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ON GENERALIZED LEAST POWER APPROXIMATION

NGUYEN QUANG DIEU AND PHUNG VAN MANH*

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Abstract. We study generalized least power approximation corresponding to certain sets of seminorms on Banach spaces. As applications, we construct sets of seminorms for trivariate harmonic polynomials and for Müntz polynomials such that the sequences of the generalized least power approximations converge uniformly.

1. Introduction

Let $\mathbb{K}$ be the real or complex field. Let $\mathscr{P}(\mathbb{K}^d)$ be the vector space of all polynomials in $\mathbb{K}^d$ and $\mathscr{P}_n(\mathbb{K}^d)$ its subspace consisting of all polynomials of total degree at most n. Given a compact subset $K \subset \mathbb{K}^d$ and a bounded function $f : K \to \mathbb{K}$ we denote by $\|f\|_K = \sup_{\mathbf{x}\in K} |f(\mathbf{x})|$ the usual supremum norm on K.

A set $A \subset \mathbb{K}^d$ is said to be *determining* for the space of functions $\mathscr{F}$, or, for short, $\mathscr{F}$-determining, if $p \in \mathscr{F}$ and $p|_A = 0$ force $p \equiv 0$. Here $p|_A$ is restriction of p to A.

Let A be determining for $\mathscr{P}_n(\mathbb{K}^d)$ and $f : A \to \mathbb{K}$. Then according to Calvi and Levenberg in [12, Theorem 1] there exists a unique polynomial $p \in \mathscr{P}_n(\mathbb{K}^d)$ which minimizes the quantity

$$\Phi_{f,A}(q) := \sum_{\mathbf{a}\in A} |q(\mathbf{a}) - f(\mathbf{a})|^2, \quad q \in \mathscr{P}_n(\mathbb{K}^d). \tag{1}$$

The polynomial p is called the discrete least square approximation polynomial. Moreover, in [12] the authors also gave a Lebesgue type inequality which leads to the theory of admissible meshes. They are defined as follows. A sequence of discrete sets $\mathbf{A} = \{A_n \subset K : n \in \mathbb{N}^*\}$ is called an *admissible mesh* for a compact set $K \subset \mathbb{K}^d$ if there exist two positive constants c_1 and c_2 not depending on n such that, for every $n \geqslant 1$ and $p \in \mathscr{P}_n(\mathbb{K}^d)$,

$$\|p\|_K \leqslant c_1 \|p\|_{A_n} \quad \text{and} \quad \#A_n \leqslant c_2 n^m,$$

where $\#A_n$ is the cardinality of A_n and $m \in \mathbb{N}^*$ not depending on n.

For a compact set K, if $\mathbf{A} = \{A_n \subset K : n \in \mathbb{N}^*\}$ is an admissible mesh, then no non-zero polynomial in $\mathscr{P}_n(\mathbb{K}^d)$ vanishes on A_n. Since $\dim \mathscr{P}_n(\mathbb{R}^d) = \binom{n+d}{d} \sim n^d$

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* Corresponding author.

as $n \to \infty$, we must have $m \geqslant d$. When the lower bound is reached, then we get an *optimal meshes*. In other words, an admissible mesh $\mathbf{A}$ is optimal if $\#A_n \leqslant c_3 n^d$ for some $c_3 > 0$ not depending on n.

In [12] the authors pointed out that admissible meshes are preserved by the operations of taking unions, product and transformation of sets under affine automorphisms. These meshes are also stable under small pertubation and analytic transformations, see [19, 22]. From computational point of views, admissible meshes are very useful. Calvi and Levenberg showed in [12] that the sequence of discrete least square approximation polynomials based on admissible meshes approximate sufficiently smooth functions (resp. holomorphic functions) uniformly, where the compact set admits a Markov inequality (resp. the compact set is regular and polynomially convex). Furthermore, in [8, 9, 11], the authors showed that discrete extremal sets of Fekete and Leja types can be extracted from admissible meshes. General construction of admissible meshes and optimal admissible meshes in compact sets in $\mathbb{R}^d$ are recently given by Kroo [14, 15, 16]. In a recent work [23], Piazzon built optimal admissible meshes on two classes of compact set in $\mathbb{R}^d$. Some relative results can be found in [21, 20, 27].

In (1) one can identify $p(a)$ with $\delta_{\mathbf{a}}(p)$, where $\delta_{\mathbf{a}}$ is the Dirac evaluation functional. This fact suggested Phung, Phan and Mai [24] to generalize the discrete least square approximation. More precisely, they replaced the $\delta_{\mathbf{a}}$'s by continuous linear functionals on the space of continuous functions $\mathscr{C}(K)$ and $\mathscr{P}_n(\mathbb{K}^d)$ by a finite dimensional subspace $\mathscr{Q}$ of $\mathscr{P}(\mathbb{K}^d)$. Let $\{\nu_1, \ldots, \nu_N\}$ be a subset of the dual space $\mathscr{Q}'$ such that $\cap_{j=1}^N \ker \nu_j = \{0\}$ and f be a function such that $\nu_j(f)$ is well-defined for $j = 1, \ldots, N$. The authors showed in [24] that there exist a unique polynomial $Q \in \mathscr{Q}$ which minimizes the quantity

$$\widetilde{\Phi}_{f,A}(q) := \sum_{j=1}^{N} |\nu_j(q) - \nu_j(f)|^2, \quad q \in \mathscr{Q}. \tag{2}$$

The polynomial Q is called the generalized least square approximation polynomial. Suppose $\mathscr{F}$ be a subset of $\mathscr{C}(K)$ that contains $\mathscr{Q}$. Let C_1 and C_2 be two positive constants such that

$$\|q\|_K \leqslant C_1 \max_{1 \leqslant j \leqslant N} |\nu_j(q)|, \quad \forall q \in \mathscr{Q}$$

and

$$\max_{1 \leqslant j \leqslant N} |\nu_j(f)| \leqslant C_2 \|f\|_K, \quad \forall f \in \mathscr{F}.$$

Then Q admits the following Lebesgue type inequality

$$\|f - Q\|_K \leqslant \left(1 + 2C_1 C_2 \sqrt{N}\right) \mathrm{dist}_K(f, \mathscr{Q}), \tag{3}$$

where $\mathrm{dist}_K(f, \mathscr{Q}) = \inf\{\|f - g\|_K : g \in \mathscr{Q}\}$. In [24] the authors constructed continuous linear functionals for the spaces of bivariate harmonic polynomials in $\mathbb{R}^2$ and univariate holomorphic polynomials such that the sequence of generalized least square approximation polynomials of bivariate harmonic functions and univariate holomorphic functions on the disks converges uniformly. They consist of sets of points and families of Radon projections. They also constructed admissible meshes on smooth curves Γ in $\mathbb{R}^d$ with cardinality $O(N_n)$, $N_n = \dim \mathscr{P}_n(\mathbb{R}^d)|_\Gamma$.

In this note, we further generalize the theory of discrete least square approximation. The space $\mathscr{P}_n(\mathbb{K}^d)$ and the Dirac evaluation functionals are replaced by a finite dimensional Banach space F and seminorms on F. The power 2 is extended to arbitrary powers λ greater than 1. We prove in Theorem 1 that there exists a unique element which minimizes the some quantity defined by the collection of seminorms and the powers λ when the seminorms satisfy certain conditions. In Theorem 2 we give a Lebesgue type inequality which generalizes (3). It establishes a bound for the error between a continuous function and its generalized least power approximation. We give two applications of the above-mentioned results. Firstly, we construct functionals represented in integral forms corresponding to the space of trivariate harmonic polynomials. We prove that the generalized least power approximation by harmonic polynomials of any function f which is harmonic in a neighborhood of the closed unit ball in $\mathbb{R}^3$ converges geometrically to f. Secondly, we consider the Müntz spaces and construct generalized admissible meshes for these spaces. We show in Proposition 9 that the generalized least power approximations of sufficiently smooth functions converge uniformly. Note that if the functions are only Lipschitz, then we still have approximation result when the power λ is large enough. On the other hand, to get similar result in the least square approximation, the functions must be sufficiently smooth. This is an advantage of the generalized least power approximation compared with the least square approximation.

