Mathematical
Inequalities
& Applications
Volume 24, Number 3 (2021), 725–736 doi:10.7153/mia-2021-24-50

APPLICATIONS OF SECTIONS AND HALF VOLUMES IN STABILITY

LUJUN GUO* AND XINJIE ZHANG

(*Communicated by M. A. Hernández Cifre*)

Abstract. It is well known that one of the applications of spherical harmonics to convexity is to the so called uniqueness results, and also to stability results. In this paper, we consider sections and half volumes $V(K\cap u^+)$ of star body K, where $u^+=\{x : x\in\mathbb{R}^d, x\cdot u\geqslant 0\}$. Using spherical harmonics, we show that the star bodies K,L are identical if they have the same volumes of their central sections and half volumes and we also prove a stability version of this result.

1. Introduction

Denote by $V_i(\cdot)$ the i-dimensional Lebesgue measure. The following classical question was posed by Busemann and Petty [8] (motivated by the theory of area in Minkowski spaces) and has become known as the Busemann-Petty problem.

If K and L are origin-symmetric convex bodies in $\mathbb{R}^d$, and for each $(d-1)$-dimensional subspace H satisfy

$$V_{d-1}(K\cap H)<V_{d-1}(L\cap H),$$

does it follow that

$$V_d(K)<V_d(L)?$$

The problem is obviously correct in $\mathbb{R}^2$. In fact, let K,L be origin-symmetric convex bodies in $\mathbb{R}^2$ and H any 1-dimensional subspace of $\mathbb{R}^2$, there is $v\in S^1\cap H$ such that

$$V_1(K\cap H)=2\rho_K(v) \quad \text{and} \quad V_1(L\cap H)=2\rho_L(v),$$

where $V_1(\cdot)$ is 1-dimension measure. Since $V_1(K\cap H)<V_1(L\cap H)$, we have $\rho_K(v)<\rho_L(v)$. From the arbitrary of the 1-dimensional subspace H, we have $K\subset L$, thus $V_2(K)<V_2(L)$.

In 1975, a negative answer was given by Larman and Rogers [22] for $n\geqslant 12$. Subsequently, a series of contributions were made to reduce the dimensions to $n\geqslant 5$

Mathematics subject classification (2020): 52A20, 52A38, 33C55.
Keywords and phrases: Spherical harmonics, stability, convex bodies, half volumes, section.
This research is supported by the National Natural Science Foundation of China (Grant No. 11801151 and No. 11871197) and Key scientific research projects in Henan colleges and Universities (Grant No. 19A110022).
The authors are very grateful to the referee who read the manuscript carefully and provided a lot of valuable suggestions and comments.
* Corresponding author.

by a number of authors (Ball [3] for $n \geqslant 10$, Giannopoulos [13] and Bourgain [7] for $n \geqslant 7$, Papadimitrakis [26], Gardner [9] and Zhang [30] for $n \geqslant 5$).

In 1994, it was proved by Gardner [9] that the problem has a positive answer for $n = 3$. In 1999, Zhang [31] proved that every origin-symmetric convex body in $\mathbb{R}^4$ is an intersection body and the Busemann-Petty problem has a positive solution in $\mathbb{R}^4$. The proof of Zhang is based on a geometric argument, similar to that of Gardner [9]. In the same year, Gardner, Koldobsky and Schlumprecht [12] derived a formula connecting the derivatives of parallel section functions of an origin-symmetric star body in $\mathbb{R}^d$ with the Fourier transform of powers of radial function of the body and applied it to confirm that the answer to the Busemann-Petty problem is affirmative for $n = 4$.

In 2018, G. Giannopoulos and A. Koldobsky [14] proved some interesting inequalities estimating the distance between volumes of two convex bodies in terms of difference between areas of their sections as follows.

$$V_d(K)^{\frac{k}{d}} - V_d(L)^{\frac{k}{d}} \leqslant r^{d-k} \max_{H \in G(d,k)} \Big(V_k(K \cap H) - V_k(L \cap H) \Big),$$

where K, L are origin-symmetric convex bodies in $\mathbb{R}^d$ such that $L \subset K$ and $G(d,k)$ is the Grassmanian of k-dimensional subspace of $\mathbb{R}^d$. Let $r_{d,k}$ be the smallest constant r in the above equation. The question that whether there exist an absolute constant C such that $r_{d,k} \leqslant C$ was discussed in detail by G. Giannopoulos and A. Koldobsky [14] and this question in fact is stronger than the slicing problem, a major open problem in convex geometry (see e.g., [2], [5], [6], [24] for details).

Instead of comparing the volumes between convex bodies, we are more interested in considering the Hausdorff distance between convex bodies. In this paper, we are interested in studying the stability and determination of convex bodies as follows.

