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THE UPPER BOUNDARY FOR THE RATIO BETWEEN n–VARIABLE OPERATOR POWER MEANS

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Abstract. In this paper, we show estimates of the upper boundary for the ratio between n-variable operator power means $P_t(\omega;\mathbb{A})$ due to Lawson-Lim-Pálfia by terms of a generalized condition number in the sense of Turing, which are partial improvements of the known results: Let $\mathbb{A}=(A_1,\dots,A_n)$ be a n-tuple of positive invertible operators such that $mI \leqslant A_j \leqslant MI$ for $j=1,\dots,n$ and $h=M/m$, and ω a weight vector. Then

$$P_t(\omega;\mathbb{A}) \leqslant \left(\frac{h^t+h^{-t}}{2}\right)^{1/t} G_K(\omega;\mathbb{A})$$

for all $t\in(0,1]$, where $G_K(\omega;\mathbb{A})$ is the Karcher mean.

1. Introduction

Let $B(\mathscr{H})$ be the space of all bounded linear operators on a Hilbert space $\mathscr{H}$, and I stands for the identity operator on $\mathscr{H}$. An operator A in $B(\mathscr{H})$ is said to be positive (in symbol: $A \geqslant 0$) if $\langle Ax,x\rangle \geqslant 0$ for all $x\in\mathscr{H}$. In particular, $A>0$ means that A is positive and invertible. Let $\mathbb{P}(\mathscr{H})$ be the open convex cone of all positive invertible operators. For selfadjoint operators A and B, the order relation $A \geqslant B$ means that $A-B$ is positive. The condition number $h=h(A)$ of an invertible operator A is defined by $h(A)=\|A\|\,\|A^{-1}\|$ in [7]. If a positive invertible operator A satisfies the condition $mI \leqslant A \leqslant MI$ for some scalars $0<m<M$, then it may be thought as $M=\|A\|$ and $m=\|A^{-1}\|^{-1}$, so that $h=h(A)=M/m$ and we call it a generalized condition number of A.

In this paper, we study estimates of the upper boundary for the ratio between n-variable operator power means by terms of a generalized condition number. For this, we recall the notion of the Karcher mean and operator power means due to Lawson-Lim-Pálfia [3, 4], which are n-variable extensions of the operator geometric mean and 2-variable operator power means, respectively: Let $\mathbb{A}=(A_1,\dots,A_n)$ be a n-tuple of positive invertible operators on a Hilbert space and $\omega=(\omega_1,\dots,\omega_n)$ a weight vector

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such that $\omega_j \geqslant 0$ for all $j = 1,\dots,n$ and $\sum_{j=1}^n \omega_j = 1$. The Karcher mean of $A_1,\dots,A_n$ is the unique positive invertible solution of the Karcher equation

$$\sum_{j=1}^{n} \omega_j \log(X^{-1/2}A_jX^{-1/2}) = 0$$

and we denote it by $G_K(\omega;\mathbb{A})$. For each $t \in (0,1]$, the operator power mean of $A_1,\dots,A_n$ is the unique positive invertible solution of a non-linear operator equation

$$X = \sum_{j=1}^{n} \omega_j (X \,\sharp_t\, A_j)$$

and we denote it by $P_t(\omega;\mathbb{A})$, where the operator geometric mean of positive invertible operators A and B is defined by

$$A \,\sharp_t\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2} \qquad \text{for } t \in [0,1].$$

For $t \in [-1,0)$, we define $P_t(\omega;\mathbb{A}) = P_{-t}(\omega;\mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1},\dots,A_n^{-1})$. Note that the Karcher mean coincides with the strong operator limit of operator power means as $t \to 0$, and in the case of $t = 1$ and $t = -1$, we have

$$P_1(\omega;\mathbb{A}) = \sum_{j=1}^{n} \omega_j A_j \quad \text{and} \quad P_{-1}(\omega;\mathbb{A}) = \left(\sum_{j=1}^{n} \omega_j A_j^{-1}\right)^{-1}$$

and the operator power means $P_t(\omega;\mathbb{A})$ have a monotone increasing property for $-1 < t < 1$;

$$-1 < t \leqslant s < 1 \qquad \Longrightarrow \qquad P_t(\omega;\mathbb{A}) \leqslant P_s(\omega;\mathbb{A}).$$

In particular, we have the following inequalities:

$$\left(\sum_{j=1}^{n} \omega_j A_j^{-1}\right)^{-1} \leqslant P_s(\omega;\mathbb{A}) \leqslant G_K(\omega;\mathbb{A}) \leqslant P_t(\omega;\mathbb{A}) \leqslant \sum_{j=1}^{n} \omega_j A_j$$

for all $-1 \leqslant s < 0 < t \leqslant 1$. Thus, the operator power means $P_t(\omega;\mathbb{A})$ of order $t \in [-1,1]\backslash\{0\}$ is a path from the arithmetic mean $\sum_{j=1}^n \omega_j A_j$ to the harmonic mean $\left(\sum_{j=1}^n \omega_j A_j^{-1}\right)^{-1}$ via the Karcher mean $G_K(\omega;\mathbb{A})$.