2. Generalized least power approximation

Our main theorem (Theorem 1) is a sharpening of a key result in [24]. The point is to implement a collection of suitable seminorms and a closed subset H of a finite dimension space F. We start with the following notion which is an extension of the notion $\mathscr{F}$-determining.

DEFINITION 1. Let F be a finite dimensional normed space and m be a positive integer. We say that a collection of seminorms $\mathscr{M} := \{\mu_1, \dots, \mu_m\}$ on F is determining if

$$f \in F, \quad \mu_1(f) = \cdots = \mu_m(f) = 0 \Rightarrow f = 0.$$

Moreover, a determining set $\mathscr{M}$ is called special if, for any $e_0, e_1, \dots, e_m \in F$, all the functions $t \mapsto \mu_j(te_0 + e_j), 1 \leqslant j \leqslant m$ are constant on $[0,1]$ forces $e_0 = 0$.

In what follows we will present a substantial class of special determining set of seminorms on a finite dimension vector space F.

PROPOSITION 1. *Let $\mathscr{M} := \{\mu_1, \dots, \mu_m\}$ be a determining set of seminorms on F such that $\mu_j := |\nu_j|$, where ν_j is a continuous linear functional on F. Then $\mathscr{M}$ is special. We will call such collection $\mathscr{M}$ very special.*

Proof. Let $\{e_0, e_1, \dots, e_m\}$ be vectors in F such that all the functions $\phi_j(t) = \mu_j(te_0 + e_j)$, $1 \leqslant j \leqslant m$, are constant on $[0,1]$. By linearity of ν_j, we have

$$|\nu_j(te_0 + e_j)| = |\nu_j(e_0)t + \nu_j(e_j)|, \quad t \in [0,1].$$

It follows that $\nu_j(e_0) = 0$ for every $j = 1, \dots, m$. Since $\mathscr{M}$ is a determining set, we have $e_0 = 0$. $\square$

The next result is somewhat less trivial to show. It gives us examples of special determining families without being very special.

PROPOSITION 2. *Let K be a fat compact subset of $\mathbb{K}^d$, i.e. $K = \overline{\operatorname{int}(K)}$, and F be a finite dimensional subspace of $\mathscr{C}(K)$. For a given subset $\{f_1, \dots, f_m\}$ of $\mathscr{C}(K)$ we define*

$$\mu_j(f) := \Big(\int\limits_K |f(\mathbf{s}) f_j(\mathbf{s})|^2 \mathrm{d}\lambda(\mathbf{s}) \Big)^{1/2}, \quad f \in F,\ 1 \leqslant j \leqslant m,$$

where λ is the Lebesgue measure on K. Suppose that $f_1, \dots, f_m$ do not vanish simultaneously at any point of K. Then $\mathscr{M} := \{\mu_1, \dots, \mu_m\}$ is a special determining set of seminorms for F.

Proof. Assume that $\mu_j(f) = 0$ for every $j = 1, \dots, m$. Then $f(\mathbf{s}) f_j(\mathbf{s}) = 0$ for a.e. $\mathbf{s} \in K$, $j = 1, \dots, m$. Since the common zero set $\{f_1 = \cdots = f_m = 0\}$ is empty, it follows that $f(\mathbf{s}) = 0$ a.e. $\mathbf{s} \in K$. The hypothesis that K is fat gives $f \equiv 0$.

Now we let $g_0, g_1, \cdots, g_m \in F$ be such that for $1 \leqslant j \leqslant m$ the function

$$\varphi_j(t) := \Big(\int\limits_K |(tg_0 + g_j) f_j|^2 \mathrm{d}\lambda \Big)^{1/2}$$

is constant on $[0,1]$. It follows that the coefficient of t^2 in $\varphi_j^2(t)$ vanishes, and hence

$$\mu_j^2(g_0) = \int_K |g_0 f_j|^2 \mathrm{d}\lambda = 0, \quad \forall j = 1, \dots, m.$$

Since $\mathscr{M}$ is determining we get $g_0 = 0$. The proof is complete. $\square$

THEOREM 1. *Let E be a normed space and H be a closed subset of a finite dimensional subspace of F of E. Let $\mathscr{M} := \{\mu_1, \dots, \mu_m\}$ be a collection of continuous seminorms on E which is determining for F. Then for any assigned elements $e_1, \dots, e_m \in F$ and any set of numbers $\lambda_1 > 0, \dots, \lambda_m > 0$ there exists an element (possibly not unique) $f^* \in H$ such that*

$$\Phi(f) \geqslant \Phi(f^*), \quad \forall f \in H,$$

where

$$\Phi(f) := \sum_{j=1}^m \mu_j(f - e_j)^{\lambda_j}.$$

If $\mathscr{M}$ is special, $\lambda_j > 1$ for all j and if H is convex then such an element f^* is unique. Moreover, if $\mathscr{M}$ is very special, i.e., $\mathscr{M} = \{|\nu_1|, \dots, |\nu_m|\}$ where the ν_i are continuous linear functionals, then f^* depends continuously on $(\nu_1, \dots, \nu_m, e_1, \dots, e_m)$.

Proof. Set $\mu := \max_{1 \leqslant j \leqslant m} \mu_j$. Then μ is a *norm* on F. Thus, since $\dim F < \infty$ there exists a constant $C > 0$ such that

$$\mu(f) \geqslant C\|f\|, \quad \forall f \in F.$$

Now we choose a sequence $\{f_n\} \subset H$ such that

$$\lim_{n \to \infty} \Phi(f_n) = d := \inf_{g \in H} \Phi(g).$$

Since $\lambda > 0$, it is easy to see that

$$\sup_{n \geqslant 1} \mu(f_n) < \infty.$$

Hence $\{f_n\}$ is bounded in the original topology of F (induced by the norm on E). Using again the fact that F is finite dimensional we may extract a subsequence $f_{n_k} \to f^*$. It is then clear that $f^* \in H$ and so $\Phi(f^*) = d$, by continuity of μ_j.

Now we deal with uniqueness of f^* (under some restrictions on μ_j and H). Assume for the sake of seeking a contradiction that there exists $\tilde{f} \in H$ with $\Phi(f^*) = \Phi(\tilde{f})$. Define for $t \in \mathbb{R}$ the function

$$\varphi(t) := \sum_{j=1}^{m} \mu_j(f^* + t(\tilde{f} - f^*) - e_j)^{\lambda_j}.$$

It is easy to see that the function $\varphi_j(t) := \mu_j\big(t(\tilde{f} - f^*) + (f^* - e_j)\big)$ is convex on $\mathbb{R}$. Since $\lambda_j > 1$ we infer that $F_j := \varphi_j^{\lambda_j}$ is convex on $\mathbb{R}$ as well. Hence $\varphi = F_1 + \cdots + F_m$ is convex on $\mathbb{R}$. Since H is convex we get $t(\tilde{f} - f^*) + f^* \in H$. It follows that

$$\varphi(0) = \varphi(1) = d = \min_{0 \leqslant t \leqslant 1} \varphi(t).$$

Now convexity of φ implies that $\varphi \equiv d$ on $[0,1]$. Therefore each convex function F_j must be affine on $[0,1]$. So for $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned} \varphi_j(t_1)^{\lambda_j} + \varphi_j(t_2)^{\lambda_j} &= F_j(t_1) + F_j(t_2) = 2F_j\Big(\frac{t_1 + t_2}{2}\Big) \\ &= 2\varphi_j\Big(\frac{t_1 + t_2}{2}\Big)^{\lambda_j} \leqslant 2\Big(\frac{\varphi_j(t_1) + \varphi_j(t_2)}{2}\Big)^{\lambda_j}. \end{aligned}$$

Here the last inequality follows from convexity of φ_j. Since $\lambda_j > 1$, it is elementary to see that $\varphi_j(t_1) = \varphi_j(t_2)$. Hence each φ_j must be constant on $[0,1]$. Since the collection $\mathscr{M}$ is special we conclude that $\tilde{f} = f^*$.

