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MAXIMAL OPERATORS OF T MEANS WITH RESPECT TO WALSH–KACZMARZ SYSTEM

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Abstract. In this paper we prove and discuss some new $(H_p, L_{p,\infty})$ type inequalities of the maximal operators of T means with monotone coefficients with respect to Walsh-Kaczmarz system. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out. In particular, we apply these results to prove a.e. convergence of such T means.

1. Introduction

Concerning definitions and notations used in this introduction we refer to Sections 2.

In 1948, Šneider [37] introduced the Walsh-Kaczmarz system and showed that the inequality $\limsup_{n\to\infty} D_n^{\kappa}(x)/\log n \geqslant C > 0$ holds a.e. In 1974 Schipp [32] and Young [47] proved that the Walsh-Kaczmarz system is a convergence system.

In 1981, Skvortsov [36] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [9] proved that, for any integrable functions, the Fejér means with respect to the Walsh-Kaczmarz system converges almost everywhere to the function. He showed that the maximal operator $\sigma^{*,\kappa}$ of Walsh-Kaczmarz-Fejér means is of weak type $(1,1)$ and of type (p,p) for all $1 < p \leqslant \infty$. Gát's result was generalized by Simon [34], who showed that the maximal operator $\sigma^{*,\kappa}$ is of type (H_p, L_p) for $p > 1/2$. In the endpoint case $p = 1/2$ Goginava [11] (see also [30, 39, 40, 41]) proved that maximal operator $\sigma^{*,\kappa}$ of Walsh-Kaczmarz-Fejér means is not of type $(H_{1/2}, L_{1/2})$. Weisz [50] showed that the following is true:

THEOREM W1. *The maximal operator $\sigma^{*,\kappa}$ of Walsh-Kaczmarz-Fejér means is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$.*

In [17] Goginava and Nagy proved that the maximal operator $R^{*,\kappa}$ of Riesz means with respect to Walsh-Kaczmarz system is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$, but is not bounded from the Hardy space H_p to the space L_p, when

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$0 < p \leqslant 1/2$. In [38] it was proved that there exists a martingale $f \in H_p$, $(0 < p \leqslant 1)$, such that the maximal operator $L^{*,\kappa}$ of Nörlund logarithmic means with respect to Walsh-Kaczmarz system is not bounded in the Lebesgue space L_p. The Logarithmic means with respect to the Walsh and Vilenkin systems systems were studied by Blahota and Gát [4], Lukkassen, Persson, Tephnadze and Tutberidze [14] (see also [8], [13], [29], [31], [42], [44] and [46]), Simon [35].

Gát and Goginava [10] proved that the maximal operator $\sigma^{\alpha,*,\kappa}$ of (C,α) $(0 < \alpha < 1)$ means with respect Walsh-Kaczmarz system is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha),\infty}$. Goginava and Nagy [12] proved that $\sigma^{\alpha,*,\kappa}$ is not bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)}$. Móricz and Siddiqi [19] investigated the approximation properties of some special Nörlund means of L_p function in norm. These means in the martingale Hardy spaces were discussed in Blahota, Tephnadze [5, 6]. In [28] and [43] (see also [1, 7, 16]) it was proved some (H_p, L_p)-type inequalities for the maximal operators of Nörlund means, with respect to Walsh-Kaczmarz and Vilenkin systems, when $0 < p \leqslant 1$. In the two dimensional case approximation properties of Nörlund and Cesáro means were considered by Nagy [20, 21, 22] and by Nagy and Tephnadze [23, 24, 25, 26, 27]. Some boundedness results of so called T, Θ and θ means in the Lebesgue and martingale Hardy spaces can be found in Blahota and Nagy [2], in Blahota, Nagy and Tephnadze [3], Tutberidze [45] and Weisz [51, 52, 53, 54].

The main aim of this paper is to investigate $(H_p, L_{p,\infty})$-type inequalities for the maximal operators of T means with monotone coefficients of the one-dimensional Kaczmarz-Fourier series.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary results of independent interest. Also these results are presented in Section 3. The detailed proofs are given in Section 4.