If K and L are convex bodies containing the origin in $\mathbb{R}^d$, and for each $(d-1)$-dimensional subspace H and some $0 < \varepsilon < 1$ satisfy

$$|V_{d-1}(K \cap H) - V_{d-1}(L \cap H)| < \varepsilon,$$

does there exist a constant C such that the Hausdorff distance between K and L satisfies

$$\delta(K, L) < C\varepsilon?$$

For centrally symmetric star bodies, the above problem is affirmative (see [18] and [23]). However, without the symmetry assumption, the answer is negative.

Using half-sections, Groemer [18] proved a corresponding stability result for arbitrary star bodies in $\mathbb{R}^3$ and his proof can immediately be extended to arbitrary dimensions. Furthermore, Goodey and Weil considered in [15] directed section functions $s_k(K;\cdot)$, and they showed that $s_k(K;\cdot)$ determines the convex body K uniquely. Böröczky and Schneider [4] proved that a star body is uniquely determined by the volumes and centroids of its hyperplane sections through origin o and they obtained a stability result for convex bodies. Some more results on stability and determination of convex bodies can be found in [16], [19], [20], [21], [28]. In this paper, without the symmetry assumption, we show that a star body K is uniquely determined by the volumes of sections and its half volumes $V(K \cap u^+)$, for $u \in S^{d-1}$ and $u^+ = \{x : x \in \mathbb{R}^d, x \cdot u \geqslant 0\}$, i.e., THEOREM 1. (Main) *Let K and L be two star bodies with respect to the origin in $\mathbb{R}^d$. If they have the same volumes of their central sections and half volumes, then $K = L$.*

We also prove a stability version of this result for convex bodies, i.e.,

THEOREM 2. (Main) *Let $K, L \in \mathscr{K}^d(r,R)$, $d \geqslant 3$. If, for any $u \in S^{d-1}$ and some $0 \leqslant \varepsilon \leqslant 1$,*

$$\|V(K \cap u^+) - V(L \cap u^+)\| \leqslant \varepsilon \quad \textit{and} \quad \|V(K \cap u^\perp) - V(L \cap u^\perp)\| \leqslant \varepsilon,$$

then

$$\delta(K,L) \leqslant c(d,r,R)\varepsilon^{\frac{4}{(d+1)(d+4)}}$$

with an explicit constant $c(d,r,R)$ depending only on d,r,R.

2. Notations and preliminaries

For quick later reference we introduce some notations and basic facts about convex bodies. Good general references for the theory of convex bodies are provided by the books of Gardner [11] and Schneider [28].

Let $\mathbb{R}^d$ denote the Euclidean d-dimensional space with corresponding Euclidean norm $|\cdot|$. Let B^d denote the Euclidean ball of radius one in $\mathbb{R}^d$. The set S^{d-1} is the unit sphere of $\mathbb{R}^d$ and σ is its spherical Lebesgue measure. For $u \in S^{d-1}$, $u^+ := \{x : x \in \mathbb{R}^d, x \cdot u \geqslant 0\}$ and $u^\perp := \{x : x \in \mathbb{R}^d, x \cdot u = 0\}$, where $x \cdot u$ is the scalar product.

A star body K in Euclidean space $\mathbb{R}^d$ is a nonempty compact set which is star shaped with respect to the origin o and has a continuous positive radial function, defined by

$$\rho_K(v) := \max\{\lambda \geqslant 0 : \lambda v \in K\} \tag{1}$$

for $v \in \mathbb{R}^d \setminus \{o\}$.

Let $V_d(K)$ denote the volume of K and $\mathscr{S}^d$ denote the set of star bodies with respect to the origin in $\mathbb{R}^d$. Write κ_d for $V_d(B^d)$ the volume of B^d, thus $d\kappa_d = \sigma(S^{d-1})$. Let $\mathscr{S}_c^d$ be the set of star bodies which are origin central symmetric.

A convex body K is a compact, convex set with non-empty interiors. Let $\mathscr{K}^d(r,R)$ be the set of convex bodies K satisfying $rB^d \subset K \subset RB^d$. Associated with a convex body K is its support function h_K defined, for $x \in \mathbb{R}^d$, by

$$h_K(x) := \max\{x \cdot y : y \in K\}. \tag{2}$$

The function h_K is positively homogeneous of degree 1. We will usually be concerned with the restriction of the support function to the unit sphere S^{d-1}.