Now, suppose that $\mathscr{M}$ is very special, we will prove the continuity of f^* with respect to $(\nu_1,\dots,\nu_m,e_1,\dots,e_m)$. For this, we consider a sequence $\{(\nu_{1,k},\dots,\nu_{m,k},e_{1,k},\dots,e_{m,k})\}_{k\geqslant 1}$ with

$$\nu_{j,k}\to\nu_j\in F', e_{j,k}\to e_j\in F, \quad \forall 1\leqslant j\leqslant m.$$

We can find $f^*\in H, f_k^*\in H$ such that

$$\sum_{j=1}^m |\nu_j(f-e_j)|^{\lambda_j}\geqslant \sum_{j=1}^m |\nu_j(f^*-e_j)|^{\lambda_j}, \quad \forall f\in H,$$

and

$$\sum_{j=1}^m |\nu_{j,k}(f-e_{j,k})|^{\lambda_j}\geqslant \sum_{j=1}^m |\nu_{j,k}(f_k^*-e_{j,k})|^{\lambda_j}, \quad \forall f\in H.$$

By taking $f=0$ and using the same reasoning as in the beginning of the proof, we can check that $\{f_k^*\}$ is *bounded* in F. Thus, it suffices to show that every cluster point of $\{f_k^*\}$ coincides with f^*. Without loss of generality we may assume that $f_k^*\to\tilde f\in H$. Then, by letting $k\to\infty$ (while keeping f fixed) in the above estimates we obtain

$$\sum_{j=1}^m |\nu_j(f-e_j)|^{\lambda_j}\geqslant \sum_{j=1}^m |\nu_j(\tilde f-e_j)|^{\lambda_j}, \quad \forall f\in H.$$

Here we apply the following easy fact:

$$g_k\to g\in F\Rightarrow \nu_{j,k}(g_k)\to\nu_j(g).$$

Finally using the uniqueness property of f^* we get $\tilde f=f^*$. So $f_k^*\to f^*$, and the proof is complete. □

Given the datum $\lambda_1=\dots=\lambda_m=\lambda>1$ and $e_1=\dots=e_m=e$, we obtain the following corollary.

COROLLARY 1. *Let E be a normed space and H be a closed convex subset of a finite dimensional subspace of F of E. Let $\mathscr{M}:=\{\mu_1,\dots,\mu_m\}$ be a collection of special seminorms on E which is determining for F. Then for any element $e\in F$ and any $\lambda>1$ there exists a unique element in H, denoted by $\mathbf{P}_H(\mathscr{M};e;\lambda)$, such that*

$$\Phi(f)\geqslant\Phi\big(\mathbf{P}_H(\mathscr{M};e;\lambda)\big), \quad \forall f\in H,$$

where

$$\Phi(f):=\sum_{j=1}^m \mu_j(f-e)^{\lambda}.$$

Let $K\subset\mathbb{K}^d$ be a compact set. Let E,F are normed subspaces of $\mathscr{C}(K)$ with $F\subset E$, $\dim F<\infty$. Assume that K is a F-determining. Equivalently, the set $\{|\delta_{\mathbf{a}}|:\mathbf{a}\in K\}$ is determining for F. It is easily check that the map $q\mapsto\|q\|_K:=\sup\limits_K|q|$ defines a norm on F. As in the proof of Theorem 1, we see that $\mu := \max_{1\leqslant j\leqslant m} \mu_j$ is a norm on F. Now the hypothesis that F is finite dimensional enable us to find a positive constant C_1 such that

$$\|q\|_K \leqslant C_1\mu(q) = \max_{1\leqslant j\leqslant m} \mu_j(q), \quad \forall q \in F. \tag{4}$$

THEOREM 2. *Let $K \subset \mathbb{K}^d$ be a compact set. Let E, F are normed subspaces of $\mathscr{C}(K)$ with $F \subset E$, $\dim F < \infty$. Assume that K is F-determining. Let H be a closed convex subset of F and $\mu_1,\ldots,\mu_m$ be a set of special seminorms on E which is determining for F. Assume that there exists a positive constant C_2 such that*

$$\max_{1\leqslant j\leqslant m} \mu_j(f) \leqslant C_2\|f\|_K, \quad \forall f \in E. \tag{5}$$

Then for every $\lambda > 1$ and $f \in E$ we have the following estimate

$$\|f - \mathbf{P}_H(\mathscr{M}; f; \lambda)\|_K \leqslant (1 + 2C_1C_2m^{\frac{1}{\lambda}})\mathrm{dist}_K(f, H),$$

where $\mathrm{dist}_K(f,H) = \inf\{\|f-g\|_K : g \in H\}$ *and C_1 is given in (4).*

Proof. Since H is a closed subset of the finite dimensional normed space F, we may choose $h \in H$ such that

$$\|f - h\|_K = \mathrm{dist}_K(f, H).$$

For simplicity of notation we let $g := \mathbf{P}_H(\mathscr{M}; f; \lambda)$. Then we have $\Phi(\tilde{g}) \geqslant \Phi(g)$ for all $\tilde{g} \in H$, where

$$\Phi(\tilde{g}) := \sum_{j=1}^{m} \mu_j(\tilde{g} - f)^{\lambda}.$$

Observe the obvious estimate

$$\|f - g\|_K \leqslant \|f - h\|_K + \|g - h\|_K. \tag{6}$$

Since $g - h \in F$ we have

$$\|g - h\|_K \leqslant C_1\mu(g - h). \tag{7}$$

Note that

$$\begin{aligned}
\mu(g-h) \leqslant \Phi(g-h)^{\frac{1}{\lambda}} &= [\sum_{j=1}^{m} \mu_j(g-h)^{\lambda}]^{\frac{1}{\lambda}} \leqslant [\sum_{j=1}^{m} \big(\mu_j(g-f) + \mu_j(f-h)\big)^{\lambda}]^{\frac{1}{\lambda}} \\
&\leqslant [\sum_{j=1}^{m} \mu_j(h-f)^{\lambda}]^{\frac{1}{\lambda}} + [\sum_{j=1}^{m} \mu_j(g-f)^{\lambda}]^{\frac{1}{\lambda}} \leqslant 2[\sum_{j=1}^{m} \mu_j(h-f)^{\lambda}]^{\frac{1}{\lambda}} \\
&\leqslant 2m^{\frac{1}{\lambda}}\mu(h-f) \leqslant 2C_2m^{\frac{1}{\lambda}}\|f-h\|_K.
\end{aligned} \tag{8}$$

Here we use the Minkowski inequality in the fourth relation and the property of g in the fifth relation. Combining (6), (7) and (8) we get

$$\|f - g\|_K \leqslant (1 + 2C_1C_2m^{\frac{1}{\lambda}})\|f - h\|_K.$$

This proves our theorem. $\square$

We consider the special case where $\mu_j = |\delta_{\mathbf{a}_j}|$, $\mathbf{a}_j \in K$ for $j = 1,\dots,m$. In such a case (5) holds for $C_2 = 1$.

COROLLARY 2. *Let $K \subset \mathbb{K}^d$ be a compact set. Let E, F are normed subspaces of $\mathscr{C}(K)$ with $F \subset E$, $\dim F < \infty$. Let $\{\mathbf{a}_1,\dots,\mathbf{a}_m\} \subset K$ be F-determining. Then for every $\lambda > 1$ and $f \in E$ we have the following estimate*

$$\|f - \mathbf{P}_F(\mathscr{M}; f; \lambda)\|_K \leqslant (1 + 2C_1 m^{\frac{1}{\lambda}}) \mathrm{dist}_K(f, F),$$

where $\mathscr{M} = \{|\delta_{\mathbf{a}_j}| : j = 1,\dots,m\}$ and C_1 is defined by

$$\|q\|_K \leqslant C_1 \max_{1 \leqslant j \leqslant m} |q(\mathbf{a}_j)|, \quad \forall q \in F. \tag{9}$$

REMARK 1. The estimate in Corollary 2 is slightly different from the estimate in [12, Theorem 2] in which the authors consider the case $\lambda = 2$. In this special case, the map $f \mapsto \mathbf{P}_F(\mathscr{M}; f; 2)$ is a linear projection onto F. This fact may not be true in the general case. So we must modify the proof of [12, Theorem 2], and hence the extra factor 2 appears.