2. Definitions and Notations

Now, we give a brief introduction to the theory of dyadic analysis [33]. Let $\mathbf{N}_+$ denote the set of positive integers, $\mathbf{N} := \mathbf{N}_+ \cup \{0\}$.

Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form

$$x = (x_0, x_1, \ldots, x_k, \ldots), \quad x_k = 0, 1, \ (k \in \mathbf{N}).$$

The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology.The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G, \quad I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},$$

$(x \in G, n \in \mathbf{N})$. These sets are called dyadic intervals. Denote by $0 = (0 : i \in \mathbf{N}) \in G$ the null element of G. Let $I_n := I_n(0)$, $\overline{I_n} := G \backslash I_n$ $(n \in \mathbf{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$, the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\}$ $\quad (i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \leqslant n < 2^{|n|+1}$.

For $k \in \mathbf{N}$ and $x \in G$ let us denote the k-th Rademacher function, by

$$r_k(x) := (-1)^{x_k}.$$

Now, define the Walsh system $w := (w_n : n \in \mathbf{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbf{N}).$$

The Walsh-Kaczmarz functions are defined by

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} \left(r_{|n|-1-k}(x)\right)^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Dirichlet kernels are defined by

$$D_0 := 0, \quad D_n^{\psi} := \sum_{i=0}^{n-1} \psi_i, \ (\psi = w \text{ or } \psi = \kappa).$$

The 2^n-th Dirichlet kernels have a closed form (see e.g. [33])

$$D_{2^n}^{w}(x) = D_{2^n}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n & x \in I_n, \\ 0 & x \notin I_n. \end{cases} \tag{1}$$

The norm (or quasi-norm) of the spaces $L_p(G)$ and $L_{p,\infty}(G)$ are respectively defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu, \quad \|f\|_{L_{p,\infty}(G)}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda), \quad (0 < p < \infty).$$

The partial sums with respect to Walsh and Walsh-Kaczmarz series are defined as follows:

$$S_M^{\psi} f := \sum_{i=0}^{M-1} \widehat{f}(i) \psi_i, \quad (\psi = w \text{ or } \psi = \kappa).$$

Let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers. The n-th Nörlund and T means for a Fourier series of f are respectively defined by

$$t_n^{\psi} f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k^{\psi} f, \quad (\psi = w \text{ or } \psi = \kappa)$$

and

$$T_n^{\psi} f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k^{\psi} f, \quad (\psi = w \text{ or } \psi = \kappa). \tag{2}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$. It is obvious that

$$T_n^{\psi} f(x) = \int_G f(t) F_n^{\psi}(x-t)\, d\mu(t)$$

where $F_n^{\psi} := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k^{\psi}$ is called T kernel.

We always assume that $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers and $q_0 > 0$. Then the summability method (2) generated by $\{q_k : k \geqslant 0\}$ is regular if and only if $\lim_{n\to\infty} Q_n = \infty$.

Let consider some class of T means with monotone and bounded sequence $\{q_k : k \in \mathbf{N}\}$, such that

$$q := \lim_{n\to\infty} q_n > c > 0.$$

Then, it easy to check that

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{3}$$

The n-th Fejér means of a function f is given by

$$\sigma_n^{\psi} f := \frac{1}{n} \sum_{k=1}^{n} S_k^{\psi} f, \quad (\psi = w \text{ or } \psi = \kappa).$$

Fejér kernel is defined in the usual manner

$$K_n^{\psi} := \frac{1}{n} \sum_{k=1}^{n} D_k^{\psi}, \quad (\psi = w \text{ or } \psi = \kappa).$$

If we invoke Abel transformation

$$\sum_{j=1}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_0 b_1 + \sum_{j=1}^{n-1} A_j (b_j - b_{j+1}), \;\; a_j = A_j - A_{j-1}, \;\; j = 1, \ldots, m,$$

for $b_j = q_j$, $a_j = 1$ and $A_j = j$ for any $j = 0, 1, \ldots, n$ we get the following identity:

$$Q_n := q_0 + \sum_{j=1}^{n-1} q_j = q_0 + \sum_{j=1}^{n-2} \left(q_j - q_{j+1}\right) j + q_{n-1}(n-1) \tag{4}$$