If A_j mutually commute for $j = 1,\dots,n$, then it follows that

$$P_t(\omega;\mathbb{A}) = \left(\sum_{j=1}^{n} \omega_j A_j^t\right)^{1/t},$$

and in the case of $n = 2$, $P_t((1-\alpha,\alpha);A,B)$ coincides with 2-variable operator power means $A\, m_{t,\alpha}\, B$ defined by

$$A\, m_{t,\alpha}\, B = A^{1/2}\left((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^t\right)^{1/t} A^{1/2}$$

for all $t \in [-1,1]$ and $\alpha \in [0,1]$. For each $\alpha \in [0,1]$, $A \ \mathrm{m}_{t,\alpha} \ B$ ($t \in [-1,1]$) is a path from the arithmetic mean $A \nabla_\alpha B$ to the harmonic mean $A \,!_\alpha\, B$ via the operator geometric mean $A \sharp_\alpha B$. Moreover, the upper boundary for the ratio between $m_{t,\alpha}$ is known in [2, Chapter 5]. Thus, it is natural to consider a reverse relation between the n-variable operator power means. However, the reverse relationship has not been well studied in the case of n-variable operator power means. In present, we know only the following result: For positive invertible operators $A_1,\dots,A_n$ such as $mI \leqslant A_j \leqslant MI$ for all $j=1,\dots,n$ and $h=M/m$,

$$\sum_{j=1}^{n} \omega_j A_j \leqslant \frac{(h+1)^2}{4h} \left(\sum_{j=1}^{n} \omega_j A_j^{-1} \right)^{-1}. \tag{1.1}$$

The constant $\frac{(h+1)^2}{4h}$ is called the Kantorovich constant. Then it follows from (1.1) that

$$P_s(\omega;\mathbb{A}) \leqslant \frac{(h+1)^2}{4h} P_t(\omega;\mathbb{A}) \tag{1.2}$$

for all $-1 < t < s < 1$. In particular, if we put $s=1$ and $t \to 0$ in (1.2), then we have the ratio type reverse inequality of the n-variable arithmetic-geometric mean one:

$$\sum_{j=1}^{n} \omega_j A_j \leqslant \frac{(h+1)^2}{4h} G_K(\omega;\mathbb{A}), \tag{1.3}$$

also see [1]. Though that's a rough estimate, we do not know better estimates than (1.3).

In this paper, we show estimates of the upper boundary for the ratio between n-variable operator power means by terms of a generalized condition number in the sense of Turing, which are partial improvements of the result (1.2).

2. Results

We are in a position to show the main theorem:

THEOREM 1. *Let $\mathbb{A}=(A_1,\dots,A_n)$ be a n-tuple of positive invertible operators such that $mI \leqslant A_j \leqslant MI$ for all $j=1,\dots,n$ and some scalars $0<m<M$, and $\omega=(\omega_1,\dots,\omega_n)$ a weight vector. Put a generalized condition number $h=M/m(>1)$. Then for $0<t<s<1$*

$$P_s(\omega;\mathbb{A}) \leqslant \left(\frac{h^s - h^{-s} + h^{t-s} - h^{s-t}}{h^t - h^{-t}} \right)^{1/s} P_t(\omega;\mathbb{A}). \tag{2.1}$$

In particular, as $t \to 0$,

$$P_s(\omega;\mathbb{A}) \leqslant \left(\frac{h^s + h^{-s}}{2} \right)^{1/s} G_K(\omega;\mathbb{A}) \tag{2.2}$$

for all $0 < s < 1$. *Moreover, there exists* $s_0 \in (0,1)$ *such that*

$$P_s(\omega;\mathbb{A}) \leqslant \left(\frac{h^s+h^{-s}}{2}\right)^{1/s} G_K(\omega;\mathbb{A}) \leqslant \frac{(h+1)^2}{4h} G_K(\omega;\mathbb{A}) \tag{2.3}$$

for all $0 < s < s_0$.

