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CONTINUITY OF GENERALIZED RIESZ POTENTIALS FOR DOUBLE PHASE FUNCTIONALS

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Abstract. In this note, we are concerned with the continuity of generalized Riesz potentials $I_{\rho,\mu,\tau}f$ of functions in Morrey spaces $L^{\Phi,\nu,\kappa}(X)$ of double phase functionals over bounded non-doubling metric measure spaces.

1. Introduction

The double phase functional introduced by Zhikov ([27]) is studied intensively by many mathematichans. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [1, 4, 5] studied a double phase functional

$$\tilde{\Phi}(x,t) = t^p + a(x)t^q,\ x \in \mathbf{R}^N,\ t \geqslant 0$$

where $1 \leqslant p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$. We refer to [10, 26] for Calderón-Zygmund estimates, [12, 15] for the Sobolev's inequality and e.g. [3, 7, 8, 9] for other double phase problems.

In the present note, relaxing the continuity of $a(\cdot)$, we consider the case $\Phi(x,t)$ is a double phase functional given by

$$\Phi(x,t) = t^p + (b(x)t)^q,$$

where $1 < p < q$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ (cf. [4]).

For $0 < \alpha < N$ and a locally integrable function f on $\mathbf{R}^N$ the Riesz potential $I_\alpha f$ of order α is defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\,dy.$$

In [13] we discussed the continuity of Riesz potentials $I_\alpha f$ of functions in Morrey spaces $L^{\Phi,\nu}(\mathbf{R}^N)$ of the double phase functionals $\Phi(x,t)$ in the case $\alpha p < \nu < (\alpha +$

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$\theta)p$ and $(\alpha-1)q<\nu<\alpha q$. We refer to [15, Section 5] for the L^Φ case and [14] for the $L^{p,\nu}$ case.

In the present note we shall extend [13, Theorem 4.1] from the Euclidean case to a non-doubling metric measure setting. We denote by (X,d,μ) a metric measure space, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. We often write X instead of (X,d,μ). For $x\in X$ and $r>0$, we denote by $B(x,r)$ the open ball in X centered at x with radius r and $d_X=\sup\{d(x,y):x,y\in X\}$. We assume that

$$\mu(\{x\})=0$$

for $x\in X$ and $0<\mu(B(x,r))<\infty$ for $x\in X$ and $r>0$ for simplicity. We do not assume that μ has a so-called doubling condition. So our results are for non-doubling metric measure spaces. Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0>0$ such that $\mu(B(x,2r))\leqslant c_0\mu(B(x,r))$ for all $x\in\operatorname{supp}(\mu)(=X)$ and $r>0$ (see [2]). Otherwise μ is said to be non-doubling. For examples of non-doubling metric measure spaces we refer to [19, 22].

To obtain general results, we consider the family (ρ) of all functions ρ satisfying the following conditions: $\rho:(0,\infty)\to(0,\infty)$ is a measurable function such that

$$\int_0^r \rho(s)\frac{ds}{s}<+\infty$$

for all sufficiently small $r>0$ and there exists constants $0<k<1$, $0<k_1<k_2$ and $C_\rho>0$ such that

$$\sup_{kr\leqslant s\leqslant r}\rho(s)\leqslant C_\rho\int_{k_1r}^{k_2r}\rho(s)\frac{ds}{s} \tag{1}$$

for all $r>0$ (e.g. [6, 23]). We do not postulate the doubling condition on ρ.

EXAMPLE 1. If ρ satisfies the doubling condition, that is, there exists a constant $C>0$ such that $C^{-1}\leqslant\rho(r)/\rho(s)\leqslant C$ for $1/2\leqslant r/s\leqslant 2$, then ρ satisfies (1) whenever $k=1/2$ and $2k_1=k_2$. If ρ is increasing, then ρ satisfies (1) with $k=1/2$, $k_1=1$ and $k_2=2$. If $\alpha>0$ such that

$$\rho(r)=\begin{cases} r^\alpha & (0<r<1)\\ e^{-(r-1)} & (r\geqslant 1),\end{cases}$$

then ρ satisfies (1) with $k=1/2$, $k_1=1/4$ and $k_2=1/2$. See also [18, Lemma 2.5], [20, 23] and [25, Remark 2.2].

