex $L_p$ projection body operator $\Pi_{C,p}$ and the complex $L_p$ centroid body operator $\Gamma_{C,p}$. When $p \geqslant 1$ and $C$ is a complex $L_p$ zonoid in the complex plane, we establish the complex extension of the $L_p$ Busemann-Petty centroid inequality and the $L_p$ Petty projection inequality.

## 1. Introduction

Let $\mathbb{R}^m, \mathbb{C}^n$ be the $m$-dimensional Euclidean space and $n$-dimensional complex space respectively. For $x, y \in \mathbb{R}^m$, we denote the standard Euclidean inner product of $x$ and $y$ by "$x \cdot y$". For $x, y \in \mathbb{C}^n$, "$x \cdot y$" denote the standard Hermitian inner product of $x$ and $y$ (see Section 2 for details). Let $S^{m-1}$ and $B_m$ be the unit sphere and the unit ball in $\mathbb{R}^m$ respectively. Let $\mathbb{S}^n$ and $\mathbb{B}_n$ denote the complex unit sphere $\{c \in \mathbb{C}^n : c \cdot c = 1\}$ and the complex unit ball $\{c \in \mathbb{C}^n : c \cdot c \leqslant 1\}$ in $\mathbb{C}^n$ respectively.

A nonempty compact convex set in $\mathbb{R}^m$ is called a convex body. A set $K \subset \mathbb{C}^n$ is called a complex convex body if $\imath K$ is a convex body in $\mathbb{R}^{2n}$, where $\imath$ is the canonical isomorphism between $\mathbb{C}^n$ (viewed as a real vector space) and $\mathbb{R}^{2n}$, i.e.,

$$\imath(c) = (\Re[c_1], \ldots, \Re[c_n], \Im[c_1], \ldots, \Im[c_n]), \quad c = (c_1, c_2, \cdots, c_n) \in \mathbb{C}^n.$$

Here, $\Re$ and $\Im$ are the real part and imaginary part, respectively. It is easy to check that

$$\Re[x \cdot y] = \imath x \cdot \imath y \tag{1}$$

for all $x, y \in \mathbb{C}^n$.

Let $\mathscr{K}(\mathbb{R}^m)$ denote the set of convex bodies in $\mathbb{R}^m$ and $\mathscr{K}_o(\mathbb{R}^m)$ denote the set of convex bodies that contain the origin in their interiors. The convex body $K \in \mathscr{K}(\mathbb{R}^m)$ is uniquely determined by its support function $h_K : \mathbb{R}^m \to \mathbb{R}$, where

$$h_K(x) = \max\{x \cdot y : y \in K\} \quad \forall x \in \mathbb{R}^m. \tag{2}$$

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See [49, Theorem 1.7.1] for details.

The radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n\backslash\{0\} \to [0,\infty)$, of a compact star-shaped (about the origin) $K \subset \mathbb{R}^m$, is defined, for $x \neq 0$, by

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}. \tag{3}$$

A star body (about the origin) in $\mathbb{R}^m$ is a compact star-shaped (about the origin) set whose radial function is positive and continuous. Obviously, a convex body containing the origin in its interior is a star body about the origin.

$L_p$ centroid bodies were introduced by Lutwak et al. [40]. Given a star body about the origin $K \subset \mathbb{R}^m$ and $p \geqslant 1$, its $L_p$ centroid body is the convex body $\Gamma_p K$ with support function

$$h_{\Gamma_p K}(u) = \left(\frac{1}{|K|}\int_K |u\cdot x|^p dx\right)^{\frac{1}{p}} \qquad \forall u \in S^{m-1}. \tag{4}$$

Here, integration is with respect to the Lebesgue measure. For a real number $t \in \mathbb{R}$, $|t|$ is the norm of $t$, and for a measurable set $M \subset \mathbb{R}^m$, $|M|$ stands for the volume of $M$, i.e., the $m$-dimensional Lebesgue measure of $M$.

When $p = 1$, the $L_1$ centroid body is just the classical centroid body, which was attributed by Blaschke to Dupin (see, e.g., Section 10.8 in [49] for references).

Lutwak et al. [40] prove the following real $L_p$ Busemann-Petty centroid inequality (it should be mentioned that the coefficient in the definition of the $L_p$ centroid body in this paper is different from that in [40]):

THEOREM 1.1. [40, Theorem 1] *Let $K \subset \mathbb{R}^m$ be a star body about the origin. Then, for $p \geqslant 1$,*

$$|K|^{-1}\left|\Gamma_p K\right| \geqslant |B_m|^{-1}\left|\Gamma_p B_m\right|, \tag{5}$$

*with equality if and only if $K$ is an origin-symmetric ellipsoid.*

For the $L_p$ Busemann-Petty centroid inequality and its applications, we refer to [11, 19, 22, 37, 40, 41, 43, 45, 46].

Complex convex geometry has been studied in [1, 2, 3, 4, 5, 6, 7, 8, 18, 23, 30, 31, 32, 33, 53, 54]. Inspired by Haberl [18], we first introduce the definition of the complex $L_p$ centroid body.

Let $\mathscr{K}(\mathbb{C}^n)$, $\mathscr{K}_o(\mathbb{C}^n)$ and $\mathscr{S}_o(\mathbb{C}^n)$ denote the set of complex convex bodies, the set of complex convex bodies containing the origin in their interiors, and the set of complex star bodies about the origin, respectively. Here, a set $K \subset \mathbb{C}^n$ is called a complex star body (about the origin) if $\imath K$ is a star body (about the origin) in $\mathbb{R}^{2n}$.

The volume of a complex measurable set $M \subset \mathbb{C}^n$, $|M|$, is defined as the $2n$-dimensional Lebesgue measure of $\imath M$, i.e., $|M| := |\imath M|$. The complex convex body $K \in \mathscr{K}(\mathbb{C}^n)$ is uniquely determined by its support function $h_K : \mathbb{C}^n \to \mathbb{R}$, where

$$h_K(x) = \max\{\Re[x\cdot y] : y \in K\}.$$

The uniqueness can be deduced from the fact that a real convex body in $\mathbb{R}^{2n}$ is uniquely determined by its real support function and the relation

$$h_K = h_{\iota K} \circ \iota, \tag{6}$$

which follows from (1) and (2).

For $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$, the complex support function of the complex $L_p$ centroid body $\Gamma_{C,p}K$ of $K \in \mathscr{S}_o(\mathbb{C}^n)$ is defined as

$$h_{\Gamma_{C,p}K}(u) = \left(\frac{1}{|K|}\int_K h_{Cu}^p(x)dx\right)^{\frac{1}{p}} \quad \forall u \in \mathbb{S}^n, \tag{7}$$

where the integration is with respect to the push forward of the Lebesgue measure under the canonical isomorphism between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ (see Section 2 for the definition of $Cu$).

When $p = 1$, the complex $L_1$ centroid body is just the complex centroid body introduced by Haberl [18]. When $K \in \mathscr{S}_o(\mathbb{C}^n)$ and $C = [-1,1]$, i.e., the line segment between the points $-1$ and $1$ in the complex plane, $\Gamma_{[-1,1],p}K$ is denoted by $\Gamma_p K$ for short. It follows from (6), (4) and (7) that

$$\Gamma_p K = \iota^{-1}(\Gamma_p \iota K). \tag{8}$$

We will prove the following complex $L_p$ Busemann-Petty centroid inequality (see Section 2 for the definition of complex $L_p$ zonoid).

THEOREM 1.2. *Let $p \geqslant 1$, $K \in \mathscr{S}_o(\mathbb{C}^n)$, and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid. Then,*

$$|K|^{-1}\left|\Gamma_{C,p}K\right| \geqslant |\mathbb{B}_n|^{-1}\left|\Gamma_{C,p}\mathbb{B}_n\right|. \tag{9}$$

*If* $\dim C = 1$, *equality holds if and only if $K$ is an origin-symmetric ellipsoid. If* $\dim C = 2$ *and $p \in [1,\infty)$ is not an even integer, equality holds if and only if $K$ is an origin-symmetric Hermitian ellipsoid.*

Here, $\dim C$ denotes the dimension of $\iota C$ in $\mathbb{R}^2$.

When $C = [-1,1]$, by (8), Theorem 1.2 generalizes the real $L_p$ Busemann-Petty centroid inequality (5) in $\mathbb{R}^{2n}$.

By Theorem 7.3 of [18], if $K \in \mathscr{K}_o(\mathbb{C}^n)$ is origin-symmetric, then $\Gamma_C K = \Gamma_{\Delta C}K$, where $\Delta C$, the central symmetral of $C$, is an origin-symmetric convex body in the complex plane (see Section 2 of [18] for details). In that section, Haberl also points out that every origin-symmetric planar complex convex body is a complex $L_1$ zonoid. Therefore, the complex $L_p$ Busemann-Petty centroid inequality (9) for $p = 1$ implies the following complex Busemann-Petty centroid inequality.