3. Some applications

In this section we give two applications of the theory in Section 2. The first one focuses on the generalized least power approximation corresponding to trivariate harmonic functions. Here the seminorms are of integral forms over slides of the unit ball. The second one deals with the Müntz polynomials. We build generalized admissible meshes such that the generalized least power approximation functions converge uniformly. We hope that our results have other applications.

3.1. Special seminorms for trivariate harmonic polynomials

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. For $\mathbf{a} \in \mathbb{S}^2$ and $0 < r \leqslant 1$, we denote by $\Delta(\mathbf{a}; r)$ the disk given by the intersection between the closed unit ball $\overline{\mathbb{B}^3}$ and the hyperplane $\{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{a}, \mathbf{x} \rangle = \sqrt{1 - r^2}\}$. For each $p \in \mathscr{P}_n(\mathbb{R}^3)$, the element $p|_{\mathbb{S}^2}$ is called a spherical polynomial. Note that the unique solution of the Dirichlet problem

$$\begin{cases} u \in \mathscr{C}^2(\mathbb{B}^3) \cap \mathscr{C}(\overline{\mathbb{B}^3}) \\ \Delta u = 0 \quad \text{on} \quad \mathbb{B}^3 \\ u|_{\mathbb{S}^2} = p|_{\mathbb{S}^2} \end{cases} \tag{10}$$

is a harmonic polynomial of degree at most n in $\mathbb{R}^3$, say $u \in \mathscr{H}_n(\mathbb{R}^3)$ (see [1, Theorem 5.1]). Hence, any spherical polynomial can be identified with the restriction of a harmonic polynomial on $\mathbb{S}^2$. Moreover,

$$\dim \mathscr{H}_n(\mathbb{R}^3) = \dim \mathscr{P}_n(\mathbb{S}^2) = (n+1)^2.$$

DEFINITION 2. We say that $\mathbf{A} = \{A_n \subset \mathbb{S}^2 : n \in \mathbb{N}^*\}$ is an optimal mesh on the unit sphere $\mathbb{S}^2$ if there exist $c_1, c_2 > 0$ such that

$$\|p\|_{\mathbb{S}^2} \leqslant c_1 \|p\|_{A_n}, \quad \#A_n \leqslant c_2 n^2, \quad \forall n \geqslant 1, \quad p \in \mathscr{P}_n(\mathbb{S}^2).$$

We also call $\mathbf{A}$ an optimal mesh corresponding to two constants c_1, c_2.

Note that optimal meshes on $\mathbb{S}^2$ are constructed in [5, 7, 18]. We first investigate the spacing of optimal meshes.

PROPOSITION 3. *Let $\mathbf{A} = \{A_n \subset \mathbb{S}^2 : n \in \mathbb{N}^*\}$ be an optimal mesh on the unit sphere corresponding to two constants c_1, c_2. Then, for any $\mathbf{b} \in \mathbb{S}^2$, we can find $\mathbf{a} \in A_n$ such that*

$$\|\mathbf{a} - \mathbf{b}\| \leqslant \frac{3\pi\sqrt[3]{c_1}}{n}.$$

Proof. To get the precise constant, we repeat the arguments in the proof of [6, Lemma 1]. Let $\phi \in [0, \pi]$ be the angle between $\mathbf{x} \in \mathbb{S}^2$ and the point $\mathbf{b}$ such that $\cos\phi = \langle \mathbf{x}, \mathbf{b} \rangle$. Note that ϕ is the geodesic distance between $\mathbf{x}$ and $\mathbf{b}$, and hence $\phi \geqslant \|\mathbf{x} - \mathbf{b}\|$. Set $m := [n/3]$, the integer part of $n/3$, and define

$$Q(\mathbf{x}) = \frac{2}{2m+1}\left(\frac{1}{2} + \cos\phi + \cos 2\phi + \cdots + \cos m\phi\right) = \frac{1}{2m+1} \frac{\sin\left(\frac{2m+1}{2}\phi\right)}{\sin\frac{\phi}{2}}.$$

We can also write

$$Q(\mathbf{x}) = \frac{1}{2m+1} U_{2m}\left(\sqrt{\frac{\langle \mathbf{x}, \mathbf{b} \rangle + 1}{2}}\right),$$

where U_{2m} is the Chebyshev polynomial of the second kind of degree $2m$. It follows that Q is a polynomial of degree at most m in $\mathbb{R}^3$, and hence $P(\mathbf{x}) := Q^3(\mathbf{x})$ belongs to $\mathscr{P}_n(\mathbb{R}^3)$. Evidently, $P(\mathbf{b}) = 1$ since $Q(\mathbf{b}) = U_{2m}(1)/(2m+1) = 1$. Moreover, we see that

$$2m+1 = 2\left[\frac{n}{3}\right] + 1 \geqslant \frac{n}{3} \quad \text{and} \quad \sin\frac{\phi}{2} \geqslant \frac{2}{\pi} \cdot \frac{\phi}{2} = \frac{\phi}{\pi}.$$

It follows that

$$|P(\mathbf{x})| = \frac{1}{(2m+1)^3} \left| \frac{\sin\left(\frac{2m+1}{2}\phi\right)}{\sin\frac{\phi}{2}} \right|^3 \leqslant \frac{1}{\frac{n^3}{27}} \cdot \frac{1}{\frac{\phi^3}{\pi^3}} = \frac{27\pi^3}{n^3\phi^3} \leqslant \frac{27\pi^3}{n^3\|\mathbf{x} - \mathbf{b}\|^3}.$$

By definition we can find $\mathbf{a} \in A_n$ such that

$$\|P(\mathbf{a})\| \geqslant \frac{1}{c_1}\|P\|_{\mathbb{S}^2} \geqslant \frac{1}{c_1}|P(\mathbf{b})| = \frac{1}{c_1}.$$

Combining the above estimates we obtain

$$\frac{1}{c_1} \leqslant \frac{27\pi^3}{n^3\|\mathbf{a}-\mathbf{b}\|^3}.$$

Hence

$$\|\mathbf{a}-\mathbf{b}\| \leqslant \frac{3\pi\sqrt[3]{c_1}}{n}.$$

The proof is complete. $\square$

Note that $2\left[\frac{n}{3}\right]+1 \geqslant \frac{2n-1}{3}$. In the following result we add points to an optimal mesh to get another optimal mesh with smaller distance. It is used in our next results (e.g. Proposition 5).

PROPOSITION 4. *Let* $\mathbf{A} = \{A_n \subset \mathbb{S}^2 : n \in \mathbb{N}^*\}$ *be an optimal mesh on the unit sphere. Then, for any* $c_3 > 0$*, there exists an optimal mesh* $\widehat{\mathbf{A}} = \{\widehat{A_n} \subset \mathbb{S}^2 : n \in \mathbb{N}^*\}$ *containing* $\mathbf{A}$ *such that, for any* $\mathbf{b} \in \mathbb{S}$*, we can find* $\mathbf{c} \in \widehat{A_n}$ *satisfying*

$$\|\mathbf{c}-\mathbf{b}\| \leqslant \frac{c_3}{n}.$$

Proof. Assume that $\mathbf{A}$ is an optimal mesh corresponding to c_1, c_2. We fix a point $\mathbf{a} \in A_n$. We claim that there exists a set $B_{\mathbf{a}} \subset \mathbb{S}^2$ consisting of N distinct points such that $\mathbf{a} \in B_{\mathbf{a}}$ and

$$\Big\{\mathbf{x} \in \mathbb{S}^2 : \|\mathbf{x}-\mathbf{a}\| \leqslant \frac{3\pi\sqrt[3]{c_1}}{n}\Big\} \subset \bigcup_{\mathbf{c}\in B_{\mathbf{a}}} \Big\{\mathbf{x} \in \mathbb{S}^2 : \|\mathbf{x}-\mathbf{c}\| \leqslant \frac{c_3}{n}\Big\}, \tag{11}$$

where N depending only on c_1 and c_3. Note that the set at the left hand side of (11) is a spherical cap in which the peak point is $\mathbf{a}$ and the circle of the base is

$$\Gamma\Big(\mathbf{a}; \frac{3\pi\sqrt[3]{c_1}}{n}\Big) := \Big\{\mathbf{x} \in \mathbb{S}^2 : \|\mathbf{x}-\mathbf{a}\| = \frac{3\pi\sqrt[3]{c_1}}{n}\Big\}.$$