For a function $\rho\in(\rho)$ and $\tau\geqslant 1$, we define the generalized Riesz potential $I_{\rho,\mu,\tau}f$ of f by

$$I_{\rho,\mu,\tau}f(x)=\int_X\frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}d\mu(y),$$

where $f\in L^1(X)$. We write $I_{\rho,\mu,\tau}f=I_{\alpha,\mu,\tau}f$ when $\rho(r)=r^\alpha$ for $\alpha>0$. If $\rho(r)=r^\alpha$, $0<\alpha<N$ and $X=\mathbf{R}^N$ with the usual distance and the Lebesgue measure, then $I_{\rho,\mu,\tau}f$ is equal to $I_\alpha f$. We refer to [21, 24] etc. for the study of $I_{\rho,\mu,\tau}f$.

Our aim in this note is to discuss the continuity of generalized Riesz potential $I_{\rho,\mu,\tau}f$ of functions f in Morrey spaces $L^{\Phi,\nu,\kappa}(X)$ of the double phase functionals over bounded non-doubling metric measure spaces X (Theorem 1), as an extension of [13, Theorem 4.1].

2. Statement of the main Theorem

Throughout this paper, let C denote various constants independent of the variables in question.

For $\nu > 0$, $\kappa \geqslant 1$ and $1 \leqslant p < \infty$, Morrey space $L^{p,\nu,\kappa}(X)$ is the family of measurable functions f on X satisfying

$$\|f\|_{L^{p,\nu,\kappa}(X)} = \left(\sup_{x\in X, 0<r<d_X} \frac{r^\nu}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{1/p} < \infty$$

(cf. see [16]).

We consider a function

$$\Phi(x,t) : X \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi 1)$ and $(\Phi 2)$:

$(\Phi 1)$ $\Phi(\cdot,t)$ is measurable on X for each $t \geqslant 0$ and $\Phi(x,\cdot)$ is convex on $[0,\infty)$ for each $x \in X$;

$(\Phi 2)$ there exists a constant $A_1 \geqslant 1$ such that

$$A_1^{-1} \leqslant \Phi(x,1) \leqslant A_1 \quad \text{for all } x \in X.$$

For $\nu > 0$ and $\kappa \geqslant 1$, the Musielak-Orlicz-Morrey space $L^{\Phi,\nu,\kappa}(X)$ is defined by

$$L^{\Phi,\nu,\kappa}(X) = \left\{ f \in L^1_{\rm loc}(X) : \sup_{x\in X, 0<r<d_X} \frac{r^\nu}{\mu(B(x,\kappa r))} \int_{B(x,r)} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) d\mu(y) < \infty \text{ for some } \lambda > 0 \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi,\nu,\kappa}(X)} = \inf\left\{ \lambda > 0 : \sup_{x\in X, 0<r<d_X} \frac{r^\nu}{\mu(B(x,\kappa r))} \int_{B(x,r)} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) d\mu(y) \leqslant 1 \right\}$$

(see [11, 17]).

In what follows, set

$$\Phi(x,t) = t^p + (b(x)t)^q,$$

where $1 \leqslant p < q$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ (cf. [4]).

Our result is the following.

THEOREM 1. *Let $\rho \in (\rho)$. Assume that there are constants $\eta > 0, \iota \geqslant 1$ and $C_0 > 0$ such that*

$$\left| \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))} \right| \leqslant C_0 \left(\frac{d(x,z)}{d(x,y)} \right)^{\eta} \frac{\rho(d(x,y))}{\mu(B(x,\iota d(x,y)))} \tag{2}$$

whenever $d(x,z) \leqslant d(x,y)/2$. Abbreviate

$$\psi(r) \equiv \int_0^{6k_2 r} s^{-\nu/p+\theta} \rho(s) \frac{ds}{s} + \int_0^{6k_2 r} s^{-\nu/q} \rho(s) \frac{ds}{s} + r^{\theta} \int_{2k_1 r}^{4k_2 d_X} s^{-\nu/p} \rho(s) \frac{ds}{s}$$
$$+ r^{\eta} \int_{2k_1 r}^{4k_2 d_X} s^{-\nu/p-\eta+\theta} \rho(s) \frac{ds}{s} + r^{\eta} \int_{2k_1 r}^{4k_2 d_X} s^{-\nu/q-\eta} \rho(s) \frac{ds}{s}$$

for $x \in X$ and $0 < r \leqslant d_X$, where k_1 and k_2 are constants in (ρ). If $1 \leqslant \kappa < \min\{\tau, \iota\}$, then there exists a constant $C > 0$ such that

$$\left| b(x) I_{\rho,\mu,\tau} f(x) - b(z) I_{\rho,\mu,\tau} f(z) \right| \leqslant C \psi(d(x,z))$$

for all $x, z \in X$ and measurable functions f on X with $\|f\|_{L^{\Phi,\nu,\kappa}(X)} \leqslant 1$.

When $\rho(r) = r^{\alpha}$, we obtain the following corollary.