THEOREM 1.3. [18, Theorem 1.2] *Let $C \in \mathscr{K}(\mathbb{C})$ and $K \in \mathscr{K}_o(\mathbb{C}^n)$. If $K$ is origin-symmetric, then*

$$|K|^{-1}|\Gamma_C K| \geqslant |\mathbb{B}_n|^{-1}|\Gamma_C \mathbb{B}_n|.$$

*If* $\dim C = 1$, *equality holds if and only if $K$ is an origin symmetric ellipsoid. If* $\dim C = 2$, *equality holds if and only if $K$ is an origin symmetric Hermitian ellipsoid.*

A further class of relevant convex bodies in this context is $L_p$ projection bodies introduced in [40] for $p \geqslant 1$. Given $K \in \mathscr{K}_o(\mathbb{R}^m)$ and $p \geqslant 1$, the $L_p$ projection body of $K$ is the origin-symmetric convex body $\Pi_p K$ with support function

$$h_{\Pi_p K}(u) = \left( \int_{S^{m-1}} |u \cdot v|^p dS_p(K,v) \right)^{\frac{1}{p}} \quad \forall u \in S^{m-1}, \tag{10}$$

where $S_p(K,\cdot)$ is $L_p$ surface area measure of $K$.

There have been many relevant papers about $L_p$ projection bodies over the past few decades (see [9, 10, 13, 19, 20, 30, 36, 37, 40, 44, 47, 50, 51, 56]). In particular, $L_1$ projection bodies, i.e., projection bodies, were introduced at the turn of the previous century by Minkowski. It is worth pointing out that projection bodies are the only Minkowski valuations that are contravariant with respect to the real affine group (see [17, 34, 35]).

Lutwak et al. [40] prove the following real $L_p$ Petty projection inequality (here, $\Pi_p^* K$ denotes the polar set of $\Pi_p K$ as in [40]).

THEOREM 1.4. [40, Theorem 2] *Let $K \in \mathscr{K}_o(\mathbb{R}^m)$. Then, for $p \geqslant 1$,*

$$|K|^{\frac{n-p}{p}} \left|\Pi_p^* K\right| \leqslant |B_m|^{\frac{n-p}{p}} \left|\Pi_p^* B_m\right|, \tag{11}$$

*with equality if and only if $K$ is an origin-symmetric ellipsoid.*

For $p \geqslant 1$, $C \in \mathscr{K}(\mathbb{C})$ and $K \in \mathscr{K}_o(\mathbb{C}^n)$, we define the complex $L_p$ projection body $\Pi_{C,p}K$ as the convex body with support function

$$h_{\Pi_{C,p}K}(u) = \left( \int_{\mathbb{S}^n} h_{Cu}(v)^p dS_p(K,v) \right)^{\frac{1}{p}} \quad \forall u \in \mathbb{S}^n. \tag{12}$$

Here, $S_p(K,\cdot)$ is the complex $L_p$ surface area measure of $K$ (see Section 2 for the precise definition). The set $\Pi_{[-1,1],p}K$ is denoted by $\Pi_p K$ for short. The equalities (6), (10) and (12) give that

$$\Pi_p K = \iota^{-1}(\Pi_p \iota K). \tag{13}$$

When $p = 1$, Abardia and Bernig [3] proved that $\Pi_{C,1}$ are the only Minkowski valuations that are contravariant with respect to the complex affine group. Complex $L_1$ projection bodies, i.e., complex projection bodies, have also been studied in [33, 52, 18].

We will prove the following complex $L_p$ projection inequality.

THEOREM 1.5. *Let $p \geqslant 1$, $K \in \mathscr{K}_o(\mathbb{C}^n)$, and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid. Then,*

$$|K|^{\frac{2n-p}{p}} \left|\Pi_{C,p}^* K\right| \leqslant |\mathbb{B}_n|^{\frac{2n-p}{p}} \left|\Pi_{C,p}^* \mathbb{B}_n\right|. \tag{14}$$

*If $\dim C = 1$, equality holds if and only if K is an origin-symmetric ellipsoid. If $\dim C = 2$ and $p \in [1,\infty)$ is not an even integer, equality holds if and only if K is an origin-symmetric Hermitian ellipsoid.*

Here, $\Pi^*_{C,p}K$ denotes the polar set of the complex $L_p$ projection body of $K$ (see Section 2 for the precise definition).

When $C=[-1,1]$, by (13), Theorem 1.5 generalizes the real $L_p$ Petty projection inequality (11) in $\mathbb{R}^{2n}$.

When $p=1$, the complex $L_p$ Projection body inequality is the following complex Petty Projection inequality.

THEOREM 1.6. [18, Theorem 5.6] *Let $C\in\mathscr{K}(\mathbb{C})$ be origin-symmetric and $K\in\mathscr{K}_o(\mathbb{C}^n)$. Then,*

$$|K|^{2n-1}\,|\Pi^*_C K|\leqslant|\mathbb{B}_n|^{2n-1}\,|\Pi^*_C\mathbb{B}_n|.$$

*If* $\dim C=1$, *equality holds if and only if $K$ is an ellipsoid. If* $\dim C=2$, *equality holds if and only if $K$ is an Hermitian ellipsoid.*

Haberl [18] also proves that this complex $L_1$ Petty projection inequality strengthens and directly implies the isoperimetric inequality, and it is invariant with respect to the unitary group. Consequently, the affine inequalities are stronger than their unitary counterparts. Similar phenomenon was observed in [12, 20, 21].

This paper is organized as follows. In Section 2, some basic facts regarding complex convex bodies for quick reference are provided. In Section 3, some properties of complex $L_p$ projection bodies and complex $L_p$ centroid bodies are presented. In Section 4, we prove that the complex $L_p$ Busemann-Petty centroid inequality (9) is equivalent to the complex $L_p$ Petty projection inequality (14). Theorem 1.5 is proved in Section 5. In Section 6, we prove Theorem 1.2 by using Theorem 1.5.

## 2. Preliminaries

For a complex number $c\in\mathbb{C}$, we write $\bar{c}$ for its complex conjugate and $|c|$ for its norm. If $\phi\in\mathbb{C}^{n\times n}$ for an integer $n\geqslant 1$, then $\phi^*$ denotes the conjugate transpose of $\phi$. If $\phi$ is invertible, the inverse of $\phi$ is denoted by $\phi^{-1}$. The standard Hermitian inner product on $\mathbb{C}^n$ is conjugate linear in the first argument, i.e., $x\cdot y=x^*y \quad \forall x,y\in\mathbb{C}^n$. For a set $N$ in $\mathbb{S}^1$, let $N^c$ denotes the complement of $N$.

The general linear group of $\mathbb{C}^n$ and the special unitary group of degree $n$ of $\mathbb{C}^n$ are denoted by $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SU}(n,\mathbb{C})$, respectively. A linear transformation $\phi\in\mathrm{GL}(n,\mathbb{C})$ is called an Hermitian matrix if and only if $\phi^*=\phi$. Let $\phi\in\mathrm{GL}(n,\mathbb{C})$ be decomposed in its real part and imaginary part, i.e., $\phi=\mathfrak{R}[\phi]+i\mathfrak{I}[\phi]$. The real matrix representation $\mathbb{R}[\phi]\in\mathrm{GL}(2n,\mathbb{R})$ of $\phi$ is the block matrix

$$\mathbb{R}[\phi]=\begin{pmatrix}\mathfrak{R}[\phi] & -\mathfrak{I}[\phi]\\ \mathfrak{I}[\phi] & \mathfrak{R}[\phi]\end{pmatrix}.$$

It is not hard to show that

$$|\det\phi|^2=|\det\mathbb{R}[\phi]| \tag{15}$$

as well as

$$\iota(\phi x)=\mathbb{R}[\phi]\iota x \quad \forall x\in\mathbb{C}^n. \tag{16}$$

We present some properties of the volume of a complex set $K \subset \mathbb{C}^n$. For $\phi \in \mathrm{GL}(n,\mathbb{C})$, by (16),

$$|\phi K| = |\iota(\phi K)| = |\mathbb{R}[\phi]\iota K|.$$

Thus, relation (15) implies

$$|\phi K| = |\det \phi|^2 |K|. \tag{17}$$

In particular, we have

$$|cK| = |c|^{2n}|K| \tag{18}$$

for all $c \in \mathbb{C}$, where $cK = \{cx : x \in K\}$.

For $K, L \in \mathscr{K}(\mathbb{C}^n)$, $K$ and $L$ are real dilates if there exists $t > 0$ such that $K = tL$.

Next, we provide some properties of ellipsoids. A convex body $K \in \mathscr{K}(\mathbb{C}^n)$ is called an ellipsoid if $\iota K$ is a real ellipsoid, or equivalently, there exists a positive definite symmetric matrix $\varphi \in \mathrm{GL}(2n,\mathbb{R})$ and a $t \in \mathbb{C}^n$ such that

$$K = \{x \in \mathbb{C}^n : \iota x \cdot \varphi \iota x \leqslant 1\} + t.$$

A set $K \subset \mathbb{C}^n$ is called a Hermitian ellipsoid if

$$K = \{x \in \mathbb{C}^n : x \cdot \phi x \leqslant 1\} + t$$

for some positive definite Hermitian matrix $\phi \in \mathrm{GL}(n,\mathbb{C})$ and some $t \in \mathbb{C}^n$. The following fact is pointed out in Section 2 of [18]. For readers' convenience, we offer a short proof here.