Moreover, if use $D_0 = K_0 = 0$ for any $x \in G_m$ and invoke Abel transformation for $b_j = q_j$, $a_j = D_j$ and $A_j = jK_j$ for any $j = 0, 1, \ldots, n-1$ we get identity:

$$F_n^{\psi} = \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} \left(q_j - q_{j+1}\right) jK_j^{\psi} + q_{n-1}(n-1) K_{n-1}^{\psi} \right). \tag{5}$$

The (C,α)-means are defined as

$$\sigma_n^{\alpha,\psi} f=\frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k^\psi f, \quad (\psi=w \text{ or } \psi=\kappa),$$

where

$$A_0^\alpha=0, \qquad A_n^\alpha=\frac{(\alpha+1)\dots(\alpha+n)}{n!}, \ \alpha\neq -1,-2,\dots$$

It is known that

$$A_n^\alpha \sim n^\alpha, \ A_n^\alpha-A_{n-1}^\alpha=A_n^{\alpha-1}, \ \sum_{k=1}^{n} A_{n-k}^{\alpha-1}=A_n^\alpha .$$

We also consider "inverse" (C,α)-means, which is example of T-means:

$$U_n^{\alpha,\psi} f:=\frac{1}{A_n^\alpha}\sum_{k=0}^{n-1} A_k^{\alpha-1} S_k^\psi f,, \quad (\psi=w \text{ or } \psi=\kappa).$$

Let β_n^α denote the T mean, where

$$\left\{q_0=1, \qquad q_k=k^{\alpha-1}: k\in\mathbf{N}_+\right\},$$

that is

$$V_n^{\alpha,,\psi} f:=\frac{1}{Q_n}\sum_{k=1}^{n} k^{\alpha-1} S_k^\psi f, \quad (\psi=w \text{ or } \psi=\kappa) \qquad 0<\alpha<1.$$

The n-th Riesz's logarithmic mean R_n^ψ and Nörlund logarithmic mean L_n^ψ are defined by

$$R_n^\psi f:=\frac{1}{l_n}\sum_{k=0}^{n-1}\frac{S_k^\psi f}{k}, \ L_n^\psi f:=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k^\psi f}{n-k}, \quad (\psi=w \text{ or } \psi=\kappa)$$

respectively, where $l_n:=\sum_{k=1}^{n-1} 1/k$.

Up to now we have considered T mean in the case when the sequence $\{q_k : k\in\mathbf{N}\}$ is bounded but now we consider T means with unbounded sequence $\{q_k : k\in\mathbf{N}\}$. Let $\alpha\in\mathbb{R}_+$, $\beta\in\mathbf{N}_+$ and

$$\log^{(\beta)} x:=\overbrace{\log\dots\log}^{\beta-\text{times}} x.$$

If we define the sequence $\{q_k : k\in\mathbf{N}\}$ by

$$\left\{q_0=0 \text{ and } q_k=\log^{(\beta)} k^\alpha : k\in\mathbf{N}_+\right\},$$

then we get the class of T means:

$$B_n^{\alpha,\beta,\psi} f:=\frac{1}{Q_n}\sum_{k=1}^{n}\log^{(\beta)} k^\alpha S_k^\psi f, \quad (\psi=w \text{ or } \psi=\kappa).$$

It is obvious that $\frac{n}{2}\log^{(\beta)}\frac{n^{\alpha}}{2^{\alpha}} \leqslant Q_n \leqslant n\log^{(\beta)}n^{\alpha}$. It follows that

$$\frac{q_{n-1}}{Q_n} \leqslant \frac{c\log^{(\beta)}(n-1)^{\alpha}}{n\log^{(\beta)}n^{\alpha}} = O\left(\frac{1}{n}\right) \to 0, \quad \text{as } n\to\infty. \tag{6}$$

Let us define maximal operator of T means by

$$T^{*,\psi}f := \sup_{n\in\mathbf{N}}\left|T_n^{\psi}f\right|, \qquad (\psi = w \text{ or } \psi = \kappa).$$

The well-known example of maximal operator of T means are Fejer and Riesz logarithmic means $\sigma^{*,\psi}f$ and $R^{*,\psi}f$.