COROLLARY 1. *Assume that there are constants $\eta > 0, \iota \geqslant 1$ and $C_0 > 0$ such that*

$$\left| \frac{d(x,y)^{\alpha}}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^{\alpha}}{\mu(B(z,\tau d(z,y)))} \right| \leqslant C_0 \left(\frac{d(x,z)}{d(x,y)} \right)^{\eta} \frac{d(x,y)^{\alpha}}{\mu(B(x,\iota d(x,y)))} \tag{3}$$

whenever $d(x,z) \leqslant d(x,y)/2$. Suppose

$$\max\{\alpha p, (\alpha - \eta + \theta) p\} < \nu < (\alpha + \theta) p$$

and

$$(\alpha - \eta) q < \nu < \alpha q.$$

If $1 \leqslant \kappa < \min\{\tau, \iota\}$, then there exists a constant $C > 0$ such that

$$\left| b(x) I_{\alpha,\mu,\tau} f(x) - b(z) I_{\alpha,\mu,\tau} f(z) \right| \leqslant C \left\{ d(x,z)^{\alpha+\theta-\nu/p} + d(x,z)^{\alpha-\nu/q} \right\}$$

for all $x, z \in X$ and measurable functions f on X with $\|f\|_{L^{\Phi,\nu,\kappa}(X)} \leqslant 1$.

Compare this with [13, Theorem 4.1] and [15, Theorem 5].

REMARK 1. Assume that there are constants $\eta > 0, \iota \geqslant 1$ and $C_0 > 0$ such that (3) hollds. Suppose

$$(\alpha - \eta) p < \nu < \alpha p.$$

If $1 \leqslant \kappa < \min\{\tau, \iota\}$, then there exists a constant $C > 0$ such that

$$\left| I_{\alpha,\mu,\tau} f(x) - I_{\alpha,\mu,\tau} f(z) \right| \leqslant C d(x,z)^{\alpha-\nu/p}$$

for all $x, z \in X$ and measurable functions f on X with $\|f\|_{L^{p,\nu,\kappa}(X)} \leqslant 1$. Compare this with [13, Remark 4.2].

REMARK 2. The referee kindly suggested that the case of $\rho : X \times (0,\infty) \to (0,\infty)$ can be treated to discuss the continuity of more general Riesz potentials. But we do not go into details any more.

3. Proof of Theorem

Before giving a proof of Theorem 1, we prepare the following lemma.

LEMMA 1. *Let $\beta \in \mathbf{R}$ and $\rho \in (\rho)$. Let f be a nonnegative function on X such that $\|f\|_{L^{p,\nu,\kappa}(X)} \leqslant 1$. If $1 \leqslant \kappa < \tau$, then there exist constants $C > 0$ such that*

$$\int_{B(x,r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^\beta}\,d\mu(y) \leqslant C\int_0^{2k_2 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s} \tag{4}$$

and

$$\int_{X\setminus B(x,r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^\beta}\,d\mu(y) \leqslant C\int_{k_1 r}^{4k_2 d_X} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s} \tag{5}$$

for all $x \in X$ and $0 < r \leqslant d_X$, where k_1 and k_2 are constants in (ρ).

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p,\nu,\kappa}(X)} \leqslant 1$. Take $\gamma \in \mathbf{R}$ such that $1 < \gamma \leqslant \min\{\tau/\kappa, 1/k, 2\}$. If $y \in B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r)$ and $j \in \mathbf{Z}$, then a geometric observation and (1) show

$$\begin{aligned}
\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))d(x,y)^\beta} &\leqslant \frac{\max\{1,\gamma^{-\beta}\}}{\mu(B(x,\gamma^{j-1}\tau r))(\gamma^{j-1}r)^\beta}\sup_{\gamma^{j-1}r\leqslant s\leqslant \gamma^j r}\rho(s)\\
&\leqslant \frac{\max\{1,\gamma^{-\beta}\}}{\mu(B(x,\gamma^{j-1}\tau r))(\gamma^{j-1}r)^\beta}\sup_{k\gamma^j r\leqslant s\leqslant \gamma^j r}\rho(s)\\
&\leqslant \frac{C_\rho\max\{1,\gamma^{-\beta}\}}{\mu(B(x,\gamma^{j-1}\tau r))(\gamma^{j-1}r)^\beta}\int_{\gamma^j k_1 r}^{\gamma^j k_2 r}\rho(s)\,\frac{ds}{s}
\end{aligned}$$