Proof. For $s \in (0,1]$, let $f : \mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{H})$ defined by $f(X) = \sum_{j=1}^n \omega_j (X \sharp_s A_j)$. Then by the Löwner-Heinz inequality, f is monotone: $X \leqslant Y$ implies $f(X) \leqslant f(Y)$. Since f is a strict contraction for the Thompson metric, it follows from the Banach fixed point theorem that $P_s(\omega;\mathbb{A}) = \lim_{k\to\infty} f^k(X)$ for any $X \in \mathbb{P}(\mathscr{H})$. Since $s/t > 1$ and $y = x^{s/t}$ is convex, it follows that the inequality $x^{s/t} \leqslant \frac{h^s-h^{-s}}{h^t-h^{-t}}(x-h^t)+h^s$ holds on $[h^{-t},h^t]$. Hence $h^{-t} \leqslant (X^{-1/2}A_jX^{-1/2})^t \leqslant h^t$ for $0 < t < 1$ implies

$$\begin{aligned} X \sharp_s A_j &= X^{1/2}\left[(X^{-1/2}A_jX^{-1/2})^t\right]^{s/t} X^{1/2} \\ &\leqslant X^{1/2}\left[\frac{h^s-h^{-s}}{h^t-h^{-t}}((X^{-1/2}A_jX^{-1/2})^t - h^t)+h^s\right] X^{1/2} \\ &= \frac{h^{t-s}-h^{s-t}}{h^t-h^{-t}}X + \frac{h^s-h^{-s}}{h^t-h^{-t}} X \sharp_t A_j. \end{aligned}$$

Therefore we have

$$\begin{aligned} f(X) &= \sum_{j=1}^n \omega_j (X \sharp_s A_j) \\ &\leqslant \sum_{j=1}^n \omega_j \left[\frac{h^{t-s}-h^{s-t}}{h^t-h^{-t}}X + \frac{h^s-h^{-s}}{h^t-h^{-t}} X \sharp_t A_j\right] \\ &= \frac{h^{t-s}-h^{s-t}}{h^t-h^{-t}}X + \frac{h^s-h^{-s}}{h^t-h^{-t}} \sum_{j=1}^n \omega_j (X \sharp_t A_j). \end{aligned}$$

If we put $X_0 = P_t(\omega;\mathbb{A})$, then we have

$$\begin{aligned} f(X_0) &\leqslant \frac{h^{t-s}-h^{s-t}}{h^t-h^{-t}}X_0 + \frac{h^s-h^{-s}}{h^t-h^{-t}} \sum_{j=1}^n \omega_j (X_0 \sharp_t A_j) \\ &= \frac{h^{t-s}-h^{s-t}}{h^t-h^{-t}}X_0 + \frac{h^s-h^{-s}}{h^t-h^{-t}} X_0 \\ &= \frac{h^{t-s}-h^{s-t}+h^s-h^{-s}}{h^t-h^{-t}} X_0. \end{aligned}$$

If we put $h_0 = \frac{h^{t-s}-h^{s-t}+h^s-h^{-s}}{h^t-h^{-t}}$, then we have

$$f^2(X_0) \leqslant f(h_0X_0) = h_0^{1-s} f(X_0) \leqslant h_0^{(1-s)+1} X_0$$

and inductively we have

$$f^k(X_0) \leqslant h_0^{\frac{1-(1-s)^k}{1-(1-s)}} X_0.$$

As $k \to \infty$, we have the desired inequality (2.1):

$$P_s(\omega;\mathbb{A}) \leqslant h_0^{1/s} P_t(\omega;\mathbb{A}) \qquad \text{for } 0 < t < s < 1.$$

Since $\lim_{t\to 0} h_0^{1/s} = \left(\frac{h^s+h^{-s}}{2}\right)^{1/s}$, we have the desired inequality (2.2).

Since $g(s) = \left(\frac{h^s+h^{-s}}{2}\right)^{1/s}$ is increasing on $[0,1]$ and $g(0) = 1$, and $g(1) = \frac{h+h^{-1}}{2} \geqslant \frac{(h+1)^2}{4h}$, it follows that there exists $s_0 \in (0,1)$ such that

$$\left(\frac{h^s+h^{-s}}{2}\right)^{1/s} \leqslant \frac{(h+1)^2}{4h}$$

for all $0 < s < s_0$, and we have the desired inequality (2.3). □

REMARK 1. If we put $t = s$ in Theorem 1, then $\left(\dfrac{h^s - h^{-s} + h^{t-s} - h^{s-t}}{h^t - h^{-t}}\right)^{1/s} = 1$.