LEMMA 2.1. *Let $K \in \mathscr{K}(\mathbb{C}^n)$. Then, $K$ is an origin-symmetric Hermitian ellipsoid if and only if there exists a positive definite Hermitian matrix $\psi \in \mathrm{GL}(n,\mathbb{C})$ such that*

$$K = \psi \mathbb{B}_n.$$

*Proof.* Let

$$K = \psi\mathbb{B}_n = \{\psi x \in \mathbb{C}^n : x \cdot x \leqslant 1\} = \left\{y \in \mathbb{C}^n : y \cdot (\psi^{-1})^* \psi^{-1} y \leqslant 1\right\},$$

where $\psi \in \mathrm{GL}(n,\mathbb{C})$ is a positive definite Hermitian matrix. Since $(\psi^{-1})^*\psi^{-1}$ is a positive definite Hermitian matrix (see [55, Theorem 8.1]), $K$ is an origin-symmetric Hermitian ellipsoid.

Now, assume that $K$ is an origin-symmetric Hermitian ellipsoid, i.e.,

$$K = \{x \in \mathbb{C}^n : x \cdot \phi x \leqslant 1\}$$

for some positive definite Hermitian matrix $\phi \in \mathrm{GL}(n,\mathbb{C})$. An application of [55, Theorem 8.1] shows that there exists a positive definite Hermitian matrix $\psi \in \mathrm{GL}(n,\mathbb{C})$ such that

$$\phi = (\psi^{-1})^*\psi^{-1}.$$

Now, by the sesquilinearity of the Hermitian inner product,

$$\psi^{-1}K = \left\{\psi^{-1}x \in \mathbb{C}^n : x \cdot (\psi^{-1})^*\psi^{-1}x \leqslant 1\right\} = \{y \in \mathbb{C}^n : y \cdot y \leqslant 1\} = \mathbb{B}_n. \quad \square$$

The following lemma follows from [18, Lemma 3.1]. For readers' convenience, we offer a short proof here.

LEMMA 2.2. *If $K \in \mathscr{K}(\mathbb{C}^n)$ is an origin-symmetric Hermitian ellipsoid, then $cK = K$ for all $c \in \mathbb{S}^1$.*

*Proof.* Let $K$ be an origin-symmetric Hermitian ellipsoid. Then, there exists a positive definite Hermitian matrix $\phi \in \mathrm{GL}(n,\mathbb{C})$ such that

$$K = \{x \in \mathbb{C}^n : x \cdot \phi x \leqslant 1\}.$$

This gives that

$$cK = \left\{x \in \mathbb{C}^n : (c^{-1}x) \cdot \phi(c^{-1}x) \leqslant 1\right\}.$$

The sesquilinearity of the Hermitian inner product implies that

$$\left(c^{-1}x\right) \cdot \phi\left(c^{-1}x\right) = \left(\overline{c^{-1}}c^{-1}\right) x \cdot \phi x = \left|c^{-1}\right|^2 x \cdot \phi x.$$

Note that $|c^{-1}| = 1$. This implies that $cK = K$. $\square$

We also need the following lemma to deal with the equality cases of Theorem 1.2 and Theorem 1.5. The following lemma is an easy application of [18, Theorem 3.4], since, for any origin-symmetric convex body $K$ in $\mathbb{C}^n$, $\Delta K = \frac{1}{2}K + \frac{1}{2}(-K) = K$ (see Section 2 of [18] for details).

LEMMA 2.3. *Let $K \in \mathscr{K}(\mathbb{C}^n)$ be an origin-symmetric ellipsoid. Then, $K$ is Hermitian if and only if $cK = K$ for some $c \in \mathbb{S}^1$ with $\Im[c] \neq 0$.*

In the sequel, we collect complex reformulations of well known results from convex geometry. These complex versions can be directly deduced from their real counterparts by an appropriate application of $\iota$. The standard references for these real results are the books of Gardner [13], Gruber [16] and Schneider [49].

**Elements of complex support functions and complex radial functions**

For $K \in \mathscr{K}(\mathbb{C}^n)$, it is easy to see that

$$h_{\phi K} = h_K \circ \phi^* \quad \forall \phi \in \mathrm{GL}(n,\mathbb{C}) \tag{19}$$

and

$$h_{\lambda K} = \lambda h_K \quad \forall \lambda > 0.$$

Let $C \in \mathscr{K}(\mathbb{C})$ and $u, v \in \mathbb{S}^n$. The convex body $Cu \in \mathscr{K}(\mathbb{C}^n)$ is defined as $Cu = \{cu : c \in C\}$. By the conjugate symmetry of the Hermitian inner product and the definition of support functions,

$$\begin{aligned} h_{Cu}(v) &= \max_{c \in C}\{\Re[v \cdot (cu)]\} = \max_{c \in C}\{\Re[(v \cdot u)c]\} = \max_{c \in C}\{\Re[\overline{u \cdot v}c]\} \\ &= \max_{c \in C}\{\Re[(u \cdot v) \cdot c]\} = h_C(u \cdot v). \end{aligned} \tag{20}$$

Using a similar method, we obtain

$$h_{cK}(v) = h_K(\bar{c}v) \quad \forall c \in \mathbb{S}^1 \tag{21}$$

and

$$h_{\bar{C}u}(v) = h_{Cv}(u), \tag{22}$$

where $\overline{C} := \{\bar{c} : c \in C\}$.

The complex radial function, $\rho_K = \rho(K,\cdot) : \mathbb{C}^n \backslash \{0\} \to [0,\infty)$, of a complex star body (about the origin) $K \subset \mathbb{C}^n$, is defined, for $x \neq 0$, by

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}.$$

It is easy to see that

$$\rho_{\phi K} = \rho_K \circ \phi^{-1} \quad \forall \phi \in \mathrm{GL}(n,\mathbb{C}). \tag{23}$$

It follows from (3) that

$$\rho_K = \rho_{\iota K} \circ \iota \tag{24}$$

and

$$\rho_{\lambda K} = \lambda \rho_K \quad \forall \lambda > 0. \tag{25}$$

Given $M \subset \mathbb{C}^n$, its polar set $M^*$ is defined by

$$M^* = \{x \in \mathbb{C}^n : \Re[x \cdot y] \leqslant 1 \text{ for all } y \in M\}.$$

It is easy to see that

$$(\phi M)^* = \phi^{-*} M^* \tag{26}$$

and

$$(\lambda M)^* = \lambda^{-1} M^* \quad \forall \lambda > 0. \tag{27}$$

If $K \in \mathscr{K}_o(\mathbb{C}^n)$, it is easy to verify that

$$\rho_{K^*} = h_K^{-1}. \tag{28}$$

An application of polar coordinates to the volume of a complex star body $K \in \mathscr{S}_o(\mathbb{C}^n)$ gives that

$$|K| = \frac{1}{2n} \int_{\mathbb{S}^n} \rho_K^{2n} d\sigma, \tag{29}$$

where $\sigma$ stands for the push forward with respect to $\iota^{-1}$ of $\mathscr{H}^{2n-1}$ on the $(2n-1)$-dimensional Euclidean unit sphere. Here, $\mathscr{H}^{2n-1}$ denotes the $(2n-1)$-dimensional Hausdorff measure in $\mathbb{R}^{2n}$.

A change to polar coordinates in (7) shows

$$h^p_{\Gamma_{C,p}K}(u) = \frac{1}{(2n+p)|K|} \int_{\mathbb{S}^n} h^p_{Cu} \rho_K^{2n+p} d\sigma. \tag{30}$$

**Elements of complex $L_p$ mixed volume**

For two real numbers $c,d \geqslant 0$ and $K,L \in \mathscr{K}_o(\mathbb{R}^m)$, the real $L_p$ Minkowski combination $c\cdot K +_p d\cdot L$ for $p \geqslant 1$ is defined as

$$h^p_{c\cdot K+_p d\cdot L} = ch^p_K + dh^p_L.$$

The real $L_p$ mixed volume $V_p(K,L)$ is defined by

$$V_p(K,L) = \frac{p}{m}\lim_{\varepsilon\to 0^+}\frac{|K+_p\varepsilon\cdot L|-|K|}{\varepsilon}. \tag{31}$$

See [38] for details.