by $\gamma \leqslant 1/k$. Using $\gamma \leqslant \tau/\kappa$, we obtain

$$\begin{aligned}
&\int_{B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^\beta}\,d\mu(y)\\
&\leqslant \frac{C_\rho\max\{1,\gamma^{-\beta}\}}{(\gamma^{j-1}r)^\beta}\int_{\gamma^j k_1 r}^{\gamma^j k_2 r}\rho(s)\,\frac{ds}{s}\cdot\frac{1}{\mu(B(x,\gamma^{j-1}\tau r))}\int_{B(x,\gamma^j r)} f(y)\,d\mu(y)\\
&\leqslant \frac{C_\rho\max\{1,\gamma^{-\beta}\}}{(\gamma^{j-1}r)^\beta}\int_{\gamma^j k_1 r}^{\gamma^j k_2 r}\rho(s)\,\frac{ds}{s}\cdot\frac{1}{\mu(B(x,\kappa\gamma^j r))}\int_{B(x,\gamma^j r)} f(y)\,d\mu(y).
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
&\int_{B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^\beta}\,d\mu(y)\\
&\leqslant \frac{C_\rho\max\{1,\gamma^{-\beta}\}}{(\gamma^{j-1}r)^\beta}\int_{\gamma^j k_1 r}^{\gamma^j k_2 r}\rho(s)\,\frac{ds}{s}\left(\frac{1}{\mu(B(x,\kappa\gamma^j r))}\int_{B(x,\gamma^j r)} f(y)^p\,d\mu(y)\right)^{1/p}
\end{aligned}$$

$$\leqslant \frac{C_\rho \max\left\{1,\gamma^{-\beta}\right\}}{(\gamma^{j-1}r)^{\beta}} \int_{\gamma^j k_1 r}^{\gamma^j k_2 r} \rho(s)\,\frac{ds}{s}\cdot\left(\gamma^j r\right)^{-\nu/p}$$
$$= C_\rho \max\left\{1,\gamma^{\beta}\right\}(\gamma^j r)^{-\nu/p-\beta} \int_{\gamma^j k_1 r}^{\gamma^j k_2 r} \rho(s)\,\frac{ds}{s}$$
$$\leqslant C_\rho \max\left\{1,\gamma^{\beta}\right\} \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \int_{\gamma^j k_1 r}^{\gamma^j k_2 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}. \tag{6}$$

Let j_0 be the smallest integer such that $k_2/k_1 \leqslant \gamma^{j_0}$. Using (6), we obtain

$$\int_{B(x,r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^{\beta}}\,d\mu(y)$$
$$= \sum_{j=0}^{\infty} \int_{B(x,\gamma^{-j}r)\setminus B(x,\gamma^{-j-1}r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^{\beta}}\,d\mu(y)$$
$$\leqslant C_\rho \max\left\{1,\gamma^{\beta}\right\} \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \sum_{j=0}^{\infty} \int_{\gamma^{-j}k_1 r}^{\gamma^{-j}k_2 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}$$
$$\leqslant C_\rho \max\left\{1,\gamma^{\beta}\right\} \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \sum_{j=0}^{\infty} \int_{\gamma^{-j}k_1 r}^{\gamma^{-j+j_0}k_1 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}$$
$$\leqslant \max\left\{1,2^{\beta}\right\} C_\rho\, j_0 \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \int_0^{2k_2 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s},$$

which proves (4).

Let j_1 be the smallest integer such that $d_X \leqslant \gamma^{j_1} r$. If we use (6),

$$\int_{X\setminus B(x,r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^{\beta}}\,d\mu(y)$$
$$\leqslant \sum_{j=1}^{j_1} \int_{B(x,\gamma^{j}r)\setminus B(x,\gamma^{j-1}r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^{\beta}}\,d\mu(y)$$
$$\leqslant C_\rho \max\left\{1,\gamma^{\beta}\right\} \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \sum_{j=1}^{j_1} \int_{\gamma^{j}k_1 r}^{\gamma^{j}k_2 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}$$
$$\leqslant C_\rho \max\left\{1,\gamma^{\beta}\right\} \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \sum_{j=1}^{j_1} \int_{\gamma^{j}k_1 r}^{\gamma^{j+j_0}k_1 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}$$
$$\leqslant C_\rho \max\left\{1,\gamma^{\beta}\right\} j_0 \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \int_{\gamma k_1 r}^{2\gamma^{j_1}k_2 r} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}$$
$$\leqslant \max\left\{1,2^{\beta}\right\} C_\rho\, j_0 \max\left\{k_1^{\nu/p+\beta}, k_2^{\nu/p+\beta}\right\} \int_{k_1 r}^{4k_2 d_X} s^{-\nu/p-\beta}\rho(s)\,\frac{ds}{s}.$$