Obviously, the radius $\rho_{\mathbf{a}}$ of the circle is smaller than $\frac{3\pi\sqrt[3]{c_1}}{n}$, see Figure 1. We set $N_1 = \left[\frac{12\sqrt[3]{c_1}\pi^2}{c_3}\right]+1$ and consider the set $X_{\mathbf{a}}$ of N_1 equidistance points on the circle $\Gamma\Big(\mathbf{a}; \frac{3\pi\sqrt[3]{c_1}}{n}\Big)$. Then, for two consecutive points $\mathbf{c}, \mathbf{d} \in X_{\mathbf{a}}$, we have

$$\|\mathbf{c}-\mathbf{d}\| = \frac{2\pi\rho_{\mathbf{a}}}{N_1} < \frac{2\pi\frac{3\pi\sqrt[3]{c_1}}{n}}{\frac{12\sqrt[3]{c_1}\pi^2}{c_3}} = \frac{c_3}{2n}.$$

Since $\|\mathbf{a}-\mathbf{d}\| = \frac{3\pi\sqrt[3]{c_1}}{n}$ the geodesic curve joining $\mathbf{a}$ and any point $\mathbf{d} \in X_{\mathbf{a}}$ has the same length, that is

$$2\sin^{-1}\Big(\frac{3\pi\sqrt[3]{c_1}}{2n}\Big) \leqslant 2\cdot\frac{\pi}{2}\cdot\frac{3\pi\sqrt[3]{c_1}}{2n} = \frac{3\pi^2\sqrt[3]{c_1}}{2n}.$$

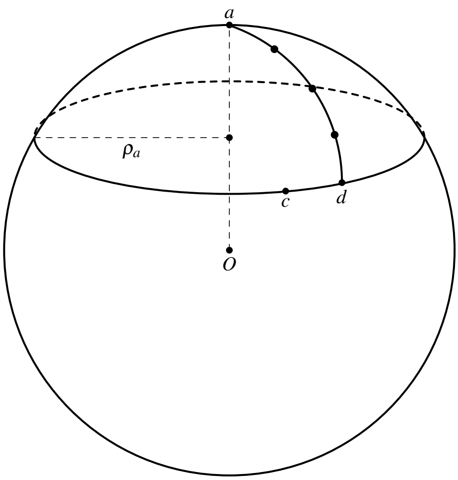

Figure 1: *An illustration of objects in the proof*

Hence, if we partition that geodesic curve into N_2 equal parts by N_2+1 points forming the set $Z_{\mathbf{a},\mathbf{d}}$, where $N_2 = \left[\frac{3\sqrt[3]{c_1}\pi^2}{c_3}\right]+1$, then the geodesic length between two consecutive points in $Z_{\mathbf{a},\mathbf{d}}$ is less than $\frac{c_3}{2n}$. The set $B_{\mathbf{a}} := \bigcup_{\mathbf{d}\in X_{\mathbf{a}}} Z_{\mathbf{a},\mathbf{d}}$ contains $N = N_1N_2+1$ points. Geometrically, the set $B_{\mathbf{a}}$ consists of the point $\mathbf{a}$ and N_1N_2 equidistance points on N_2 parallel circles. These N_2 circles and N_1 geodesic curves (joining $\mathbf{a}$ and $X_{\mathbf{a}}$) partition the spherical cap $\left\{\mathbf{x}\in\mathbb{S}^2 : \|\mathbf{x}-\mathbf{a}\| \leqslant \frac{3\pi\sqrt[3]{c_1}}{n}\right\}$ into finite many spherical zones. Each point $\mathbf{b}$ on the spherical cap must lies in one spherical zone. By the construction, the length of each curve forming the spherical zone is smaller than $\frac{c_3}{2n}$. Hence the Euclidean distance from $\mathbf{b}$ to one vertex of the zone, say a point $\mathbf{c}\in B_{\mathbf{a}}$, is less than $\frac{c_3}{n}$. It follows that $\mathbf{b}\in\left\{\mathbf{x}\in\mathbb{S}^2 : \|\mathbf{x}-\mathbf{c}\|\leqslant\frac{c_3}{n}\right\}$ and the claim follows.

Now we set $\widehat{A_n} = \bigcup_{\mathbf{a}\in A_n} B_{\mathbf{a}}$. Then

$$A_n \subset \widehat{A_n}, \quad \#\widehat{A_n} \leqslant N(\#A_n) \leqslant c_1 N n^2.$$

Since $A_n \subset \widehat{A_n}$, we have

$$\|p\|_{\mathbb{S}^2} \leqslant c_1\|p\|_{A_n} \leqslant c_1\|p\|_{\widehat{A_n}}, \quad \forall p \in \mathscr{P}_n(\mathbb{S}^2).$$

By Proposition 3, for a given point $\mathbf{b}\in\mathbb{S}^2$, we can find $\mathbf{a}\in A_n$ satisfying

$$\|\mathbf{a}-\mathbf{b}\| \leqslant \frac{3\pi\sqrt[3]{c_1}}{n}. \tag{12}$$

Combining (11) and (12) we get

$$\mathbf{b} \in \bigcup_{\mathbf{c}\in B_{\mathbf{a}}} \left\{ \mathbf{x} \in \mathbb{S}^2 : \|\mathbf{x}-\mathbf{c}\| \leqslant \frac{c_3}{n} \right\}.$$

The last relation shows that there is a point $\mathbf{c} \in B_{\mathbf{a}} \subset \widehat{A_n}$ such that $\|\mathbf{c}-\mathbf{b}\| \leqslant \dfrac{c_3}{n}$. The proof is complete. □

PROPOSITION 5. *Let $\mathbf{A} = \{A_n \subset \mathbb{S}^2 : n \in \mathbb{N}^*\}$ be an optimal mesh on the unit sphere such that, for any $\mathbf{b} \in \mathbb{S}^2$, we can find $\mathbf{c} \in A_n$ satisfying*

$$\|\mathbf{c}-\mathbf{b}\| \leqslant \frac{c_3}{n}, \quad c_3 < \frac{1}{2}.$$

Let $0 < c_4 < \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{2} - c_3\right)$. For each $n \geqslant 1$ and $\mathbf{a} \in A_n$, we take $r_{\mathbf{a}} \in \left(0, \dfrac{c_4}{n}\right]$ arbitrary. Consider the following set of seminorms

$$\mathscr{M}_n = \left\{ \mu_{\mathbf{a}} = |\nu_{\mathbf{a}}|, \nu_{\mathbf{a}} \in \mathscr{C}'\left(\overline{\mathbb{B}^3}\right) : \nu_{\mathbf{a}}(f) = \frac{1}{\pi r_{\mathbf{a}}^2} \int\limits_{\Delta(\mathbf{a};r_{\mathbf{a}})} f(\mathbf{x}) \mathrm{d}\lambda_2(\mathbf{x}),\ \mathbf{a} \in A_n \right\},$$

where λ_2 is the two dimensional Lebesgue measure. Then
(a) $\#\mathscr{M}_n = \#A_n = O(n^2)$;
(b) For any $f \in \mathscr{C}\left(\overline{\mathbb{B}^3}\right)$ we have

$$\|f\|_{\overline{\mathbb{B}^3}} \geqslant \max_{\mu_{\mathbf{a}}\in\mathscr{M}_n} \mu_{\mathbf{a}}(f);$$