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k $(k\in\mathbf{N})$. Denote by $f = \left(f^{(n)}, n\in\mathbf{N}\right)$ a martingale with respect to $(F_n, n\in\mathbf{N})$ (for details see, e. g. [48, 49]).

If $f\in L_1(G)$, then it is easy to show that the sequence $\left(S_{2^n}^{\psi}f : n\in\mathbf{N}\right)$ is a martingale.

The maximal function of a martingale f is defined by $f^* = \sup_{n\in\mathbf{N}}\left|f^{(n)}\right|$.

In case $f\in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n\in\mathbf{N}}\frac{1}{\mu(I_n(x))}\left|\int_{I_n(x)} f(u)\,d\mu(u)\right|, \quad x\in G.$$

For $0<p<\infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If f is a martingale, then the Walsh-Fourier and Walsh-Kaczmarz-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}^{\psi}(i) = \lim_{n\to\infty}\int_G f^{(n)}\psi_i\,d\mu, \quad (\psi = w, \text{ or } \psi = \kappa).$$

The Walsh-Fourier and Walsh-Kaczmarz-Fourier coefficients of $f\in L_1(G)$ are the same as the ones of the martingale $\left(S_{2^n}^{\psi}f : n\in\mathbf{N}\right)$ obtained from f.

A bounded measurable function a is p-atom, if there exists an interval I, such that

$$\int_I a\,d\mu = 0, \quad \|a\|_{\infty} \leqslant \mu(I)^{-1/p}, \quad \operatorname{supp}(a)\subset I.$$

Weisz proved that Hardy spaces H_p have atomic characterization. In particular the following is true:

PROPOSITION 1. [48] *A martingale* $f=\left(f^{(n)},n\in\mathbf{N}\right)$ *is in* $H_p\,(0<p\leqslant 1)$ *if and only if there exists sequence* $(a_k,k\in\mathbf{N})$ *of p-atoms and a sequence* $(\mu_k,k\in\mathbf{N})$, *of real numbers, such that, for every* $n\in\mathbf{N}$,

$$\sum_{k=0}^{\infty}\mu_k S_{2^n}^{\psi}a_k=f^{(n)},\qquad \sum_{k=0}^{\infty}|\mu_k|^p<\infty,\quad (\psi=w \text{ or } \psi=\kappa). \tag{7}$$

Moreover,

$$\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decomposition of f *of the form (7).*

PROPOSITION 2. [48, 49] *Suppose that an operator* T *is quasi-linear and for some* $0<p<1$

$$\|Tf\|_{L_{p,\infty}}\leqslant c_p\|f\|_{H_p}$$

and bounded on L_∞ *or on* L_q *with* $1<q<\infty$. *Then* T *is of weak type-*$(1,1)$*:*

$$\|Tf\|_{L_{1,\infty}}\leqslant c\|f\|_1 .$$

3. Main results and their some consequences

We state our main result concerning the maximal operator of the summation method (2), which we also show is in a sense sharp.

THEOREM 1. *a) The maximal operator* $T^{*,\kappa}$ *of the summability method (2) with non-increasing sequence* $\{q_k:k\geqslant 0\}$, *is bounded from the Hardy space* $H_{1/2}$ *to the space* $L_{1/2,\infty}$.

b) (Sharpness) Let $0<p<1/2$ *and* $\{q_k:k\geqslant 0\}$ *is non-decreasing sequence, satisfying the condition*

$$\frac{q_{n+1}}{Q_{n+2}}\geqslant\frac{c}{n},\quad (c\geqslant 1). \tag{8}$$

Then there exists a martingale $f\in H_p$, *such that*

$$\sup_{n\in\mathbf{N}}\|T_n^{\kappa}f\|_{L_{p,\infty}}=\infty.$$

A number of special cases of our results are of particular interest and give both well-known and new information. We just give the following examples of such T means with non-increasing sequence $\{q_k:k\geqslant 0\}$:

COROLLARY 1. *The maximal operators of* $U^{\alpha,\kappa}$, $V^{\alpha,\kappa}$ *and* R^{κ} *means are bounded from the Hardy space* $H_{1/2}$ *to the space* $L_{1/2,\infty}$ *but are not bounded from* H_p *to the space* $L_{p,\infty}$, *when* $0<p<1/2$.