Let K be a convex body with origin $o \in \text{int}K$. We define the polar body of K by

$$K^o := \{x \in \mathbb{R}^d : \langle x, y\rangle \leqslant 1 \quad for \ \ all \ \ y \in K\}.$$

From the definitions (1) and (2), we have

$$\rho_K(u) = \frac{1}{h_{K^o}(u)} \tag{3}$$

The Hausdorff distance between convex bodies K, L is defined by

$$\delta(K,L) := \max\{\max_{x\in K}\min_{y\in L}|x-y|, \max_{x\in L}\min_{y\in K}|x-y|\}$$

or, equivalently, by

$$\delta(K,L) := \min\{\lambda \geqslant 0 | K \subset L+\lambda B^d, L\subset K+\lambda B^d\}.$$

In terms of the support function, the Hausdorff distance between convex bodies K, L can also be expressed as follows (see e.g., [11] or [28]),

$$\delta(K,L) = \max_{u\in S^{d-1}} |h_K(u) - h_L(u)|. \tag{4}$$

Let $L_2(S^{d-1})$ denote the class of all real valued Lebesgue integrable functions f on S^{d-1} with the property that $\int_{S^{d-1}} f^2(u)d\sigma(u) < \infty$. If $f, g \in L_2(S^{d-1})$, the inner product $\langle f, g\rangle$ is defined by $\langle f, g\rangle = \int_{S^{d-1}} f(u)g(u)d\sigma(u)$. Let $\|\cdot\|$ denote the norm derived from this inner product. Let f^+ and f^- denote the functions

$$f^+(u) = \frac{1}{2}(f(u)+f(-u)), \qquad f^-(u) = \frac{1}{2}(f(u)-f(-u)),$$

respectively. So, $f = f^+ + f^-$, and f^+ is an even function and f^- is an odd function on S^{d-1}.

3. Determination and stability of convex bodies

In this section we first introduce the following two useful spherical integral transformations used to obtain our results. The study of spherical harmonics has a long history. Groemer's book [17] and Müller's [25] are our standard reference for basics regarding spherical harmonics. More information about spherical harmonics can also be found in references [1], [27] and [29].

One is the Radon transformation on S^{d-1} denoted by $\mathscr{R}$, e.g., for each bounded integrable function f on S^{d-1}, let $\mathscr{R}f$ be the function defined by

$$\mathscr{R}(f)(u) = \int_{S^{d-1}\cap u^\perp} f(v)d\sigma(v), \quad \text{for} \quad u\in S^{d-1}.$$

The other is the following linear integral transformation $\mathscr{T}$ of functions on S^{d-1},

$$\mathscr{T}(f)(u) = \int_{S^{d-1}} \tau(u\cdot v)f(v)d\sigma(v), \quad \text{for} \quad u\in S^{d-1}, \tag{5}$$

where $f\in L_2(S^{d-1})$, $\tau(x) = 1$ for $x \geqslant 0$ and $\tau(x) = 0$ for $x < 0$. The transformation (5) is called the hemispherical integral transformation.

Now we introduce the injectivity of the Radon transformation and the hemispherical integral transformation as follows (see, e.g., [17], p. 102 or [25]).

LEMMA 1. *Let f_1, f_2 be two bounded integrable functions on S^{d-1}. Then the equality $\mathscr{R}(f_1) = \mathscr{R}(f_2)$ holds if and only if $f_1^+ = f_2^+$ almost everywhere.*

LEMMA 2. *Let f_1, f_2 be two continuous functions on S^{d-1}. Then the equality $\mathscr{T}(f_1) = \mathscr{T}(f_2)$ holds if and only if $f_1^- = f_2^-$ and $\langle f_1, 1\rangle = \langle f_2, 1\rangle$, where $\langle f_i, 1\rangle = \int_{S^{d-1}} f_i(u)d\sigma(u)$, $i = 1,2$.*

Groemer [17] showed that spherical harmonics enable one not only to prove uniqueness results, but also to establish stability results. In other words, it is possible to estimate under suitable assumptions the L_2-distance between two functions if the distance between their spherical integral transforms is known. The following lemma (see, e.g., [17], p. 110) shows such estimates and we will use the stability results to prove the Theorem 2.

In the following lemma, the constant β_d is defined by $\beta_3 = 2^{-\frac{3}{4}}$ and, for $d \geqslant 4$, by

$$\beta_d = (d-1)^{-\frac{d-2}{4}} 1 \cdot 3 \cdot \dots \cdot (d-3), \quad \text{when } d \text{ is even}$$

and

$$\beta_d = \frac{1}{\sqrt{2}}(d-1)^{-\frac{d-2}{4}} 2 \cdot 4 \cdot \dots \cdot (d-3), \quad \text{when } d \text{ is odd.}$$

Let ∇ denote the gradient. If f is a function whose domain is a subset of $\mathbb{R}^d$ that contains S^{d-1}, we write $f^\wedge$ for the restriction of f to S^{d-1}. On the other hand, if f is defined on S^{d-1}, we let $f^\vee$ denote the radial extension of f to $\mathbb{R}^d \setminus \{0\}$. This means that $f^\vee(x) = f(\frac{x}{|x|})$. Using the above extension procedure one can transfer the gradient to the operator acting on functions on S^{d-1}. We define ∇_0 by $\nabla_0 f = (\nabla f^\vee)^\wedge$.