We turn to the complex case. Given two real numbers $c,d \geqslant 0$ and $K,L \in \mathscr{K}_o(\mathbb{C}^n)$, the complex $L_p$ Minkowski combination $c\cdot K +_p d\cdot L$ for $p \geqslant 1$ is defined as

$$h^p_{c\cdot K+_p d\cdot L} = ch^p_K + dh^p_L.$$

By (19), we have $\phi K +_p \varepsilon\cdot\phi L = \phi(K+_p\varepsilon\cdot L)$ $\forall \phi \in \mathrm{GL}(n,\mathbb{C})$, $\varepsilon > 0$. We define the complex $L_p$ mixed volume $V_p(K,L)$ by

$$V_p(K,L) = \frac{p}{2n}\lim_{\varepsilon\to 0^+}\frac{|K+_p\varepsilon\cdot L|-|K|}{\varepsilon}. \tag{32}$$

By (31), this definition gives that

$$V_p(K,L) = V_p(\imath K,\imath L). \tag{33}$$

Obviously,

$$V_p(K,K) = |K|, \tag{34}$$

and for $\phi \in \mathrm{GL}(n,\mathbb{C})$, the relation $\phi K +_p \varepsilon\cdot\phi L = \phi(K+_p\varepsilon\cdot L)$, (32) and (17) imply

$$V_p(\phi K,\phi L) = |\det\phi|^2 V_p(K,L). \tag{35}$$

The complex surface area measure $S(K,\cdot)$ of $K \in \mathscr{K}_o(\mathbb{C}^n)$ is the Borel measure on $\mathbb{S}^n$ defined in[18] for every Borel set $\omega \subset \mathbb{S}^n$ by

$$S(K,\omega) = \mathscr{H}^{2n-1}\left(\imath\left\{x\in K : \exists u\in\omega \text{ with } \Re[x\cdot u] = h_K(u)\right\}\right). \tag{36}$$

By (21) and the sesquilinearity of the Hermitian inner product, we obtain

$$S(cK,\omega) = S(K,\bar{c}\omega) \tag{37}$$

for all $c \in \mathbb{S}^1$ and each Borel set $\omega \subset \mathbb{S}^n$. For $p \geqslant 1$, we define the complex $L_p$ surface area measure $S_p(K,\cdot)$ of $K \in \mathscr{K}_o(\mathbb{C}^n)$ as

$$S_p(K,\omega) = \int_\omega h_K^{1-p}(v)\,dS(K,v). \tag{38}$$

Therefore, $S_p(K,\cdot)$ can be viewed as the push-forward of the real $L_p$ surface area measure (introduced in [38]) $S_p(\iota K,\cdot)$ with respect to $\iota^{-1}$.

By (21) and (37), we get

$$S_p(cK,\omega) = S_p(K,\bar{c}\omega) \tag{39}$$

for all $c \in \mathbb{S}^1$ and each Borel set $\omega \subset \mathbb{S}^n$.

For $K,L \in \mathscr{K}_o(\mathbb{C}^n)$, as it was shown in [38], the real $L_p$ mixed volume $V_p(\iota K,\iota L)$ has the following representation

$$V_p(\iota K,\iota L) = \frac{1}{2n}\int_{S^{2n-1}} h(\iota L,u)^p dS_p(\iota K,u),$$

where $S_p(\iota K,\cdot)$ is the $L_p$ surface area measure of $\iota K$. Thus, by (6) and (33), we get the representation

$$V_p(K,L) = \frac{1}{2n}\int_{\mathbb{S}^n} h(L,u)^p dS_p(K,u). \tag{40}$$

The complex $L_p$ Minkowski inequality states that, for $K,L \in \mathscr{K}_o(\mathbb{C}^n)$,

$$V_p(K,L) \geqslant |K|^{\frac{2n-p}{2n}}|L|^{\frac{p}{2n}}, \tag{41}$$

with equality if and only if $K$ and $L$ are real dilates. The inequality (41) follows from (33) and the real $L_p$ Minkowski inequality in $\mathbb{R}^{2n}$ proved in [38].

### Elements of complex $L_p$ dual mixed volume

For $K,L \in \mathscr{K}(\mathbb{C}^n)$, $p \geqslant 1$ and $\varepsilon > 0$, the $L_p$-harmonic radial combination $K\widetilde{+}_{-p}\varepsilon\cdot L$ is the star body defined by

$$\rho(K\widetilde{+}_{-p}\varepsilon\cdot L,\cdot)^{-p} = \rho(K,\cdot)^{-p} + \varepsilon\rho(L,\cdot)^{-p}.$$

By (23), we have $\phi(K\widetilde{+}_{-p}\varepsilon\cdot L) = \phi K\widetilde{+}_{-p}\varepsilon\cdot\phi L \quad \forall\phi \in \mathrm{GL}(n,\mathbb{C})$. The complex $L_p$ dual mixed volume $\widetilde{V}_{-p}(K,L)$ is defined by

$$\widetilde{V}_{-p}(K,L) = \frac{-p}{2n}\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}_{-p}\varepsilon\cdot L) - V(K)}{\varepsilon}. \tag{42}$$

By (29), we obtain

$$\widetilde{V}_{-p}(K,L) = \frac{1}{2n}\int_{\mathbb{S}^n} \rho_K^{2n+p}\rho_L^{-p} d\sigma \tag{43}$$

and

$$\widetilde{V}_{-p}(K,K) = |K|. \tag{44}$$

For $\phi \in \mathrm{GL}(n,\mathbb{C})$, the relation $\phi(K\widetilde{+}_{-p}\varepsilon\cdot L) = \phi K\widetilde{+}_{-p}\varepsilon\cdot\phi L$, (42) and (17) imply

$$\widetilde{V}_{-p}(\phi K,\phi L) = |\det\phi|^2\widetilde{V}_{-p}(K,L). \tag{45}$$

An application of Hölder's inequality to (43) and (29) gives that

$$\widetilde{V}_{-p}(K,L) \geqslant |K|^{\frac{2n+p}{2n}}|L|^{\frac{-p}{2n}}, \tag{46}$$

with equality if and only if $K$ and $L$ are real dilates.

For the real $L_p$-harmonic radial combination, the real $L_p$ dual mixed volume and related inequalities, we refer to [39].

### Elements of complex $L_p$ zonoids

For $p \geqslant 1$, a real origin-symmetric convex body $K \in \mathscr{K}(\mathbb{R}^m)$ is called a real $L_p$ zonoid if its support function equals the $L_p$-cosine transform (see [40, 42] for this subject) of some finite even Borel measure on the real unit sphere. Namely, there exists a finite even Borel measure $\mu$ on the sphere $S^{m-1}$ such that

$$h_K(x) = \left(\int_{S^{m-1}} |x\cdot v|^p d\mu(v)\right)^{\frac{1}{p}} \quad \forall x \in \mathbb{R}^m.$$

Note that the right-hand side of this equality is positively homogeneous and subadditive by Minkowski's inequality with respect to $x$, and thus is a support function of a convex body by [49, Theorem 1.7.1]. If $p \in [1,\infty)$ is not an even integer, $\mu$ is uniquely determined by $K$. Indeed, when $p \in [1,\infty)$ is not an even integer, Lutwak et al. (see page 178 of Section 5 of [42]) point out that the $L_p$-cosine transform is injective, i.e., Lemma 2.4. See also Goodey and Weil [14, 15], Koldobsky [24, 25, 26, 27, 28, 29], and Rubin [48].

LEMMA 2.4. *Suppose that $p \in [1,\infty)$ is not an even integer. If $\mu$ is a finite signed even Borel measure on the unit sphere $S^{m-1}$ satisfying*

$$\int_{S^{m-1}} |u\cdot v|^p d\mu(v) = 0$$

*for all $u \in S^{m-1}$, then $\mu = 0$.*

Using (1), Lemma 2.4, the uniqueness of the $L_p$ surface measure for $p > 1$ (see [38, Corollary 2.3 and Corollary 2.6]) and the fact that a convex body is uniquely determined by its surface area measure (i.e., the case of $p = 1$) up to translations (see Section 2 of [18]), one can obtain the following lemma.

LEMMA 2.5. *Suppose that $p \in [1,\infty)$ is not an even integer. If two origin-symmetric complex convex bodies $K, L \in \mathscr{K}_o(\mathbb{C}^n)$ satisfy*

$$\int_{\mathbb{S}^n} |\Re[u\cdot v]|^p dS_p(K,v) = \int_{\mathbb{S}^n} |\Re[u\cdot v]|^p dS_p(L,v)$$

*for all $u \in \mathbb{S}^n$, then $K = L$.*

Note that $K$ and $L$ are origin-symmetric in this lemma. Thus, when $p=1$, there is no translation between $K$ and $L$.

For $p \geqslant 1$, a convex body $K \in \mathscr{K}(\mathbb{C}^n)$ is called a complex $L_p$ zonoid if $\iota K$ is a real $L_p$ zonoid in $\mathbb{R}^{2n}$. That is, there exists a finite even Borel measure $\mu_{\iota K}$ on the unit sphere $S^{2n-1}$ such that

$$h_{\iota K}(x) = \left( \int_{S^{2n-1}} |x \cdot v|^p d\mu_{\iota K}(v) \right)^{\frac{1}{p}}$$

for every $x \in \mathbb{R}^{2n}$. Define the measure $\mu_K$ on $\mathbb{S}^n$ as the push-forward of $\mu_{\iota K}$ with respect to $\iota^{-1}$. By (6), we get

$$h_K(x) = \left( \int_{\mathbb{S}^n} |\Re(x \cdot v)|^p d\mu_K(v) \right)^{\frac{1}{p}} \quad \forall x \in \mathbb{C}^n. \tag{47}$$

LEMMA 2.6. *Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid. Then, there exists a finite even Borel measure $\mu_C$ on the complex unit circle $\mathbb{S}^1$ such that for all $u, v \in \mathbb{S}^n$,*

$$h_{Cu}(v) = \left( \int_{\mathbb{S}^1} |\Re[cu \cdot v]|^p d\mu_C(c) \right)^{\frac{1}{p}}.$$

*Moreover, when $p \in [1, \infty)$ is not an even integer, $\mu_C$ is uniquely determined by $C$.*

*Proof.* Let $p \geqslant 1$, $u, v \in \mathbb{S}^n$ and $C$ be a complex $L_p$ zonoid. Then, there exists a finite even Borel measure $\mu_C$ on the complex unit circle $\mathbb{S}^1$ such that

$$h_C^p(u \cdot v) = \int_{\mathbb{S}^1} |\Re[c \cdot (u \cdot v)]|^p d\mu_C(c) \quad \forall u, v \in \mathbb{S}^1.$$

It follows from the fact $c \cdot (u \cdot v) = \bar{c}(u \cdot v) = (cu) \cdot v$ and (20) that

$$h_{Cu}^p(v) = h_C^p(u \cdot v) = \int_{\mathbb{S}^1} |\Re[cu \cdot v]|^p d\mu_C(c),$$

which is the desired equality.