COROLLARY 2. *Let $f \in L_1$ and T_n^{κ} be the T means with non-increasing sequence $\{q_k : k \geqslant 0\}$. Then $T_n^{\kappa} f \to f$, a.e., as $n \to \infty$.*

COROLLARY 3. *Let $f \in L_1$. Then*

$$\begin{aligned} R_n^{\kappa} f &\to f, \quad a.e., \quad as \ n \to \infty, \\ U_n^{\alpha,\kappa} f &\to f, \quad a.e., \quad as \ n \to \infty, \\ V_n^{\alpha,\kappa} f &\to f, \quad a.e., \quad as \ n \to \infty, \end{aligned}$$

Analogously to part a) of Theorem 1 we can also prove similar result for non-decreasing sequences which is given in part a) of next theorem. Moreover, we also show that statement in b) above hold also for non-decreasing sequences and now without any restriction like (8).

THEOREM 2. *a) The maximal operator $T^{*,\kappa}$ of the summability method (2) with non-decreasing sequence $\{q_k : k \geqslant 0\}$ satisfying the condition*

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad as \quad n \to \infty, \tag{9}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$.

b) (Sharpness) Let $0 < p < 1/2$. For any non-decreasing sequence $\{q_k : k \geqslant 0\}$, there exists a martingale $f \in H_p$, such that

$$\sup_{n \in \mathbf{N}} \|T_n^{\kappa} f\|_{L_{p,\infty}} = \infty.$$

A number of special cases of our results are of particular interest and give both well-known and new information. We just give the following examples of such T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$:

COROLLARY 4. *The maximal operator of $B^{\alpha,\beta,\kappa}$ means is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$ but is not bounded from H_p to the space $L_{p,\infty}$, when $0 < p < 1/2$.*

COROLLARY 5. *Let $f \in L_1$ and T_n^{κ} be the T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$ and satisfying condition (9). Then*

$$T_n^{\kappa} f \to f, \quad a.e., \quad as \ n \to \infty.$$

COROLLARY 6. *Let $f \in L_1$. Then $B_n^{\alpha,\beta,\kappa} f \to f$, a.e., as $n \to \infty$.*

4. Proofs

Proof of Theorem 1. Let the sequence $\{q_k : k \geqslant 0\}$ be non-increasing. By combining (4) with (5) and using Abel transformation we get that

$$\begin{aligned}
|T_n^{\kappa} f| &\leqslant \left| \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j S_j^{\kappa} f \right| \\
&\leqslant \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} \left| q_j - q_{j+1} \right| j \left| \sigma_j^{\kappa} f \right| + q_{n-1}(n-1) \left| \sigma_n^{\kappa} f \right| \right) \\
&\leqslant \frac{1}{Q_n} \left(\sum_{j=1}^{n-2} \left(q_j - q_{j+1} \right) j + q_{n-1}(n-1) \right) \sigma^{*,\kappa} f \leqslant \sigma^{*,\kappa} f
\end{aligned}$$

so that

$$T^{*,\kappa} f \leqslant \sigma^{*,\kappa} f. \tag{10}$$

If we apply (10) and Theorem W1 we can conclude that the maximal operators $T^{*,\kappa}$ of all T means with non-increasing sequence $\{q_k : k \geqslant 0\}$, are bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$.