LEMMA 3. *If f_1 and f_2 are twice continuously differentiable functions on S^{d-1} $(d \geqslant 3)$, then*

$$\|f_1^+ - f_2^+\| \leqslant g_d(f_1,f_2)\|\mathscr{R}(f_1) - \mathscr{R}(f_2)\|^{\frac{2}{d}} \tag{6}$$

with

$$g_d(f_1,f_2) = \frac{1}{(d-1)\kappa_{d-1}}\Big(2(d-1)^2\kappa_{d-1}^2\beta_d^{-\frac{4}{d-2}}(\|\nabla_0 f_1\|^2 + \|\nabla_0 f_2\|^2) + \|\mathscr{R}(f_1) - \mathscr{R}(f_2)\|^2\Big)^{\frac{d-2}{2d}}. \tag{7}$$

LEMMA 4. *If f_1 and f_2 are twice continuously differentiable functions on S^{d-1} $(d \geqslant 3)$, then*

$$\|f_1^- - f_2^-\| \leqslant q_d(f_1,f_2)\|\mathscr{T}(f_1) - \mathscr{T}(f_2)\|^{\frac{2}{d+2}} \tag{8}$$

with

$$q_d(f_1,f_2) = \sqrt{2}(\sqrt{2}\kappa_{d-1}\beta_{d+2})^{-\frac{2}{d+2}}(\|\nabla_0 f_1\|^2 + \|\nabla_0 f_2\|^2)^{\frac{d}{2(d+2)}}. \tag{9}$$

We are now in the position to prove our main results.

THEOREM 1. *Let K and L be two star bodies with respect to the origin in $\mathbb{R}^d$. If they have the same volumes of their central sections and half volumes, then $K = L$.*

Proof. For $u \in S^{d-1}$, $V(K \cap u^+)$ can be expressed as follows,

$$\begin{aligned} V(K\cap u^+) &= \int_K \tau(u\cdot x)dx \\ &= \frac{1}{d}\int_{S^{d-1}} \tau(u\cdot v)\rho_K^d(v)d\sigma(v) \\ &= \frac{1}{d}\mathscr{T}(\rho_K^d)(u). \end{aligned} \tag{10}$$

In the similar way, we have

$$V(L\cap u^+) = \frac{1}{d}\mathscr{T}(\rho_L^d)(u). \tag{11}$$

Since $V(K\cap u^+) = V(L\cap u^+)$, for all $u \in S^{d-1}$, from (10), (11) and Lemma 2, we get

$$\rho_K^d(u) - \rho_L^d(u) = \rho_K^d(-u) - \rho_L^d(-u), \qquad \forall u \in S^{d-1}. \tag{12}$$

Since K and L have the same volumes of their central sections, from the polar coordinate formula of volume, we have

$$\rho_K^{d-1}(u) - \rho_L^{d-1}(u) = -(\rho_K^{d-1}(-u) - \rho_L^{d-1}(-u)), \quad \forall u \in S^{d-1}. \tag{13}$$

From (12) and (13), we have $\rho_K = \rho_L$. $\square$

REMARK 1. The ideas and techniques of Greomer [18] play a critical role in the proof of Theorem 1. However, the uniqueness proof of this theorem is not exactly the same as the uniqueness Theorem given by Greomer in [18]. To prove the uniqueness, we use the volumes of sections $V_{d-1}(K\cap u^\perp)$ ($(d-1)$-dimensional measure) of the star body K and its half volumes $V_d(K\cap u^+)$ (d-dimensional measure), and Groemer used the volumes of half-sections $V_{d-1}(K\cap H(u,w))$ ($(d-1)$-dimensional measure), where $H(u,w) = \{x : u^\perp,\ x\cdot w \geqslant 0\}$, $u \in S^{d-1}$ and $w \in S^{d-1}\cap u^\perp$.

To prove the stability version of this new uniqueness, we will have to restrict ourselves to the case of convex bodies. Let $\mathscr{K}^d(r,R)$ denote the space of convex bodies K satisfying $rB^d \subset K \subset RB^d$, where $0 < r < R$ are given numbers. We shall require the following crude estimate for the gradient of radial function. For the sake of completeness, we include a proof.

LEMMA 5. *Let $K \in \mathscr{K}^d(r,R)$ have twice continuously differentiable radial function and support function. Then, for $m > 0$,*

$$\|\nabla_o \rho_K^m\| \leqslant \frac{mR^{m+1}\sqrt{(d-1)d\kappa_d}}{r}. \tag{14}$$

Proof. Using Euler's relation

$$\sum_{i=1}^{d} x_i \frac{\partial f(x)}{\partial x_i} = mf(x),$$

we have after a straightforward calculation that

$$\Delta f\left(\frac{x}{|x|}\right) = \Delta(f(x)|x|^{-m}) = |x|^{-m}\Delta f(x) - m(m+d-2)f(x)|x|^{-m-2}, \tag{15}$$

where f is positively homogeneous of degree m on $\mathbb{R}^d \setminus \{o\}$.