Thus, (5) follows. □

Proof of Theorem 1. Let f be a nonnegative function on X such that $\|f\|_{L^{\Phi,\nu,\kappa}(X)} \leqslant 1$. First note from (2) that for $x,y \in X$ and $r = d(x,z)$

$$
\begin{aligned}
&\left|b(x)I_{\rho,\mu,\tau}f(x) - b(z)I_{\rho,\mu,\tau}f(z)\right| \\
&\leqslant b(x)\int_{B(x,2r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&\quad + b(z)\int_{B(x,2r)} \frac{\rho(d(z,y))f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y) \\
&\quad + |b(x)-b(z)|\int_{X\setminus B(x,2r)} \frac{\rho(d(z,y))f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y) \\
&\quad + b(x)\int_{X\setminus B(x,2r)} \left|\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z,\tau d(z,y)))}\right| f(y)\,d\mu(y) \\
&\leqslant C\Bigg\{ b(x)\int_{B(x,3r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&\quad + b(z)\int_{B(z,3r)} \frac{\rho(d(z,y))f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y) \\
&\quad + r^{\theta}\int_{X\setminus B(z,2r)} \frac{\rho(d(z,y))f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y) \\
&\quad + r^{\eta} b(x)\int_{X\setminus B(x,2r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\iota d(x,y)))d(x,y)^{\eta}}\,d\mu(y) \Bigg\} \\
&= C\{I_1(x) + I_1(z) + I_2(z) + I_3(x)\}.
\end{aligned}
$$

For $I_1(x)$, we have

$$
\begin{aligned}
I_1(x) &\leqslant \int_{B(x,3r)} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))}|b(x)-b(y)|f(y)\,d\mu(y) \\
&\quad + \int_{B(x,3r)} \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} b(y)f(y)\,d\mu(y) \\
&\leqslant C\int_{B(x,3r)} \frac{\rho(d(x,y))f(y)}{\mu(B(x,\tau d(x,y)))d(x,y)^{-\theta}}\,d\mu(y) + \int_{B(x,3r)} \frac{\rho(d(x,y))\{b(y)f(y)\}}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\
&= CI_{11}(x) + I_{12}(x).
\end{aligned}
$$

By (4), we obtain

$$I_{11}(x) \leqslant C\int_0^{6k_2 r} s^{-\nu/p+\theta}\rho(s)\,\frac{ds}{s},$$

and

$$I_{12}(x) \leqslant C\int_0^{6k_2 r} s^{-\nu/q}\rho(s)\,\frac{ds}{s}.$$

For $I_2(z)$, we have by (5)

$$I_2(z) \leqslant Cr^{\theta}\int_{2k_1 r}^{4k_2 d_X} s^{-\nu/p}\rho(s)\,\frac{ds}{s}.$$

Finally, for $I_3(x)$ we have

$$\begin{aligned}
I_3(x) &\leqslant r^\eta \int_{X\setminus B(x,2r)} \frac{\rho(d(x,y))}{\mu(B(x,\iota d(x,y)))d(x,y)^\eta} |b(x)-b(y)| f(y)\, d\mu(y) \\
&\quad + r^\eta \int_{X\setminus B(x,2r)} \frac{\rho(d(x,y))}{\mu(B(x,\iota d(x,y)))d(x,y)^\eta} b(y) f(y)\, d\mu(y) \\
&\leqslant Cr^\eta \int_{X\setminus B(x,2r)} \frac{\rho(d(x,y)) f(y)}{\mu(B(x,\iota d(x,y)))d(x,y)^{-\theta+\eta}}\, d\mu(y) \\
&\quad + r^\eta \int_{X\setminus B(x,2r)} \frac{\rho(d(x,y))\{b(y)f(y)\}}{\mu(B(x,\iota d(x,y)))d(x,y)^\eta}\, d\mu(y) \\
&= CI_{31}(x) + I_{32}(x).
\end{aligned}$$

Note from (5) that

$$I_{31}(x) \leqslant Cr^\eta \int_{2k_1 r}^{4k_2 d_X} s^{-\nu/p-\eta+\theta} \rho(s) \frac{ds}{s}$$

and

$$I_{32}(x) \leqslant Cr^\eta \int_{2k_1 r}^{4k_2 d_X} s^{-\nu/q-\eta} \rho(s) \frac{ds}{s}.$$

Collecting these facts, we obtain

$$\left| b(x) I_{\rho,\mu,\tau} f(x) - b(z) I_{\rho,\mu,\tau} f(z) \right| \leqslant C\psi(r).$$

Thus this theorem is proved. $\square$

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