Let $0 < p < 1/2$ and $\{\alpha_k : k \in \mathbf{N}\}$ be an increasing sequence of positive integers such that:

$$\sum_{k=0}^{\infty} 1/\alpha_k^p < \infty, \tag{11}$$

$$\sum_{\eta=0}^{k-1} \frac{2^{\alpha_\eta/p}}{\alpha_\eta} < \frac{2^{\alpha_k/p-1}}{2\alpha_k}, \tag{12}$$

$$\frac{2^{\alpha_{k-1}(1/p-1)}}{\alpha_{k-1}} < \frac{2^{\alpha_k(1/p-1)-4}}{\alpha_k}. \tag{13}$$

We note that such an increasing sequence $\{\alpha_k : k \in \mathbf{N}\}$ which satisfies conditions (11-13) can be constructed.

Let

$$f^{(A)} = \sum_{\{k;\, \lambda_k < A\}} \lambda_k a_k, \tag{14}$$

where

$$\lambda_k = \frac{1}{\alpha_k} \quad \text{and} \quad a_k = 2^{\alpha_k(1/p-1)} \left(D_{2^{\alpha_k+1}} - D_{2^{\alpha_k}} \right).$$

By using Proposition 1, it is easy to see that the martingale $f = \left(f^{(1)}, f^{(2)} \ldots f^{(A)} \ldots \right) \in H_{1/2}$. Moreover, it is easy to show that

$$\widehat{f}(j) = \begin{cases} \frac{2^{\alpha_k(1/p-1)}}{\alpha_k}, & \text{if } \; j \in \left\{ 2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1 \right\},\; k = 0,1,2\ldots, \\ 0, & \text{if } \; j \notin \bigcup_{k=1}^{\infty} \left\{ 2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1 \right\} \end{cases} \tag{15}$$

We can write

$$T^{\kappa}_{2^{\alpha_k}+2}f = \frac{1}{Q_{2^{\alpha_k}+2}} \sum_{j=0}^{2^{\alpha_k}} q_j S^{\kappa}_j f + \frac{q_{2^{\alpha_k}+1}}{Q_{2^{\alpha_k}+2}} S^{\kappa}_{2^{\alpha_k}+1} f := I + II. \tag{16}$$

Let $2^{\alpha_s} \leqslant j \leqslant 2^{\alpha_s+1}$, where $s = 0, \dots, k-1$. Moreover,

$$\left| D^{\kappa}_j - D_{2^{\alpha_s}} \right| \leqslant j - 2^{\alpha_s} \leqslant 2^{\alpha_s}, \quad (s \in \mathbf{N})$$

so that, according to (1) and (15), we have that

$$\begin{aligned}
\left|S^{\kappa}_j f\right| &= \left| \sum_{v=0}^{2^{\alpha_{s-1}+1}-1} \widehat{f}(v)\kappa_v + \sum_{v=2^{\alpha_s}}^{j-1} \widehat{f}(v)\kappa_v \right| \\
&\leqslant \left| \sum_{\eta=0}^{s-1} \sum_{v=2^{\alpha_\eta}}^{2^{\alpha_\eta+1}-1} \frac{2^{\alpha_\eta(1/p-1)}}{\alpha_\eta}\kappa_v \right| + \frac{2^{\alpha_s(1/p-1)}}{\alpha_s} \left| \left(D^{\kappa}_j - D_{2^{\alpha_s}} \right) \right| \\
&= \left| \sum_{\eta=0}^{s-1} \frac{2^{\alpha_\eta(1/p-1)}}{\alpha_\eta} \left(D_{2^{\alpha_\eta+1}} - D_{2^{\alpha_\eta}} \right) \right| + \frac{2^{\alpha_s(1/p-1)}}{\alpha_s} \left| \left(D^{\kappa}_j - D_{2^{\alpha_s}} \right) \right| \\
&\leqslant \sum_{\eta=0}^{s-1} \frac{2^{\alpha_\eta/p}}{\alpha_\eta} + \frac{2^{\alpha_s/p}}{\alpha_s} \leqslant \frac{2^{\alpha_{s-1}/p+1}}{\alpha_{s-1}} + \frac{2^{\alpha_s/p}}{\alpha_s} \leqslant \frac{2^{\alpha_{k-1}/p+1}}{\alpha_{k-1}}.
\end{aligned} \tag{17}$$