Since h_K is twice continuously differentiable, for $u \in S^{d-1}$ and $m = 1$ in (15), we have

$$\Delta_o h_K(u) = \Delta h_K(u) - (d-1)h_K(u). \tag{16}$$

From Green's formula, (16) and the fact in the theory of convex bodies (see, e.g., [28]) for twice continuously differentiable h_K

$$W_{d-2}(K) = \frac{1}{d(d-1)} \int_{S^{d-1}} h_K(u)\Delta h_K(u) d\sigma(u), \tag{17}$$

it follows that

$$\begin{aligned} W_{d-2}(K) &= \frac{1}{d(d-1)} \int_{S^{d-1}} h_K(u)((d-1)h_K(u) + \Delta_o h_K(u)) d\sigma(u) \\ &= \frac{1}{d(d-1)} \int_{S^{d-1}} ((d-1)h_K(u)^2 - |\nabla_o h_K(u)|^2) d\sigma(u) \\ &= \frac{1}{d}\|h_K\|^2 - \frac{1}{d(d-1)}\|\nabla_o h_K\|^2 \geqslant 0. \end{aligned} \tag{18}$$

From (3), (18) and the facr $K \in \mathscr{K}^d(r,R)$, we have

$$\begin{aligned} \|\nabla_o \rho_K^m\| &= m\|h_{K^o}^{-m-1}\nabla_o h_{K^o}\| \\ &= m\|\rho_K^{m+1}\nabla_o h_{K^o}\| \\ &\leqslant mR^{m+1}\|\nabla_o h_{K^o}\| \\ &\leqslant mR^{m+1}\sqrt{d-1}\|h_{K^o}\| \\ &= mR^{m+1}\sqrt{d-1}\|\rho_K^{-1}\| \\ &\leqslant \frac{mR^{m+1}\sqrt{(d-1)d\kappa_d}}{r}. \quad \square \end{aligned}$$

For convenience, we write

$$V(K,u) := V(K \cap u^+), \quad \text{for} \quad u \in S^{d-1}.$$

Now we can formulate our stability version of Theorem 1 as follows.

THEOREM 2. *Let $K,L \in \mathscr{K}^d(r,R)$, $d \geqslant 3$. If, for any $u \in S^{d-1}$ and some $0 \leqslant \varepsilon \leqslant 1$,*

$$\|V(K,\cdot)-V(L,\cdot)\| \leqslant \varepsilon \quad \textit{and} \quad \|V(K\cap u^{\perp})-V(L\cap u^{\perp})\| \leqslant \varepsilon,$$

then

$$\delta(K,L) \leqslant c(d,r,R)\varepsilon^{\frac{4}{(d+1)(d+4)}}$$

with an explicit constant $c(d,r,R)$ depending only on d,r,R.

Proof. We assume that the assumptions are satisfied and that K and L have twice continuously differentiable radial functions. If the theorem is proved under this assumption, then the general case follows by approximation.

Since

$$V(K,u)=\frac{1}{d}\mathscr{T}(\rho_K^d)(u), \quad V(L,u)=\frac{1}{d}\mathscr{T}(\rho_L^d)(u),$$

we have

$$\|\mathscr{T}(\rho_K^d)-\mathscr{T}(\rho_L^d)\| \leqslant d\|V(K,\cdot)-V(L,\cdot)\| \leqslant d\varepsilon. \tag{19}$$

Hence Lamma 4 together with (19) gives

$$\begin{aligned}\|(\rho_K^d)^- - (\rho_L^d)^-\|^2 &\leqslant q_d(\rho_K^d,\rho_L^d)^2\|\mathscr{T}(\rho_K^d)-\mathscr{T}(\rho_L^d)\|^{\frac{4}{d+2}}\\ &\leqslant q_d(\rho_K^d,\rho_L^d)^2(d\varepsilon)^{\frac{4}{d+2}} =: c_1.\end{aligned} \tag{20}$$

Since $K,L \in \mathscr{K}^d(r,R)$, we have $\rho_K(v),\rho_L(v) \geqslant r > 0$ for $v \in S^{d-1}$, hence

$$\gamma_m := \sum_{i=0}^{m-1}\rho_K^i\rho_L^{m-1-i} \geqslant mr^m \qquad \text{on} \qquad S^{d-1}.$$