(c) There exists $C_1 > 0$ such that

$$\|p\|_{\overline{\mathbb{B}^3}} \leqslant C_1 \max_{\mu_{\mathbf{a}}\in\mathscr{M}_n} \mu_{\mathbf{a}}(p), \quad \forall p \in \mathscr{H}_n(\mathbb{R}^3).$$

Proof. By definition, we can find $c_1 > 1$ such that, for any spherical polynomial p of degree at most n,

$$\|p\|_{\mathbb{S}^2} \leqslant c_1 \|p\|_{A_n}.$$

Moreover $\#A_n = O(n^2)$. Hence the first assertion follows. The second one is trivial because of the mean-value theorem for integration. It remains to prove the third assertion. We take $p \in \mathscr{H}_n(\mathbb{R}^3)$ arbitrarily. By the maximum principle, p attains its maximal value at $\mathbf{a}^* \in \mathbb{S}^2$, i.e., $|p(\mathbf{a}^*)| = \|p\|_{\overline{\mathbb{B}^3}}$. We choose $\mathbf{a} \in A_n$ such that $\|\mathbf{a}^* - \mathbf{a}\| \leqslant \dfrac{c_3}{n}$. From the mean-value theorem for integration there exists a point $\mathbf{y} \in \Delta(\mathbf{a}, r_{\mathbf{a}})$ such that

$$\nu_{\mathbf{a}}(p) = \frac{1}{\pi r_{\mathbf{a}}^2} \int\limits_{\Delta(\mathbf{a};r_{\mathbf{a}})} p(\mathbf{x}) \mathrm{d}\lambda_2(\mathbf{x}) = p(\mathbf{y}).$$

Observe that $\|\mathbf{a}-\mathbf{y}\|$ does not exceed the distance from $\mathbf{a}$ to the circle $\partial\Delta(\mathbf{a},r_{\mathbf{a}})$. In other words,

$$\begin{aligned}\|\mathbf{a}-\mathbf{y}\| &\leqslant \sqrt{r_{\mathbf{a}}^2+\left(1-\sqrt{1-r_{\mathbf{a}}^2}\right)^2}=\sqrt{2\left(1-\sqrt{1-r_{\mathbf{a}}^2}\right)}\\ &=\frac{\sqrt{2}r_{\mathbf{a}}}{\sqrt{1+\sqrt{1-r_{\mathbf{a}}^2}}}\leqslant\sqrt{2}r_{\mathbf{a}}\leqslant\frac{\sqrt{2}c_4}{n}.\end{aligned}$$

It follows that

$$\|\mathbf{a}^*-\mathbf{y}\|\leqslant\|\mathbf{a}^*-\mathbf{a}\|+\|\mathbf{a}-\mathbf{y}\|\leqslant\frac{c_3}{n}+\frac{\sqrt{2}c_4}{n}=\frac{c_3+\sqrt{2}c_4}{n}. \tag{13}$$

Next we recall the Markov inequality for trivariate harmonic polynomials in [26]

$$\|\nabla p\|_{\overline{\mathbb{B}^3}}\leqslant n\gamma_n\|p\|_{\overline{\mathbb{B}^3}}, \tag{14}$$

where

$$\gamma_n=\begin{cases}2\Big(1-\dfrac{1}{3}+\dfrac{1}{5}+\cdots+\dfrac{(-1)^{n-1}}{(2n-1)}\Big) & \text{if } n \text{ is even, } n\geqslant 4\\ 2\Big(1-\dfrac{1}{3}+\dfrac{1}{5}+\cdots+\dfrac{(-1)^{n-1}}{(2n-1)}\Big)-1 & \text{if } n \text{ is odd, } n\geqslant 5\\ \sqrt{2} & \text{if } n=2,3\\ 1 & \text{if } n=1.\end{cases}$$

In any case, we see that $\gamma_n<2$ for $n\geqslant 1$. It follows that

$$\begin{aligned}|p(\mathbf{a}^*)-p(\mathbf{y})| &\leqslant \|\nabla p\|_{\overline{\mathbb{B}^3}}\|\mathbf{a}^*-\mathbf{y}\|\leqslant 2n\|p\|_{\overline{\mathbb{B}^3}}\|\mathbf{a}^*-\mathbf{y}\|\\ &\leqslant 2n\cdot\frac{c_3+\sqrt{2}c_4}{n}\|p\|_{\overline{\mathbb{B}^3}}=2(c_3+\sqrt{2}c_4)\|p\|_{\overline{\mathbb{B}^3}}.\end{aligned}$$

Consequently

$$|p(\mathbf{y})|\geqslant|p(\mathbf{a}^*)|-2(c_3+\sqrt{2}c_4)\|p\|_{\overline{\mathbb{B}^3}}=(1-2c_3-2\sqrt{2}c_4)\|p\|_{\overline{\mathbb{B}^3}}.$$

By hypothesis we have $1-2c_3-2\sqrt{2}c_4>0$. Hence, the following relation

$$|\nu_{\mathbf{a}}(p)|=|p(\mathbf{y})|\geqslant(1-2c_3-2\sqrt{2}c_4)\|p\|_{\overline{\mathbb{B}^3}}$$

deduces the third assertion. The proof is complete. □

In Proposition 5, we use the Markov inequality (14) for trivariate harmonic polynomials to get a set of very special seminorms $\mathscr{M}_n$ such that

$$\sharp\mathscr{M}_n=O(n^2)=O\big(\dim\mathscr{H}_n(\mathbb{R}^3)\big).$$

Note that the cardinality of $\mathscr{M}_n$ is optimal. The reason why we work with the three dimensional space is that we do not known whether there is an analogous Markov inequality for $\mathscr{H}_n(\mathbb{R}^d)$, $d\geqslant 4$. Hence, we can not construct an optimal set of very special seminorms. Modifying the functionals in the above result, we get three dimensional Radon projections. Hence we obtain an analogous of [24, Proposition 3.2].

COROLLARY 3. *Under the assumptions of Proposition 5, if $0 < c_5 < c_4$ and*

$$\widetilde{\mathscr{M}_n} = \left\{ \widetilde{\mu}_{\mathbf{a}} = |\widetilde{\nu}_{\mathbf{a}}| : \widetilde{\nu}_{\mathbf{a}} \in \mathscr{C}'\left(\overline{\mathbb{B}^3}\right) : \widetilde{\nu}_{\mathbf{a}}(f) = \int\limits_{\Delta(\mathbf{a};r_{\mathbf{a}})} f(\mathbf{x}) \mathrm{d}\lambda_2(\mathbf{x}),\ \mathbf{a} \in A_n,\ \frac{c_5}{n} \leqslant r_{\mathbf{a}} \leqslant \frac{c_4}{n} \right\},$$

then
(a) $\#\widetilde{\mathscr{M}_n} = \#A_n = O(n^2)$*;*
(b) For any $f \in \mathscr{C}\left(\overline{\mathbb{B}^3}\right)$ *we have*

$$\frac{\pi c_4^2}{n^2} \|f\|_{\overline{\mathbb{B}^3}} \geqslant \max_{\widetilde{\mu}_{\mathbf{a}} \in \widetilde{\mathscr{M}_n}} \widetilde{\mu}_{\mathbf{a}}(f);$$

(c) There exists $C_1 > 0$ *such that*

$$\|p\|_{\overline{\mathbb{B}^3}} \leqslant \frac{C_1 n^2}{\pi c_5^2} \max_{\widetilde{\mu}_{\mathbf{a}} \in \widetilde{\mathscr{M}_n}} \widetilde{\mu}_{\mathbf{a}}(p), \quad \forall p \in \mathscr{H}_n(\mathbb{R}^3).$$