Let $2^{\alpha_{s-1}+1}+1 \leqslant j \leqslant 2^{\alpha_s}$, $s = 1, \dots, k$. Analogously to (17) we can prove that

$$\begin{aligned}
\left|S^{\kappa}_j f\right| &= \left| \sum_{v=0}^{2^{\alpha_{s-1}+1}-1} \widehat{f}(v)\psi_v \right| = \left| \sum_{\eta=0}^{s-1} \sum_{v=2^{\alpha_\eta}}^{2^{\alpha_\eta+1}-1} \frac{2^{\alpha_\eta(1/p-1)}}{\alpha_\eta}\kappa_v \right| \\
&= \left| \sum_{\eta=0}^{s-1} \frac{M^{1/p-1}_{\alpha_\eta}}{\alpha_\eta} \left(D_{2^{\alpha_\eta+1}} - D_{2^{\alpha_\eta}} \right) \right| \leqslant \frac{2^{\alpha_{s-1}/p+1}}{\alpha_{s-1}} \leqslant \frac{2^{\alpha_{k-1}/p+1}}{\alpha_{k-1}}.
\end{aligned}$$

Hence,

$$|I| \leqslant \frac{1}{Q_{2^{\alpha_k}+2}} \sum_{j=0}^{2^{\alpha_k}} q_j \left|S^{\kappa}_j f\right| \leqslant \frac{2^{\alpha_{k-1}/p+1}}{\alpha_{k-1}} \frac{1}{Q_{M_{\alpha_k}+1}} \sum_{j=0}^{2^{\alpha_k}} q_j \leqslant \frac{2^{\alpha_{k-1}/p+1}}{\alpha_{k-1}}. \tag{18}$$

If we now apply (15) and (17) we get that

$$\begin{aligned}
|II| &= \frac{q_{2^{\alpha_k}+1}}{Q_{2^{\alpha_k}+2}} \left| \frac{2^{\alpha_k(1/p-1)}}{\alpha_k}\kappa_{2^{\alpha_k}} + S_{2^{\alpha_k}} f \right| \\
&= \frac{q_{2^{\alpha_k}+1}}{Q_{2^{n_k}+2}} \left| \frac{2^{\alpha_k(1/p-1)}}{\alpha_k}\kappa_{2^{\alpha_k}} + S_{2^{\alpha_{k-1}+1}} f \right| \\
&\geqslant \frac{q_{2^{\alpha_k}+1}}{Q_{2^{n_k}+2}} \left(\left| \frac{2^{\alpha_k(1/p-1)}}{\alpha_k}\kappa_{2^{\alpha_k}} \right| - \left| S_{2^{\alpha_{k-1}+1}} f \right| \right) \\
&\geqslant \frac{q_{2^{\alpha_k}+1}}{Q_{2^{\alpha_k}+2}} \left(\frac{2^{\alpha_k(1/p-1)}}{\alpha_k} - \frac{2^{\alpha_{k-1}/p+2}}{\alpha_{k-1}} \right) \geqslant \frac{q_{2^{\alpha_k}+1}}{Q_{2^{\alpha_k}+2}} \frac{2^{\alpha_k(1/p-1)-2}}{\alpha_k}.
\end{aligned} \tag{19}$$

Without lost the generality we may assume that $c=1$ in (8). By combining (18) and (19) we get

$$\left|T^{\kappa}_{2^{\alpha_k}+2}f\right| \geqslant |II|-|I| \tag{20}$$
$$\geqslant \frac{q_{2^{\alpha_k}+1}}{Q_{2^{\alpha_k}+2}}\frac{2^{\alpha_k(1/p-1)-2}}{\alpha_k}-\frac{2^{\alpha_{k-1}/p+1}}{\alpha_{k-1}}$$
$$\geqslant \frac{2^{\alpha_k(1/p-2)}}{4\alpha_k}-\frac{2^{\alpha_{k-1}/p+1}}{\alpha_{k-1}} \geqslant \frac{2^{\alpha_k(1/p-2)}}{16\alpha_k}.$$

On the other hand,

$$\mu\left\{x\in G:\left|T^{\kappa}_{2^{\alpha_k}+2}f(x)\right|\geqslant \frac{2^{\alpha_k(1/p-2)}}{16\alpha_k}\right\}=\mu(G)=1. \tag{21}$$