Putting $\alpha(v) := \frac{\rho_K^d(-v)-\rho_L^d(-v)}{\gamma_d(v)}$, from (20), we have

$$\begin{aligned}&\int_{S^{d-1}}|\rho_K(v)-\rho_L(v)-\alpha(v)|^2 d\sigma(v)\\ &=\int_{S^{d-1}}\Big|\frac{\rho_K^d(v)-\rho_L^d(v)-\big(\rho_K^d(-v)-\rho_L^d(-v)\big)}{\gamma_d(v)}\Big|^2 d\sigma(v)\\ &\leqslant \frac{4}{(dr^d)^2}\|\big(\rho_K^d(\cdot)\big)^- - \big(\rho_L^d(\cdot)\big)^-\|^2 \leqslant \frac{4c_1}{(dr^d)^2}\end{aligned} \tag{21}$$

In the similar way, we have

$$\|\mathscr{R}(\rho_K^{d-1})-\mathscr{R}(\rho_L^{d-1})\|^2=(d-1)^2\|V(K\cap u^{\perp})-V(L\cap u^{\perp})\|^2 \leqslant (d-1)^2\varepsilon^2, \tag{22}$$

and Lamma 3 together with (22) gives

$$\begin{aligned}\|(\rho_K^{d-1})^+ - (\rho_L^{d-1})^+\|^2 &\leqslant g_d(\rho_K^{d-1},\rho_L^{d-1})^2\|\mathscr{R}(\rho_K^{d-1})-\mathscr{R}(\rho_L^{d-1})\|^{\frac{2}{d-1}}\\ &= g_d(\rho_K^{d-1},\rho_L^{d-1})^2((d-1)\varepsilon)^{\frac{4}{d-1}} =: c_2.\end{aligned} \tag{23}$$

Putting $\beta(v) := \frac{\rho_K^{d-1}(-v)-\rho_L^{d-1}(-v)}{\gamma_{d-1}(v)}$, from (23), we have

$$\begin{aligned}
&\int_{S^{d-1}} |\rho_K(v) - \rho_L(v) + \beta(v)|^2 d\sigma(v) \\
&= \int_{S^{d-1}} |\frac{\rho_K^{d-1}(v) - \rho_L^{d-1}(v) + \big(\rho_K^{d-1}(-v) - \rho_L^{d-1}(-v)\big)}{\gamma_{d-1}(v)}|^2 d\sigma(v) \\
&\leqslant \frac{4}{((d-1)r^{d-1})^2} \|\big(\rho_K^{d-1}(\cdot)\big)^+ - \big(\rho_L^{d-1}(\cdot)\big)^+\|^2 \leqslant \frac{4c_2}{((d-1)r^{d-1})^2}
\end{aligned} \tag{24}$$

Let

$$S^- := \{v \in S^{d-1} | \big(\rho_K(v) - \rho_L(v)\big)\alpha(v) \leqslant 0\},$$

and

$$S^+ := \{v \in S^{d-1} | \big(\rho_K(v) - \rho_L(v)\big)\beta(v) \geqslant 0\}.$$

Then we get

$$\begin{aligned}
&\int_{S^-} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) \\
&\leqslant \int_{S^-} |\rho_K(v) - \rho_L(v) - \alpha(v)|^2 d\sigma(v) \\
&\leqslant \frac{4c_1}{(dr^d)^2} \leqslant \frac{4c_1}{((d-1)r^{d-1})^2}
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\int_{S^+} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) \\
&\leqslant \int_{S^+} |\rho_K(v) - \rho_L(v) + \beta(v)|^2 d\sigma(v) \\
&\leqslant \frac{4c_2}{((d-1)r^{d-1})^2}.
\end{aligned} \tag{26}$$

Since $S^+ \cup S^- = S^{d-1}$, it follows that

$$\begin{aligned}
&\int_{S^{d-1}} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) \\
&\leqslant \int_{S^-} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) + \int_{S^+} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) \\
&\leqslant \frac{4}{((d-1)r^{d-1})^2}(c_1 + c_2).
\end{aligned} \tag{27}$$

From (20), (9) and Lemma 5, we have

$$\begin{aligned}
c_1 &= q_d(\rho_K^d, \rho_L^d)^2 (d\varepsilon)^{\frac{4}{d+2}} \\
&= 2(\sqrt{2}\kappa_{d-1}\beta_{d+2})^{-\frac{4}{d+2}} (\|\nabla_0 \rho_K^d\|^2 + \|\nabla_0 \rho_L^d\|^2)^{\frac{d}{(d+2)}} (d\varepsilon)^{\frac{4}{d+2}} \\
&\leqslant 2(\sqrt{2}\kappa_{d-1}\beta_{d+2})^{-\frac{4}{d+2}} 2(d\sqrt{(d-1)d\kappa_d}\frac{R^{d+1}}{r})^{\frac{2d}{(d+2)}} (d\varepsilon)^{\frac{4}{d+2}}.
\end{aligned} \tag{28}$$