Proof. By definition we see that $\widetilde{\mu}_{\mathbf{a}} = \pi r_{\mathbf{a}}^2 \mu_{\mathbf{a}}$. Hence, from Proposition 5(b) we get

$$\|f\|_{\overline{\mathbb{B}^3}} \geqslant \max_{\mu_{\mathbf{a}} \in \mathscr{M}_n} \mu_{\mathbf{a}}(f) = \max_{\mu_{\mathbf{a}} \in \widetilde{\mathscr{M}_n}} \frac{1}{\pi r_{\mathbf{a}}^2} \widetilde{\mu}_{\mathbf{a}}(f) \geqslant \frac{n^2}{\pi c_4^2} \max_{\mu_{\mathbf{a}} \in \widetilde{\mathscr{M}_n}} \widetilde{\mu}_{\mathbf{a}}(f).$$

Moreover, for every $p \in \mathscr{H}_n(\mathbb{R}^3)$, applying Proposition 5(c) we obtain

$$\|p\|_{\overline{\mathbb{B}^3}} \leqslant C_1 \max_{\mu_{\mathbf{a}} \in \mathscr{M}_n} \mu_{\mathbf{a}}(p) = C_1 \max_{\widetilde{\mu}_{\mathbf{a}} \in \widetilde{\mathscr{M}_n}} \frac{1}{\pi r_{\mathbf{a}}^2} \widetilde{\mu}_{\mathbf{a}}(p) \leqslant \frac{C_1 n^2}{\pi c_5^2} \max_{\widetilde{\mu}_{\mathbf{a}} \in \widetilde{\mathscr{M}_n}} \widetilde{\mu}_{\mathbf{a}}(p).$$

The proof is complete. $\square$

PROPOSITION 6. *Let $\mathscr{M}_n$ be sets of functionals defined in Proposition 5. Let $\lambda > 1$ and f be harmonic in a neighborhood of $\overline{\mathbb{B}^3}$. Then there exists $0 < \rho < 1$ such that the generalized least power approximation element $\mathbf{P}_{\mathscr{H}_n}(\mathscr{M}_n; f; \lambda) \in \mathscr{H}_n$ with $\mathscr{H}_n := \mathscr{H}_n(\mathbb{R}^3)$ satisfies the following estimate*

$$\limsup_{n \to \infty} \left(\|f - \mathbf{P}_{\mathscr{H}_n}(\mathscr{M}_n; f; \lambda)\|_{\overline{\mathbb{B}^3}} \right)^{\frac{1}{n}} \leqslant \rho.$$

Proof. Using Theorem 2 and Proposition 5 we get

$$\|f - \mathbf{P}_{\mathscr{H}_n}(\mathscr{M}_n; f; \lambda)\|_{\overline{\mathbb{B}^3}} \leqslant \left(1 + 2C_1 O(n^{\frac{2}{\lambda}})\right) \operatorname{dist}_{\overline{\mathbb{B}^3}}(f, \mathscr{H}_n(\mathbb{R}^3))$$

Applying the main theorem in [2], there exists $\rho \in (0,1)$ such that

$$\limsup_{n \to \infty} \left(\operatorname{dist}_{\overline{\mathbb{B}^3}}(f, \mathscr{H}_n(\mathbb{R}^3)) \right)^{\frac{1}{n}} \leqslant \rho.$$

Combining the above two relations we obtain the desired estimate. $\square$

PROPOSITION 7. *Let $\widetilde{\mathscr{M}_n}$ be sets of functionals defined in Corollary 3. Let $\lambda > 1$ and f be harmonic in a neighborhood of $\overline{\mathbb{B}^3}$. Then there exists $0 < \rho < 1$ such that the generalized least power approximation element $\mathbf{P}_{\mathscr{H}_n}(\widetilde{\mathscr{M}_n}; f; \lambda) \in \mathscr{H}_n$ with $\mathscr{H}_n := \mathscr{H}_n(\mathbb{R}^3)$ satisfies the following estimate*

$$\limsup_{n\to\infty} \Big(\|f - \mathbf{P}_{\mathscr{H}_n}(\widetilde{\mathscr{M}_n}; f; \lambda)\|_{\overline{\mathbb{B}^3}} \Big)^{\frac{1}{n}} \leqslant \rho.$$

Proof. Using Theorem 2 and Corollary 3 we get

$$\begin{aligned}\|f - \mathbf{P}_{\mathscr{H}_n}(\widetilde{\mathscr{M}_n}; f; \lambda)\|_{\overline{\mathbb{B}^3}} &\leqslant \Big(1 + 2 \cdot \frac{\pi c_4^2}{n^2} \cdot \frac{C_1 n^2}{\pi c_5^2} \cdot O(n^{\frac{2}{\lambda}})\Big) \mathrm{dist}_{\overline{\mathbb{B}^3}}\big(f, \mathscr{H}_n(\mathbb{R}^3)\big) \\ &= \Big(1 + O(n^{\frac{2}{\lambda}})\Big) \mathrm{dist}_{\overline{\mathbb{B}^3}}\big(f, \mathscr{H}_n(\mathbb{R}^3)\big).\end{aligned}$$

Using the main theorem in [2] again get the desired estimate. $\square$

3.2. Generalized meshes for Müntz polynomials

Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of real numbers with $\lambda_0 = 0$ and set

$$\Pi_n(\Lambda) = \mathrm{span}\{x^{\lambda_i} : 0 \leqslant i \leqslant n\}.$$

The Müntz polynomials have many properties compared with the original polynomials. In particular, they admit Markov type inequalities and the Jackson type properties. For simplicity of reasoning, we work with the interval $[0,1]$. The general case is discussed in Remark 3. The following result can be found in [4]:

THEOREM 3. *Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of real numbers with $\lambda_0 = 0$ and $\lambda_k \geqslant k$ for every k. Then*

$$\|p'\|_{[0,1]} \leqslant 18\big(\sum_{k=0}^{n} \lambda_k\big)\|p\|_{[0,1]}, \quad \forall p \in \Pi_n(\Lambda).$$

PROPOSITION 8. *Under the assumptions of Theorem 3, there exists a set $A_n \subset [0,1]$ such that $\#A_n = O(\sum_{k=0}^{n} \lambda_k)$ and*

$$\|p\|_{[0,1]} \leqslant 2\|p\|_{A_n}, \quad \forall p \in \Pi_n(\Lambda).$$

Proof. We take $N_n = 2[18(\sum_{k=0}^{n} \lambda_k)] + 2$ and consider the set

$$A_n = \Big\{x_k = \frac{k}{N_n} : k = 0, 1, \ldots, N_n - 1\Big\}.$$

For $p \in \Pi_n(\Lambda)$, we choose a point x^* such that $|p(x^*)| = \|p\|_{[0,1]}$. We see that $x^* \in [x_k, x_{k+1}]$ for some $0 \leqslant k \leqslant N_n - 1$, and hence $|x^* - x_k| \leqslant \dfrac{1}{N_n}$. Using Theorem 3 we get

$$|p(x_k) - p(x^*)| \leqslant \|p'\|_{[0,1]}|x_k - x^*| \leqslant \frac{18}{N_n}\big(\sum_{k=0}^{n} \lambda_k\big)\|p\|_{[0,1]} \leqslant \frac{1}{2}\|p\|_{[0,1]}.$$

It follows that

$$\|p\|_{A_n} \geqslant |p(x_k)| \geqslant |p(x^*)| - \|p\|_{[0,1]} = \frac{1}{2}\|p\|_{[0,1]}.$$

The proof is complete. $\square$

The Müntz theorem asserts that $\Pi(\Lambda) := \bigcup_{n=0}^{\infty} \Pi_n(\Lambda)$ is dense in $C[0,1]$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

When f is smooth, the Jackson-Müntz theorems gives the estimates for the quantity $\mathrm{dist}_{[0,1]}\big(f, \Pi_n(\Lambda)\big)$. These estimates can be found in [13, 17]. Hence we can find conditions on Λ such that $\mathrm{dist}_{[0,1]}\big(f, \Pi_n(\Lambda)\big)$ tends to 0 fast enough such that the generalized least square power approximation functions converge uniformly. Here we only consider a simple case where $\lambda_k = \delta k$ for $k \geqslant 0$. We recall a theorem of von Golitschek [13].