When $p \in [1, \infty)$ is not an even integer, by Lemma 2.4, the measure $\mu_{\iota C}$ on the unit sphere $S^1$ is uniquely determined by $\iota C$. Therefore, $\mu_C$ is uniquely determined by $C$. $\square$

## 3. Some properties of the complex $L_p$ projection body and the complex $L_p$ centroid body

LEMMA 3.1. *Let $p \geqslant 1$, $\phi \in \mathrm{GL}(n, \mathbb{C})$ and $C \in \mathscr{K}(\mathbb{C})$. Then,*

$$\Gamma_{C,p}(\phi K) = \phi \Gamma_{C,p} K \quad \forall K \in \mathscr{S}_o(\mathbb{C}^n).$$

*Proof.* Let $K \in \mathscr{S}_o(\mathbb{C}^n)$, $p \geqslant 1$, $\phi \in \mathrm{GL}(n,\mathbb{C})$ and $C \in \mathscr{K}(\mathbb{C})$. The definition of $\Gamma_{C,p}$, (17) together with the transformation formula, (19) and the equality $\phi^* Cu = C(\phi^* u)$ yield

$$\begin{aligned} h^p_{\Gamma_{C,p}(\phi K)}(u) &= \frac{1}{|\phi K|}\int_{\phi K} h^p_{Cu}(x)dx = \frac{1}{|K|}\int_K h^p_{Cu}(\phi x)dx \\ &= \frac{1}{|K|}\int_K h^p_{\phi^* Cu}(x)dx = \frac{1}{|K|}\int_K h^p_{C(\phi^* u)}(x)dx \\ &= h^p_{\Gamma_C K}(\phi^* u) = h^p_{\phi\Gamma_C K}(u) \end{aligned}$$

for all $u \in \mathbb{S}^n$. Thus, we have $\Gamma_{C,p}(\phi K) = \phi\Gamma_{C,p}K$. $\square$

LEMMA 3.2. *Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ with* $\dim C > 0$. *Then, $\Gamma_{C,p}$ maps origin-symmetric balls to origin-symmetric balls. That is, for $r > 0$, $\Gamma_{C,p}(r\mathbb{B}_n)$ is an origin-symmetric ball.*

*Proof.* Let $p \geqslant 1$, $r > 0$ and $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$. By (30) and (18), we get

$$h^p_{\Gamma_{C,p}(r\mathbb{B}_n)}(u) = \frac{1}{(2n+p)|r\mathbb{B}_n|}\int_{\mathbb{S}^n} h^p_{Cu} r^{2n+p} d\sigma = \frac{r^p}{(2n+p)|\mathbb{B}_n|}\int_{\mathbb{S}^n} h^p_{Cu} d\sigma. \tag{48}$$

Now, fix some $u_0 \in \mathbb{S}^n$. Then, for each $u \in \mathbb{S}^n$, there exists a $\vartheta_u \in \mathrm{SU}(n)$ with $\vartheta_u u_0 = u$. Note that $Cu = \vartheta_u C u_0$,

$$h^p_{\Gamma_{C,p}(r\mathbb{B}_n)}(u) = \frac{r^p}{(2n+p)|\mathbb{B}_n|}\int_{\mathbb{S}^n} h^p_{Cu_0} \circ \vartheta^*_u d\sigma$$

Noting that $\sigma$ is $\mathrm{SU}(n)$-invariant, the right-hand side is independent of $u$. Meanwhile, it is greater than zero since $\dim C > 0$. Therefore, $\Gamma_{C,p}(r\mathbb{B}_n)$ is an origin-symmetric ball. $\square$

REMARK 3.1. If $K$ is an origin-symmetric Hermitian ellipsoid, by Lemma 2.1, Lemma 3.1 and Lemma 3.2, $\Gamma_{C,p}K$ is also an origin-symmetric Hermitian ellipsoid.

LEMMA 3.3. *Let $p \geqslant 1$. If $C \subset \mathbb{C}$ is origin-symmetric with* $\dim C = 1$, *then there exists some $c \in \mathbb{C}$ such that*

$$\Gamma_{C,p}K = c\Gamma_p K \quad \forall K \in \mathscr{S}_o(\mathbb{C}^n).$$

*Proof.* Let $K \in \mathscr{S}_o(\mathbb{C}^n)$ and $p \geqslant 1$. By our assumption, $C$ is an origin-symmetric line segment. Therefore, there exists a $d \in \mathbb{C}\backslash\{0\}$ with $C = [-d,d]$ and thus

$$h^p_{\Gamma_C K}(u) = \frac{1}{|K|}\int_K h_{[-1,1](du)}(x)dx = h^p_{\Gamma K}(du).$$

So, (21) implies $h_{\Gamma_C K} = h_{\bar{d}\Gamma K}$. If we set $c := \bar{d}$, the assertion is proved. $\square$

LEMMA 3.4. *Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ with* $\dim C > 0$. *Then, $\Pi_{C,p}$ maps origin-symmetric balls to origin-symmetric balls. That is, for $r > 0$, $\Pi_{C,p}(r\mathbb{B}_n)$ is an origin-symmetric ball.*

*Proof.* Let $p \geqslant 1$, $r > 0$ and $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$. By the definition of the surface area measure (36), we get $S(r\mathbb{B}_n, \cdot) = r^{2n-1}\sigma$. Thus, by (38), the $L_p$ surface area measure $S_p(r\mathbb{B}_n, \cdot) = r^{2n-p}\sigma$. By (12),

$$h^p_{\Pi_{C,p}(r\mathbb{B}_n)}(u) = r^{2n-p}\int_{\mathbb{S}^n} h^p_{Cu}\,d\sigma. \tag{49}$$

A similar method to Lemma 3.2 shows that the right hand side is independent of $u$ and greater than zero. Therefore, $\Pi_{C,p}(r\mathbb{B}_n)$ is an origin-symmetric ball. □

The following lemma connects the complex $L_p$ Petty projection inequality (14) and the complex $L_p$ Busemann-Petty centroid inequality (9). For details, see Section 4.

LEMMA 3.5. *Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ with* $\dim C > 0$. *Then,*

$$V_p\left(K, \Gamma_{\bar{C},p}L\right) = \frac{1}{(2n+p)|L|}\widetilde{V}_{-p}\left(L, \Pi^*_{C,p}K\right)$$

*for all $K \in \mathscr{K}_o(\mathbb{C}^n)$ and all $L \in \mathscr{S}_o(\mathbb{C}^n)$.*

*Proof.* Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$. By (40), (30), (22), Fubini's theorem, (28), (12) and (43), we get

$$\begin{aligned}
V_p\left(K, \Gamma_{\bar{C},p}L\right) &= \frac{1}{2n}\int_{\mathbb{S}^n} h^p_{\Gamma_{\bar{C},p}L}(u)\,dS_p(K,u)\\
&= \frac{1}{2n(2n+p)|L|}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} h^p_{\bar{C}u}(v)\rho_L(v)^{2n+p}\,d\sigma(v)\,dS_p(K,u).\\
&= \frac{1}{2n(2n+p)|L|}\int_{\mathbb{S}^n}\rho_L(v)^{2n+p}\int_{\mathbb{S}^n} h^p_{Cv}(u)\,dS_p(K,u)\,d\sigma(v)\\
&= \frac{1}{2n(2n+p)|L|}\int_{\mathbb{S}^n}\rho_L(v)^{2n+p} h^p_{\Pi_{C,p}K}(v)\,d\sigma(v)\\
&= \frac{1}{2n(2n+p)|L|}\int_{\mathbb{S}^n}\rho_L(v)^{2n+p}\rho_{\Pi^*_{C,p}K}(v)^{-p}\,d\sigma(v)\\
&= \frac{1}{(2n+p)|L|}\widetilde{V}_{-p}\left(L, \Pi^*_{C,p}K\right)
\end{aligned}$$

for all $K \in \mathscr{K}_o(\mathbb{C}^n)$ and all $L \in \mathscr{S}_o(\mathbb{C}^n)$. □

LEMMA 3.6. *Let $p \geqslant 1$, $\phi \in \mathrm{GL}(n,\mathbb{C})$ and $K \in \mathscr{K}_o(\mathbb{C}^n)$. Then, $\Pi_{C,p}(\phi K) = |\det\phi|^{\frac{2}{p}}\phi^{-*}\Pi_{C,p}K$.*