Similarly, from (23), (7) and Lemma 5, we have

$$
\begin{aligned}
c_2 &= p_d(\rho_K^{d+1},\rho_L^{d+1})^2\Big((d+1)R^d\kappa_d\sqrt{d\kappa_d}\Big)^{\frac{4}{d+4}}\varepsilon^{\frac{4}{d+4}}\\
&= \frac{1}{(2\kappa_{d-1})^2}\Big(8\kappa_{d-1}^2\big((d-1)\beta_d\big)^{-\frac{4}{d+2}}(\|\nabla_0\rho_K^{d+1}\|^2+\|\nabla_0\rho_L^{d+1}\|^2)\\
&\quad+\|\mathscr{C}(\rho_K^{d+1})-\mathscr{C}(\rho_L^{d+1})\|^2\Big)^{\frac{d+2}{d+4}}\Big((d+1)R^d\kappa_d\sqrt{d\kappa_d}\Big)^{\frac{4}{d+4}}\varepsilon^{\frac{4}{d+4}}\\
&\leqslant \frac{1}{(2\kappa_{d-1})^2}\Big(16\kappa_{d-1}^2\big((d-1)\beta_d\big)^{-\frac{4}{d+2}}((d+1)\sqrt{(d-1)d\kappa_d}\frac{R^{d+2}}{r})^2+1\Big)^{\frac{d+2}{d+4}}\\
&\quad\cdot\Big((d+1)R^d\kappa_d\sqrt{d\kappa_d}\Big)^{\frac{4}{d+4}}\varepsilon^{\frac{4}{d+4}}. \qquad (29)
\end{aligned}
$$

Since $0\leqslant\varepsilon\leqslant 1$, then an explicit constant $c_1(d,r,R)$ depending only on d,r,R in the theorem can be read off from (27), (28) and (29), i.e.,

$$\|\rho_K-\rho_L\|^2=\int_{S^{d-1}}|\rho_K(v)-\rho_L(v)|^2d\sigma(v)\leqslant c_1(d,r,R)\varepsilon^{\frac{4}{d+4}}. \qquad (30)$$

For convex bodies $K,L\in\mathscr{K}^d(r,R)$, the Hausdorff distance $\delta(K,L)$ can be estimated in terms of the radial L_2-metric by

$$\delta(K,L)\leqslant c_dR^2r^{-\frac{d+3}{d+1}}\|\rho_K-\rho_L\|^{\frac{2}{d+1}} \qquad (31)$$

with an explicit constant c_d depending only on the dimension d (see Groemer [17], Lemma 2.3.2).

The conclusion can be obtained from (30) and (31).

In general case, the result follows by approximation (see, e.g., [28], p. 157). □

REMARK 2. The convexity in Theorem 2 is necessary. When K is a convex body containing the origin, we have

$$\rho_K(u)=h_{K^o}(u)^{-1},\quad \text{for all } u\in S^{d-1}. \qquad (32)$$

The estimate in Lemma 5, inside the proof of Theorem 2, depends on the differentiability of the support function h_{K^o} and the relation (32) (see [17], p. 234, for reference). However, the equation (32) does not hold in the case that K is not convex.

REMARK 3. In [18], the stability and determination of convex bodies were obtained from the uniqueness results and estimates of the new spherical integral transformation $\mathscr{B}$ defined as follows.

$$\mathscr{B}f(u,w)=\int_{S^{d-1}\cap H(u,w)}f(v)d\sigma(v)$$

where f is a continuous function on S^{d-1}, $H(u,w) = \{x : x \in u^{\perp},\ x\cdot w \geqslant 0\}$, $u \in S^{d-1}$ and $w \in S^{d-1} \cap u^{\perp}$. The transformation $\mathscr{B}$ can be express in terms of hemispherical transformation $\mathscr{T}_{d-2}$ (on $(d-2)$-dimensional unit sphere $S^{d-1}\cap u^{\perp}$),i.e.

$$\mathscr{B}f(u,w) = \int_{S^{d-1}\cap u^{\perp}} \tau(v\cdot w)f(v)d\sigma(v) =: (\mathscr{T}_{d-2}f)(w)$$

where $w \in S^{d-1}\cap u^{\perp}$. In our paper, Theorem 1 and Theorem 2 are proved by using spherical Radon transformation $\mathscr{R}$ (on $(d-1)$-dimensional unit sphere S^{d-1}) and hemispherical transformation $\mathscr{T}$ (on $(d-1)$-dimensional unit sphere S^{d-1} and different from $\mathscr{T}_{d-2}$ in [18]) together. Hence, our results do not follow from the results of Greomer.

REMARK 4. In Theorem 2, we estimate the distance between convex bodies by Hausdorff distance different from that of [14] in which G. Giannopoulos and A. Koldobsky proved some inequalities estimating the distance between volumes of two convex bodies.