THEOREM 4. *Let $f \in C[0,1]$ have a continuous derivative $f^{(k)}$ of order $k \geqslant 0$ in $[0,1]$ and $f^{(k)} \in \mathrm{Lip}\alpha$, $0 < \alpha \leqslant 1$. Let $\delta > 0$ such that $1/\delta \notin \mathbb{N}$ and let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ with $\lambda_k = k\delta$, $k \geqslant 0$.*
a) If $\delta \geqslant 2$ then

$$\mathrm{dist}_{[0,1]}\big(f, \Pi_n(\Lambda)\big) = O\big(n^{-\min\{(k+\alpha)\frac{2}{\delta}, \frac{2}{\delta}\}}\big).$$

b) If $0 < \delta < 2$ then

$$\mathrm{dist}_{[0,1]}\big(f, \Pi_n(\Lambda)\big) = O\big(n^{-\min\{k+\alpha, \frac{2}{\delta}\}}\big).$$

PROPOSITION 9. *Let $f \in C[0,1]$ have a continuous derivative $f^{(k)}$ of order $k \geqslant 0$ in $[0,1]$ and $f^{(k)} \in \mathrm{Lip}\alpha$, $0 < \alpha \leqslant 1$. Let $\delta > 1$ and let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ with $\lambda_k = k\delta$, $k \geqslant 0$. Let $\mathbf{A} = \{A_n\}$ be the mesh of points in $[0,1]$ constructed in Proposition 8. Let $\lambda > 1$ and $\mathbf{P}_{\Pi_n}(A_n; f; \lambda) \in \Pi_n$ be the generalized least square approximation function with $\Pi_n := \Pi_n(\Lambda)$. Then*

$$\|f - \mathbf{P}_{\Pi_n}(A_n; f; \lambda)\|_{[0,1]} = \begin{cases} O\big(n^{\frac{2}{\lambda} - \min\{(k+\alpha)\frac{2}{\delta}, \frac{2}{\delta}\}}\big) & \text{if} \quad \delta \geqslant 2 \\ O\big(n^{\frac{2}{\lambda} - \min\{k+\alpha, \frac{2}{\delta}\}}\big) & \text{if} \quad 1 < \delta < 2. \end{cases} \tag{15}$$

Consequently, $\mathbf{P}_{\Pi_n}(A_n; f; \lambda)$ converges to f uniformly on $[0,1]$ if

$$\begin{cases} \frac{2}{\lambda} - \min\{(k+\alpha)\frac{2}{\delta}, \frac{2}{\delta}\}\big) < 0 & \text{when} \quad \delta \geqslant 2 \\ \frac{2}{\lambda} - \min\{k+\alpha, \frac{2}{\delta}\}\big) < 0 & \text{when} \quad 1 < \delta < 2. \end{cases}$$

Proof. By construction we have

$$\#A_n = O\big(\sum_{k=0}^{n} \lambda_k\big) = O\big(\sum_{k=0}^{n} \delta k\big) = O(n^2).$$

Using Corollary 2 we get

$$\|f-\mathbf{P}_{\Pi_n}(A_n;f;\lambda)\|_K \leqslant (1+4(\#A_n)^{\frac{1}{\lambda}})\mathrm{dist}_{[0,1]}\big(f,\Pi_n(\Lambda)\big) = O(n^{\frac{2}{\lambda}})\mathrm{dist}_{[0,1]}\big(f,\Pi_n(\Lambda)\big).$$

Combining the last estimate with Theorem 4, we obtain the estimate in (15). □

REMARK 2. In view of the above proposition we conclude that $\mathbf{P}_{\Pi_n}(A_n;f;\lambda)$ converges to f uniformly on $[0,1]$ when λ is large enough. In particular, if f belongs to the class $\mathrm{Lip}\alpha$, $0<\alpha\leqslant 1$, then we only need the condition

$$\lambda > \begin{cases} \frac{\delta}{\alpha} & \text{when} \quad \delta \geqslant 2 \\ \frac{2}{\alpha} & \text{when} \quad 1<\delta<2. \end{cases}$$

REMARK 3. A referee pointed out to us a paper of Benko, Erdélyi and Szabados [3], where one finds a sharper version of Theorem 3 without the gap condition. Using it, we can generalize Proposition 8, where the gap condition on Λ is removed and $[0,1]$ is replaced by $[a,b]$. The new and generalized proposition along with an estimate for the quantity $\mathrm{dist}_{[a,b]}\big(f,\Pi_n(\Lambda)\big)$ would give a convergent result similar to Proposition 9.

It is known that $[-1,1]$ possesses optimal meshes (see [10], [14, p. 1109]). Using affine automorphism of $\mathbb{R}$ of the form $t\mapsto \frac{a+b}{2}+\frac{b-a}{2}t$ we obtain optimal meshes in $[a,b]$.

PROPOSITION 10. *Let $\mathbf{A}=\{A_n\subset[a,b]:n\in\mathbb{N}^*\}$ be an optimal mesh. Let $f\in C[a,b]$ have a continuous derivative $f^{(k)}$ of order $k\geqslant 0$ in $[a,b]$ and $f^{(k)}\in\mathrm{Lip}\alpha$, $0<\alpha\leqslant 1$. Let $\lambda>1$ and $\mathbf{P}_{\mathscr{P}_n}(A_n;f;\lambda)\in\mathscr{P}_n$ be the least square approximation polynomial with $\mathscr{P}_n:=\mathscr{P}_n(\mathbb{R})$. Then*

$$\|f-\mathbf{P}_{\mathscr{P}_n}(A_n;f;\lambda)\|_{[a,b]} = O\big(n^{\frac{1}{\lambda}-k-\alpha}\big).$$

Proof. By hypothesis we have $\#A_n=O(n)$ and

$$\|p\|_{[a,b]}\leqslant C_1\|p\|_{A_n}, \quad \forall p\in\mathscr{P}_n(\mathbb{R}).$$

From the Jackson theorem in [25, Theorem 1.5], we have

$$\mathrm{dist}_{[a,b]}\big(f,\mathscr{P}_n(\mathbb{R})\big) = O\Big(\frac{1}{n^{k+\alpha}}\Big).$$

Using Corollary 2 we obtain

$$\begin{aligned}\|f-\mathbf{P}_{\mathscr{P}_n}(A_n;f;\lambda)\|_K &\leqslant (1+2C_1(\#A_n)^{\frac{1}{\lambda}})\mathrm{dist}_{[a,b]}\big(f,\mathscr{P}_n(\mathbb{R})\big)\\ &= O(n^{\frac{1}{\lambda}})O\Big(\frac{1}{n^{k+\alpha}}\Big)\\ &= O\big(n^{\frac{1}{\lambda}-k-\alpha}\big).\end{aligned}$$

The desired estimate is proved. □

COROLLARY 4. *Under the assumptions of Proposition 10, $\mathbf{P}_{\mathscr{P}_n}(A_n; f; \lambda)$ converges to f uniformly on $[a,b]$ when $k \geqslant 1$ or $k = 0$, $\lambda > \dfrac{1}{\alpha}$.*

Open questions.

1. Extend Proposition 5 to the case of arbitrary (finite) dimension.
2. Construct other sets of special seminorms such that the corresponding generalized least power approximations converge uniformly.
3. Find the asymptotic behavior f^* in Theorem 1 when all continuous linear functionals v_i converge to a unique element $v \in F'$. It is analogous to a problem in polynomial interpolation, where we find the limit of Lagrange interpolation polynomials when the interpolation points coalesce. In general, the limit is a kind of Hermite interpolation polynomial.

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Nguyen Quang Dieu
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy street, Cau Giay, Hanoi, Vietnam
and
Thang Long Institute of Mathematics and Applied Sciences
Thang Long University
Nghiem Xuan Yem street, Hoang Mai, Hanoi, Vietnam
e-mail: ngquang.dieu@hnue.edu.vn

Phung Van Manh
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy street, Cau Giay, Hanoi, Vietnam
e-mail: manhpv@hnue.edu.vn