*Proof.* Let $p \geqslant 1$, $\phi \in \mathrm{GL}(n,\mathbb{C})$ and $K \in \mathscr{K}_o(\mathbb{C}^n)$. By Lemma 3.5, Lemma 3.1, (35), (45) and (17), we get

$$\begin{aligned}\widetilde{V}_{-p}\left(L,\Pi^*_{C,p}\phi K\right) &= (2n+p)|L|V_p\left(\phi K,\Gamma_{\bar{C},p}L\right)\\ &= (2n+p)|L||\det\phi|^2V_p\left(K,\Gamma_{\bar{C},p}\phi^{-1}L\right)\\ &= |L|\frac{|\det\phi|^2}{|\det\phi^{-1}L|}\widetilde{V}_{-p}\left(\phi^{-1}L,\Pi^*_{C,p}K\right)\\ &= |\det\phi|^2\widetilde{V}_{-p}\left(L,\phi\Pi^*_{C,p}K\right)\end{aligned}$$

for each $L \in \mathscr{S}_o(\mathbb{C}^n)$. Therefore, by (43) and (25),

$$\widetilde{V}_{-p}\left(L,\Pi^*_{C,p}\phi K\right) = \widetilde{V}_{-p}\left(L,|\det\phi|^{-\frac{2}{p}}\phi\Pi^*_{C,p}K\right)$$

for each $L \in \mathscr{S}_o(\mathbb{C}^n)$. Now, an application of (46) and its equality case imply that

$$\Pi^*_{C,p}\phi K = |\det\phi|^{-\frac{2}{p}}\phi\Pi^*_{C,p}K.$$

Thus, (26) and (27) give the desired conclusion. □

REMARK 3.2. If $K$ is an origin-symmetric Hermitian ellipsoid, by Lemma 2.1, Lemma 3.4 and Lemma 3.6, $\Pi_{C,p}K$ is also an origin-symmetric Hermitian ellipsoid.

## 4. The complex $L_p$ Busemann-Petty inequality is equivalent to the complex $L_p$ Petty projection inequality

Let

$$\mathrm{p}_p(C,K) = \left(|K|^{\frac{2n-p}{p}}\left|\Pi^*_{C,p}K\right|\right)^{-1}\left(|\mathbb{B}_n|^{\frac{2n-p}{p}}\left|\Pi^*_{C,p}\mathbb{B}_n\right|\right)$$

for $K \in \mathscr{K}_o(\mathbb{C}^n)$ and

$$\mathrm{b}_p(C,K) = \left(|K|^{-1}\left|\Gamma_{C,p}K\right|\right)\left(|\mathbb{B}_n|^{-1}\left|\Gamma_{C,p}\mathbb{B}_n\right|\right)^{-1}$$

for $K \in \mathscr{S}_o(\mathbb{C}^n)$.

Note that the complex $L_p$ Petty projection inequality (14) is equivalent to $\mathrm{p}_p(C,K) \geqslant 1$, whereas the complex $L_p$ Busemann-Petty centroid inequality (9) is equivalent to $\mathrm{b}_p(C,K) \geqslant 1$.

LEMMA 4.1. *Let $p \geqslant 1$, $K \in \mathscr{K}_o(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$. Then,*

$$\mathrm{p}_p(C,K) \geqslant \mathrm{b}_p\left(\bar{C},\Pi^*_{C,p}K\right),$$

*with equality if and only if $K$ and $\Gamma_{\bar{C},p}\Pi^*_{C,p}K$ are real dilates.*

*Proof.* Let $p \geqslant 1$, $K \in \mathscr{K}_o(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$. By Lemma 3.2 and Lemma 3.4, $\Gamma_{C,p}\mathbb{B}_n$ and $\Pi_{C,p}\mathbb{B}_n$ are origin-symmetric balls. Thus, $\Pi^*_{C,p}\mathbb{B}_n$ is also an origin-symmetric ball. Furthermore, by (48) and (49), the ratio between the radius of $\Pi_{C,p}\mathbb{B}_n$ and the radius of $\Gamma_{C,p}\mathbb{B}_n$ is $(2n+p)^{\frac{1}{p}}|\mathbb{B}_n|^{\frac{1}{p}}$. Thus,

$$\frac{\left|\Pi_{C,p}\mathbb{B}_n\right|}{\left|\Gamma_{C,p}\mathbb{B}_n\right|} = (2n+p)^{\frac{2n}{p}}|\mathbb{B}_n|^{\frac{2n}{p}},$$

which implies

$$\frac{\left|\Gamma_{C,p}\mathbb{B}_n\right|^{-1}}{\left|\Pi^*_{C,p}\mathbb{B}_n\right|} = (2n+p)^{\frac{2n}{p}}|\mathbb{B}_n|^{\frac{2n}{p}}.$$

Therefore, it suffices to prove that

$$\left(|K|^{\frac{2n}{p}-1}\left|\Pi^*_{C,p}K\right|\right)^{-1} \geqslant (2n+p)^{\frac{2n}{p}}\left(\left|\Pi^*_{C,p}K\right|^{-1}\left|\Gamma_{\bar{C},p}\Pi^*_{C,p}K\right|\right), \tag{50}$$

with equality if and only if $K$ and $\Gamma_{\bar{C},p}\Pi^*_{C,p}K$ are real dilates.

Since $\dim C > 0$ and $K \in \mathscr{K}_o(\mathbb{C}^n)$, $\Pi_{C,p}K$ contains the origin in its interior and thus $\Pi^*_{C,p}K \in \mathscr{K}_o(\mathbb{C}^n)$. By (44) and Lemma 3.5,

$$|\Pi^*_{C,p}K| = \widetilde{V}_{-p}\left(\Pi^*_{C,p}K, \Pi^*_{C,p}K\right) = (2n+p)\left|\Pi^*_{C,p}K\right|V_p\left(K, \Gamma_{\bar{C},p}\Pi^*_{C,p}K\right).$$

Applying (41), we get that

$$1 \geqslant (2n+p)|K|^{\frac{2n-p}{2n}}|\Gamma_{\bar{C},p}\Pi^*_{C,p}K|^{\frac{p}{2n}},$$

with equality if and only if $K$ and $\Gamma_{\bar{C},p}\Pi^*_{C,p}K$ are real dilates. This is equivalent to (50). □

LEMMA 4.2. *Let $p \geqslant 1$. For $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$ and $K \in \mathscr{S}_o(\mathbb{C}^n)$,*

$$\mathrm{b}_p(C,K) \geqslant \mathrm{p}_p\left(\bar{C}, \Gamma_{C,p}K\right),$$

*with equality if and only if $K$ and $\Pi^*_{\bar{C},p}\Gamma_{C,p}K$ are real dilates.*

*Proof.* Let $p \geqslant 1$, $K \in \mathscr{S}_o(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ with $\dim C > 0$. Similar to the proof of Lemma 4.1, we need to prove that

$$|K|^{-1}\left|\Gamma_{C,p}K\right| \geqslant \left(\frac{1}{2n+p}\right)^{\frac{2n}{p}}\left(\left|\Gamma_{C,p}K\right|^{\frac{2n-p}{p}}\left|\Pi^*_{\bar{C},p}\Gamma_{C,p}K\right|\right)^{-1}. \tag{51}$$

Since $\dim C > 0$ and $K \in \mathscr{S}_o(\mathbb{C}^n)$, $\Gamma_{C,p}K$ contains the origin in its interior. By (34) and Lemma 3.5,

$$\left|\Gamma_{C,p}K\right| = V_p\left(\Gamma_{C,p}K, \Gamma_{C,p}K\right) = \frac{1}{(2n+p)|K|}\tilde{V}_{-p}\left(K, \Pi^*_{\bar{C}}\Gamma_{C,p}K\right).$$

The inequality (46) applied to the right-hand side gives

$$\left|\Gamma_{C,p}K\right| \geqslant \frac{1}{2n+p}\,|K|^{\frac{p}{2n}}\left|\Pi^*_{\bar{C}}\Gamma_{C,p}K\right|^{\frac{-p}{2n}},$$

with equality if and only if $K$ and $\Pi^*_{\bar{C},p}\Gamma_{C,p}K$ are real dilates. Rearranging terms yields (51). $\square$

## 5. Proof of the complex $L_p$ Petty projection inequality

We need the following two lemmas to prove the $L_p$ complex Petty projection inequality (14).