REFERENCES

[1] YU. E. ANIKONOV AND V. N. STEPANOV, *An inversion formula in a problem of integral geometry*, (Russian) Dokl. Akad. Nauk SSSR 318, no. 2 (1991), 265–266; translation in Soviet Math. Dokl. 43, no. 3 (1992), 678–679.

[2] K. BALL, *Isometric problems in l_p and sections of convex sets*, Ph. D. dissertation, Trinity College, Cambridge (1986).

[3] K. BALL, *Some remarks on the geometry of convex sets*, in Geometric Aspects of Functional Analysis (J. Lindenstrauss and V. D. Milman, eds.), Lecture Notes in Math. **1317**, Springer-Verlag, New York (1988), 224–231.

[4] K. J. BÖRÖCZKY AND R. SCHNEIDER, *Stable determination of convex bodies from sections*, Studia Sci. Math. Hung. **46** (2009), 367–376.

[5] J. BOURGAIN, *On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math., **108** (1986), 1467–1476.

[6] J. BOURGAIN, *Geometry of Banach spaces and harmonic analysis*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, (1987), 871–878.

[7] J. BOURGAIN, *On the Busemann-Petty problem for perturbations of the ball*, Geom. Funct. Anal. **1** (1991), 1–13.

[8] H. BUSEMANN AND C. M. PETTY, *Problems on convex bodies*, Math. Scand. **4** (1956), 88–94.

[9] R. J. GARDNER, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. **140** (1994) 435–447.

[10] R. J. GARDNER, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994) 435–445.

[11] R. J. GARDNER, *Geometric Tomography*, 2th edition, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.

[12] R. J. GARDNER, A. KOLDOBSKY AND T. SCHLUMPRECHT, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math., **149** (1999), 691–703.

[13] A. GIANNOPOULOS, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika **37** (1990), 239–244.

[14] A. GIANNOPOULOS AND A. KOLDOBSKY, *Volum difference inequlities*, Trans. Amer. Math. Soc. **370** (2018), 4351–4372.

[15] P. GOODEY AND W. WEIL, *Average section functions for star-shaped sets*, Adv. Math. **36** (2006), 70–84.

[16] P. GOODEY, V. YASKIN AND M. YASKINA, *Fourier transforms and the Funk-Hecke theorem in convex geometry*, J. London Math. Soc. (2) **80** (2009), 388–404.
[17] H. GROEMER, *Geometric Applications of Fourier Series and Spherical Harmonics*, Cambridge University Press, New York, 1996.
[18] H. GROEMER, *On a spherical integral transform and sections of star bodies*, Monatsh. Math. **126**(1998), 117–124.
[19] L. GUO AND G. LENG, *Determination of star bodies from p-centroid bodies*, Proc. Indian Acad. Sci. Math. Sci., **123** (2013), 577–586.
[20] L. GUO AND G. LENG, *Stable determination of convex bodies from centroid bodies*, Houston J. Math., **40** (2014), 395–406.
[21] A. KOLDOBSKY, *An applications of the Fourier transform to sections of star bodies*, Israel J. Math., **106** (1998), 157–164.
[22] D. G. LARMAN AND C. A. ROGERS, *The existence of a centrally symmetric convex body with central sections that are unexpectedly small*, Mathematika, **22** (1975), 164–175.
[23] E. LUTWAK, *Intersection bodies and dual mixed volumes*, Adv. Math., **71** (1988), 232–261.
[24] V. D. MILMAN AND A. PAJOR, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, 64–104.
[25] C. MÜLLER, *Spherical Harmonics*, Springer, Berlin/Heidelberg/New York, 1966.
[26] M. PAPADIMITRAKIS, *On the Busemann-Petty problem about convex, centrally symmetric bodies in $\mathbb{R}^n$*, Mathematika, **39** (1992), 258–266.
[27] R. SCHNEIDER, *The use of spherical harmonics in convex geometry*, `http://home.mathematik.uni-freiburg.de/rschnei/Vortr%E4ge.Kent.pdf`.
[28] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia Math. Appl., vol. 151, expanded edn. Cambridge University Press, Cambridge, 2014.
[29] V. N. STEPANOV, *The method of spherical harmonics for integral transforms on a sphere*, Mat. Strukt. Model. **42** (2017), 36–48.
[30] G. ZHANG, *Intersection bodies and the Busemann-Petty inequalities in $\mathbb{R}^4$*, Ann. of Math., **140** (1994), 331–346.
[31] G. ZHANG, *A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$*, Ann. of Math., **149** (1999), 535–543.

(Received June 17, 2020)

Lujun Guo
Henan Engineering Laboratory for
Big Data Statistical Analysis and Optimal Control
College of Mathematics and Information Science
Henan Normal University, Henan, 453007, P. R. China
e-mail: lujunguo0301@163.com

Xinjie Zhang
Henan Normal University
Henan, 453007, P. R. China
e-mail: 1584648191@qq.com