Let $0<p<1/2$. Then

$$\frac{2^{\alpha_k(1/p-2)}}{16\alpha_k}\cdot\left(\mu\left\{x\in G:\left|T^{\kappa}_{M_{\alpha_k}+2}f(x)\right|\geqslant \frac{2^{\alpha_k(1/p-2)}}{16\alpha_k}\right\}\right)^{1/p} \tag{22}$$
$$=\frac{2^{\alpha_k(1/p-2)}}{16\alpha_k}\to\infty,\ \text{ as } k\to\infty.$$

The proof is complete.

Proof of Corollary 1. Since $R^{\kappa}_n, U^{\alpha,\kappa}_n$ and $V^{\alpha,\kappa}_n$ are T means with non-increasing sequence $\{q_k : k\geqslant 0\}$, then the proof is direct consequence of Theorem 1.

Proof of Corollary 2. According to Theorem 1 a) and Proposition 2 we also have weak $(1,1)$ type inequality and by well-known density argument due to Marcinkiewicz and Zygmund [15] we have $T^{\kappa}_n f\to f$, a.e., for all $f\in L_1$. Which follows proof of Corollary 2.

Proof of Corollary 3. Since $R^{\kappa}_n, U^{\alpha,\kappa}_n$ and $V^{\alpha,\kappa}_n$ are T means with non-increasing sequence $\{q_k : k\geqslant 0\}$, then the proof is direct consequence of Corollary 2.

Proof of Theorem 2. Let the sequence $\{q_k : k\geqslant 0\}$ be non-decreasing. By combining (4) with (5) and using Abel transformation we get that

$$|T_n f| \leqslant \left|\frac{1}{Q_n}\sum_{j=1}^{n-1}q_jS_jf\right|$$
$$\leqslant \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}\left|q_j-q_{j+1}\right|j\left|\sigma_jf\right|+q_{n-1}(n-1)\left|\sigma_nf\right|\right)$$
$$\leqslant \frac{1}{Q_n}\left(\sum_{j=1}^{n-2}-\left(q_j-q_{j+1}\right)j-q_{n-1}(n-1)+2q_{n-1}(n-1)\right)\sigma^*f$$
$$\leqslant \frac{1}{Q_n}\left(2q_{n-1}(n-1)-Q_n\right)\sigma^*f\leqslant c\sigma^*f$$

so that

$$T^*f \leqslant c\sigma^*f. \tag{23}$$

If we apply (23) and Theorem W1 we can conclude that the maximal operators T^* is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$.

To prove part b) of theorem 2 we use the martingale, defined by (14) where α_k satisfy conditions (11-13). It is easy to show that for every non-increasing sequence $\{q_k : k \geqslant 0\}$ it automatically holds that $q_{2^{\alpha_k}+1}/Q_{2^{\alpha_k}+2} \geqslant c/2^{\alpha_k}$. According to (16-20) we can conclude that

$$\left|T^{\kappa}_{2^{\alpha_k}+2}f\right| \geqslant |II| - |I| \geqslant \frac{2^{\alpha_k(1/p-2)}}{16\alpha_k}.$$

Analogously to (21) and (22) we then get that $\sup_{k\in\mathbf{N}}\left\|T^{\kappa}_{2^{\alpha_k}+2}f\right\|_{L_{p,\infty}} = \infty$ and the proof is complete.

Proof of Corollary 4. Since $B^{\alpha,\beta,*,\kappa}$ are the T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$, then the proof is direct consequence of Theorem 2.

Proof of Corollary 5. According to Proposition 2 we can conclude that $T^{*,\kappa}$ has weak type-(1,1) and by well-known density argument due to Marcinkiewicz and Zygmund [15] we also have $T_n^{\kappa}f \to f$, a.e.. Which follows proof of Corollary 5.

Proof of Corollary 6. Since $B^{\alpha,\beta,*,\kappa}$ are the T means with non-decreasing sequence $\{q_k : k \geqslant 0\}$, then the proof is direct consequence of Corollary 5.

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