LEMMA 5.1. *Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid. Then, for each $K \in \mathscr{K}(\mathbb{C}^n)$, there is a finite even Borel measure $\mu_C$ on the unit circle $\mathbb{S}^1$ such that*

$$h_{\Pi_{C,p}K}(u) = \left(\int_{\mathbb{S}^1} h^p_{\bar{c}\Pi_pK}(u)\,d\mu_C(c)\right)^{\frac{1}{p}} \quad \forall u \in \mathbb{S}^n. \tag{52}$$

*Moreover, the total mass $|\mu_C| := \mu_C(\mathbb{S}^1)$ satisfies*

$$|\mu_C| = \left(\frac{\left|\Pi^*_p\mathbb{B}_n\right|}{\left|\Pi^*_{C,p}\mathbb{B}_n\right|}\right)^{\frac{p}{2n}}.$$

*Proof.* Let $p \geqslant 1$, $K \in \mathscr{K}(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid. Then, (12), Lemma 2.6, Fubini's theorem and (21) yield that there is a finite even Borel measure $\mu_C$ on the unit circle $\mathbb{S}^1$ such that

$$\begin{aligned}
h^p_{\Pi_{C,p}K}(u) &= \int_{\mathbb{S}^n}\int_{\mathbb{S}^1} |\Re[cu\cdot v]|^p\,d\mu_C(c)\,dS_p(K,v)\\
&= \int_{\mathbb{S}^1}\int_{\mathbb{S}^n} |\Re[(cu)\cdot v]|^p\,dS_p(K,v)\,d\mu_C(c)\\
&= \int_{\mathbb{S}^1} h^p_{\Pi_pK}(cu)\,d\mu_C(c)\\
&= \int_{\mathbb{S}^1} h^p_{\bar{c}\Pi_pK}(u)\,d\mu_C(c) \quad \forall u \in \mathbb{S}^n.
\end{aligned}$$

By Lemma 3.4, $h_{\Pi_p\mathbb{B}}$ is constant on $\mathbb{S}^n$. Thus, by (21), $h_{\bar{c}\Pi_p\mathbb{B}}$ is also constant and $h_{\bar{c}\Pi_p\mathbb{B}} = h_{\Pi_p\mathbb{B}}$ for all $c \in \mathbb{S}^1$. The equality (52) and the homogeneity property of support function give

$$\begin{aligned}
h_{\Pi_{C,p}\mathbb{B}}(u) &= \left(\int_{\mathbb{S}^1} h^p_{\bar{c}\Pi_p\mathbb{B}}(u)\,d\mu_C(c)\right)^{\frac{1}{p}}\\
&= h_{\bar{c}\Pi_p\mathbb{B}}(u)\left(\int_{\mathbb{S}^1} d\mu_C(c)\right)^{\frac{1}{p}}\\
&= |\mu_C|^{\frac{1}{p}}h_{\Pi_p\mathbb{B}}(u) = h_{|\mu_C|^{\frac{1}{p}}\Pi_p\mathbb{B}}(u)
\end{aligned}$$

for all $u \in \mathbb{S}^n$. Thus, $\Pi_{C,p}\mathbb{B} = |\mu_C|^{\frac{1}{p}} \Pi_p \mathbb{B}$. By (27) and polarizing, we have $\Pi_{C,p}^* \mathbb{B} = |\mu_C|^{-\frac{1}{p}} \Pi_p^* B$. Therefore, taking the volume on both sides and using (18), we obtain the desired result. $\square$

LEMMA 5.2. *Let $p \geqslant 1$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid with* $\dim C > 0$. *Then, for each $K \in \mathscr{S}_o(\mathbb{C}^n)$, there is a finite even Borel measure $\mu_C$ on the unit circle $\mathbb{S}^1$ such that*

$$\left|\Pi_{C,p}^* K\right| \leqslant |\mu_C|^{-\frac{2n}{p}} \left|\Pi_p^* K\right|, \tag{53}$$

*with equality if and only if there exists a point $d \in \mathbb{S}^1$ with $\bar{c}\Pi_p K = d\Pi_p K$ for $\mu_C$-almost every $c \in \mathbb{S}^1$.*

*Proof.* Let $p \geqslant 1$, $K \in \mathscr{S}_o(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid with $\dim C > 0$. By (29), (28), Lemma 5.1, Jensen's inequality and Fubini's theorem, we obtain

$$\begin{aligned}
\left|\Pi_{C,p}^* K\right| &= \frac{1}{2n} \int_{\mathbb{S}^n} h_{\Pi_{C,p} K}(u)^{-2n} d\sigma(u) \\
&= \frac{|\mu_C|^{-\frac{2n}{p}}}{2n} \int_{\mathbb{S}^n} \left[ \frac{1}{|\mu_C|} \int_{\mathbb{S}^1} h_{\bar{c}\Pi_p K}^p(u) d\mu_C(c) \right]^{-\frac{2n}{p}} d\sigma(u) \\
&\leqslant \frac{|\mu_c|^{-\frac{2n}{p}-1}}{2n} \int_{\mathbb{S}^n} \int_{\mathbb{S}^1} h_{\bar{c}\Pi_p K}(u)^{-2n} d\mu_C(c) d\sigma(u) \\
&= \frac{|\mu_C|^{-\frac{2n}{p}-1}}{2n} \int_{\mathbb{S}^1} \int_{\mathbb{S}^n} h_{\bar{c}\Pi_p K}(u)^{-2n} d\sigma(u) d\mu_C(c) \\
&= |\mu_C|^{-\frac{2n}{p}-1} \int_{\mathbb{S}^1} \left|\bar{c}\Pi_p^* K\right| d\mu_C(c) \\
&= |\mu_C|^{-\frac{2n}{p}-1} \int_{\mathbb{S}^1} \left|\Pi_p^* K\right| d\mu_C(c) \\
&= |\mu_C|^{-\frac{2n}{p}} \left|\Pi_p^* K\right|
\end{aligned}$$

for some finite even Borel measure $\mu_C$ on the unit circle $\mathbb{S}^1$.

It remains to establish the equality condition. To do so, let us first prove the following equivalence for $K \in \mathscr{S}_o(\mathbb{C}^n)$.

$$\text{For each } u \in \mathbb{S}^n : c \mapsto h_{\bar{c}\Pi_p K}(u) \text{ is constant } \mu_C - \text{almost everywhere} \tag{54}$$

$$\Longleftrightarrow$$

$$\exists c_0 \in \mathbb{S}^1 : h_{\bar{c}\Pi_p K}(u) = h_{\bar{c}_0 \Pi_p K}(u) \text{ holds for all } u \in \mathbb{S}^n \text{ and } \mu_C\text{-almost all } c \in \mathbb{S}^1. \tag{55}$$

Note that (55) implies (54). Next, we prove that (54) implies (55).

Assume that (54) holds. Then, for each $u \in \mathbb{S}^n$, there exist $c_u \in \mathbb{S}^1$ and a Borel set $N_u \subset \mathbb{S}^1$ with

$$\mu_C(N_u) = 0 \quad \text{and} \quad h_{\bar{c}\Pi_p K}(u) = h_{\bar{c}_u \Pi_p K}(u) \text{ for all } c \in N_u^c. \tag{56}$$

Let $u \in \mathbb{S}^n$ and $b \in \operatorname{supp}(\mu_C)$, i.e., $b$ belongs to the support set of $\mu_C$. By definition, each open neighborhood of $b$ has positive $\mu_C$ measure and therefore non-empty intersection with $N_u^c$. So, we can find a sequence $\{b_k\}_{k\in N}$ with $b_k \in N_u^c$ and $b_k \to b$. By the continuity of $c \to h_{\bar{c}\Pi_p K}(u)$ and (56), we get

$$h_{\bar{b}\Pi_p K}(u) = \lim_{k\to\infty} h_{\bar{b}_k\Pi_p K}(u) = h_{\bar{c}_u\Pi_p K}(u) = h_{\bar{c}\Pi_p K}(u)$$

for all $c \in N_u^c$. Since $\dim C > 0$, we have $\operatorname{supp}(\mu_C) \neq \emptyset$ and $N_u^c \neq \emptyset$. Thus, there is a $c_0 \in \operatorname{supp}(\mu_C)$ such that, for all $u \in \mathbb{S}^n$ and $c \in \operatorname{supp}(\mu_C)$,

$$h_{\bar{c}\Pi_p K}(u) = h_{\bar{c}_0\Pi_p K}(u).$$

This concludes the proof of the equivalence of (54) and (55), since $\mu_C(\operatorname{supp}(\mu_C)^c) = 0$.

Now, we turn to deal with the equality case. Assume that the equality in (53) holds. Inspecting the above derivation of (53), we know that this happens if and only if, for all $u \in \mathbb{S}^n$, equality holds when Jensen's inequality is applied. Therefore, the equality in (53) holds if and only if, for all $u \in \mathbb{S}^n$, the map $c \to h_{\bar{c}\Pi_p K}(u)$ is constant $\mu_C$-almost everywhere. From the equivalence of (54) and (55), we get that this happens if and only if there exists a $c_0 \in \mathbb{S}^1$ with $h_{\bar{c}\Pi_p K}(u) = h_{\bar{c}_0\Pi_p K}(u)$ for $\mu_C$-almost every $c \in \mathbb{S}^1$. That is, $\bar{c}\Pi_p K = \bar{c}_0\Pi_p K$ for $\mu_C$-almost every $c \in \mathbb{S}^1$. Set $d := \bar{c}_0$, we conclude the proof of equality condition. $\square$

Next, we turn to prove Theorem 1.5.

*Proof of Theorem* 1.5. Let $p \geqslant 1$, $K \in \mathscr{K}_o(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid.

Assume that $\dim C = 0$. Since $C$ is a complex $L_p$ zonoid, by (47), $C$ is origin-symmetric and thus $C = \{o\}$. By (12), we have $\Pi_{C,p}K = \Pi_{C,p}\mathbb{B}_n = \{o\}$. Thus, $\Pi^*_{C,p}K = \Pi^*_{C,p}\mathbb{B}_n = \mathbb{C}^n$ and $|\Pi^*_{C,p}K| = |\Pi^*_{C,p}\mathbb{B}_n| = \infty$. Therefore, the inequality (14) is trivial.

Now, assume that $\dim C > 0$. By Lemma 5.2, (13) and Theorem 1.4, we have

$$\begin{aligned}
|K|^{\frac{2n-p}{p}}\left|\Pi^*_{C,p}K\right| &\leqslant |\mu_C|^{-\frac{2n}{p}}|K|^{\frac{2n-p}{p}}\left|\Pi^*_p K\right| \\
&= |\mu_C|^{-\frac{2n}{p}}|K|^{\frac{2n-p}{p}}\left|\iota\Pi^*_p K\right| \\
&= |\mu_C|^{-\frac{2n}{p}}|\iota K|^{\frac{2n-p}{p}}\left|\Pi^*_p \iota K\right| \\
&\leqslant |\mu_C|^{-\frac{2n}{p}}|\iota\mathbb{B}_n|^{\frac{2n-p}{p}}\left|\Pi^*_p \iota\mathbb{B}_n\right| \\
&= |\mu_C|^{-\frac{2n}{p}}|\mathbb{B}_n|^{\frac{2n-p}{p}}\left|\Pi^*_p \mathbb{B}_n\right|.
\end{aligned}$$

Plugging in the value of the total mass of $\mu_C$ from Lemma 5.1 proves (14).

By Lemma 5.2 and the equality conditions of Theorem 1.4, the equality in (14) holds if and only if the following two conditions hold simultaneously.

(I) there exists a point $d \in \mathbb{S}^1$ with $\bar{c}\Pi_p K = d\Pi_p K$ for $\mu_C$-almost every $c \in \mathbb{S}^1$.

(II) $\imath K$ is an origin-symmetric ellipsoid in $\mathbb{R}^{2n}$, i.e, $K$ is an origin-symmetric ellipsoid in $\mathbb{C}^n$.

We turn to deal with the two equality cases, since the range of $c$ in condition (I) depends on the dimension of $C$.

*Case* 1. When $\dim C = 1$, we prove that the equality in (14) holds if and only if $K$ is an origin-symmetric ellipsoid in $\mathbb{C}^n$.

First, suppose that the equality in (14) holds, which implies that condition (II) holds. Thus, $K$ is an origin-symmetric ellipsoid in $\mathbb{C}^n$.

Now, suppose that $K$ is an origin-symmetric ellipsoid in $\mathbb{C}^n$. We need to prove that the equality in (14) holds, which happens if and only if condition (I) and condition (II) hold. Note that condition (II) holds obviously. It remains to prove that condition (I) holds.

Since $C$ is a complex $L_p$ zonoid, $C$ is an origin-symmetric convex body. Thus, $\dim C = 1$ implies that $C$ is a line segment $[-c_0, c_0]$ for some $c_0 \in \mathbb{C}\setminus\{0\}$. Therefore, by (47), we obtain

$$\mu_C = \frac{|c_0|}{2}\left(\delta_{-\langle c_0\rangle} + \delta_{\langle c_0\rangle}\right),$$

where $\delta$ denotes the Dirac measure and $\langle c_0\rangle := c_0\,|c_0|^{-1}$ stands for the spherical projection of $c_0$ to the unit circle. Therefore, condition (I) holds if and only if $c_0\Pi_p K = -c_0\Pi_p K$. This is always true, since $\Pi_p K$ is origin-symmetric.

*Case* 2. When $\dim C = 2$ and $p \in [1,\infty)$ is not an even integer, we prove that the equality in (14) holds if and only if $K$ is an origin-symmetric Hermitian ellipsoid in $\mathbb{C}^n$.

First, suppose that $K$ is an origin-symmetric Hermitian ellipsoid. By Remark 3.2 and Lemma 2.2, we obtain that condition (I) holds. Note that condition (II) holds obviously. Thus, condition (I) and condition (II) hold, which implies that the equality in (14) holds.

Now, suppose that the equality in (14) holds, which implies that condition (I) and condition (II) hold. We need to show that $K$ is an origin-symmetric Hermitian ellipsoid.

By (21), (12) and (39),

$$\begin{aligned} h^p_{\bar{c}\Pi_p K}(u) = h^p_{\Pi_p K}(cu) &= \int_{\mathbb{S}^n} |\Re[cu\cdot v]|^p dS_p(K,v) \\ &= \int_{\mathbb{S}^n} |\Re[u\cdot \bar{c}v]|^p dS_p(K,v) \\ &= \int_{\mathbb{S}^n} |\Re[u\cdot v]|^p dS_p(\bar{c}K,v). \end{aligned}$$

Thus, condition (I) implies that there exists a point $d \in \mathbb{S}^1$ such that

$$\int_{\mathbb{S}^n} |\Re[u\cdot v]|^p dS_p(\bar{c}K,v) = \int_{\mathbb{S}^n} |\Re[u\cdot v]|^p dS_p(dK,v)$$

holds for $\mu_C$-almost every $c \in \mathbb{S}^1$. By Lemma 2.5 and the fact that $K$ is origin-symmetric, we have

$$\bar{c}K = dK$$

for $\mu_C$-almost every $c \in \mathbb{S}^1$. This implies the existence of a Borel set $N \subset \mathbb{S}^1$ with $\mu_C(N) = 0$ such that $\bar{c}K = dK$ for all $c \in N^c$.

Since $\dim C = 2, N^c$ contains two non-antipodal points, i.e., there exist $c_0, c_1 \in N^c$ such that $c_0 \neq -c_1$ and $\bar{c}_0 K = \bar{c}_1 K$. Clearly, $\bar{c}_0$ and $\bar{c}_1$ are also non-antipodal. So, for $c := \bar{c}_0 \bar{c}_1^{-1}$ we have

$$cK = K, \text{ where } c \in \mathbb{S}^1 \text{ with } \Im[c] \neq 0.$$

Note that $K$ is an origin-symmetric ellipsoid (by condition (II)). Thus, Lemma 2.3 gives that $K$ is an origin-symmetric Hermitian ellipsoid. $\square$

## 6. Proof of the complex $L_p$ Busemann-Petty centroid inequality

In this section, we will prove the complex $L_p$ Busemann-Petty centroid inequality (9) by the complex $L_p$ Petty projection inequality (14) and Theorem 1.1.

*Proof of Theorem* 1.2. Let $p \geqslant 1$, $K \in \mathscr{S}_o(\mathbb{C}^n)$ and $C \in \mathscr{K}(\mathbb{C})$ be a complex $L_p$ zonoid.

Assume that $\dim C = 0$. Since $C$ is a complex $L_p$ zonoid, by (47), $C$ is origin-symmetric and thus $C = \{o\}$. Consequently, by (7), we have $\Gamma_{C,p}K = \Gamma_{C,p}\mathbb{B}_n = \{o\}$ and thus inequality (9) holds trivially true.

Assume that $\dim C = 1$. Since $C$ is a complex $L_p$ zonoid, $C$ is an origin-symmetric convex body. Thus, by $\dim C = 1$, $C$ is an origin-symmetric line segment in the complex plane. By Lemma 3.3 and (18), the inequality (9) is equivalent to

$$|K|^{-1}\left|\Gamma_p K\right| \geqslant |\mathbb{B}_n|^{-1}\left|\Gamma_p \mathbb{B}_n\right|.$$

By (8), this inequality is equivalent to the classical $L_p$ Busemann-Petty centroid inequality. Thus, Theorem 1.1 for $m = 2n$ settles the case where $\dim C = 1$.

Assume that $\dim C = 2$. The inequality (9) follows from the complex $L_p$ Petty projection inequality (14) and Lemma 4.2. That is,

$$\mathrm{b}_b(C,K) \geqslant \mathrm{p}_p\left(\bar{C}, \Gamma_{C,p}K\right) \geqslant 1. \tag{57}$$

We turn to deal the equality case for $\dim C = 2$. We first prove the 'if' part. If $p \in [1,\infty)$ is not an even integer and these equalities in (57) hold, then by Lemma 4.2 and Theorem 1.5, $K$ is a real dilate of $\Pi^*_{\bar{C},p}\Gamma_{C,p}K$ and $\Gamma_{C,p}K$ must be an origin-symmetric Hermitian ellipsoid. Thus, by Remark 3.2, $\Pi_{\bar{C},p}\Gamma_{C,p}K$ is also an origin-symmetric Hermitian ellipsoid. Thus, by Lemma 2.1 and (26), $\Pi^*_{\bar{C},p}\Gamma_{C,p}K$ is also an origin-symmetric Hermitian ellipsoid. Therefore, $K$, being a real dilate of $\Pi^*_{\bar{C},p}\Gamma_{C,p}K$, is an origin-symmetric Hermitian ellipsoid as well.

It remains to show that the equality condition is also sufficient. So, assume that $K$ is an origin-symmetric Hermitian ellipsoid. By Lemma 2.1, $K = \phi\mathbb{B}_n$ for some $\phi \in \mathrm{GL}(n,\mathbb{C})$. Thus, by Lemma 3.1 and (17),

$$|K|^{-1}\left|\Gamma_{C,p}K\right| = |\phi\mathbb{B}_n|^{-1}\left|\Gamma_{C,p}\phi\mathbb{B}_n\right| = |\mathbb{B}_n|^{-1}\left|\Gamma_{C,p}\mathbb{B}_n\right|. \quad \square$$

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