In the case of $n = 2$, Tominaga [6] showed the following Specht type inequality, which is regarded as a ratio type reverse inequality of the arithmetic-geometric mean inequality:

$$(A \sharp_\alpha B \leqslant) \quad (1-\alpha)A + \alpha B \leqslant S(h) A \sharp_\alpha B \qquad \text{for } \alpha \in [0,1],$$

where the Specht ratio $S(h)$ in [5] is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h \neq 1) \quad \text{and} \quad S(1) = 1.$$

We would expect that the n-variable Specht type inequality

$$\sum_{j=1}^{n} \omega_j A_j \leqslant S(h) G_K(\omega;\mathbb{A}) \tag{2.4}$$

holds. However, we do not know whether the inequality (2.4) holds or not. We know only the inequality (1.3) though $S(h) \leqslant \frac{(h+1)^2}{4h}$ for $h \geqslant 1$.

If we put $s = 1$ and $t \to 0$ in (2.1) of Theorem 1, then we have an n-variable Specht type inequality

$$\sum_{j=1}^{n} \omega_j A_j \leqslant \frac{h+h^{-1}}{2} G_K(\omega;\mathbb{A}). \tag{2.5}$$

Unfortunately, since $\frac{h+h^{-1}}{2} > \frac{(h+1)^2}{4h}$, the inequality (2.5) is not better than (1.3). By Theorem 1, we obtain a partial improvement (2.3) of the inequality (1.2) as in the proof of (2.3).

Since $P_t(\omega;\mathbb{A}) = P_{-t}(\omega;\mathbb{A}^{-1})^{-1}$ for $-1 < t < 0$, we have the negative order version of Theorem 1:

THEOREM 2. *Let $\mathbb{A} = (A_1,\ldots,A_n)$ be a n-tuple of positive invertible operators such that $mI \leqslant A_j \leqslant MI$ for all $j = 1,\ldots,n$ and some scalars $0 < m < M$, and $\omega = (\omega_1,\ldots,\omega_n)$ a weight vector. Put $h = M/m$. Then for $-1 < t < s < 0$*

$$P_s(\omega;\mathbb{A}) \leqslant \left(\frac{h^t - h^{-t} + h^{s-t} - h^{t-s}}{h^s - h^{-s}}\right)^{-1/t} P_t(\omega;\mathbb{A}). \tag{2.6}$$

In particular, as $s \to 0$,

$$G_K(\omega;\mathbb{A}) \leqslant \left(\frac{h^t + h^{-t}}{2}\right)^{-1/t} P_t(\omega;\mathbb{A}) \tag{2.7}$$

for all $-1 < t < 0$. Moreover, there exists $t_0 \in (-1,0)$ such that

$$G_K(\omega;\mathbb{A}) \leqslant \left(\frac{h^t + h^{-t}}{2}\right)^{-1/t} P_t(\omega;\mathbb{A}) \leqslant \frac{(h+1)^2}{4h} P_t(\omega;\mathbb{A}) \tag{2.8}$$

for all $-1 < t_0 < t < 0$.

Proof. For $-1 < t < s < 0$, we put $t' = -s$ and $s' = -t$. Since $0 < t' < s' < 1$ and $M^{-1}I \leqslant A_j^{-1} \leqslant m^{-1}I$ for $j = 1,\ldots,n$, it follows from a generalized condition number $\frac{m^{-1}}{M^{-1}} = \frac{M}{m} = h$ and Theorem 1 that

$$P_{s'}(\omega;\mathbb{A}^{-1}) \leqslant \left(\frac{h^{s'} - h^{-s'} + h^{t'-s'} - h^{s'-t'}}{h^{t'} - h^{-t'}}\right)^{1/s'} P_{t'}(\omega;\mathbb{A}^{-1}).$$

By taking the inverse of the both sides, we have

$$P_{-t}(\omega;\mathbb{A}^{-1})^{-1} \geqslant \left(\frac{h^{-t} - h^{t} + h^{-s+t} - h^{-t+s}}{h^{-s} - h^{s}}\right)^{1/t} P_{-s}(\omega;\mathbb{A}^{-1})^{-1}$$

and hence we have the desired inequality (2.6). Similarly, as $s \to 0$ we have the desired inequality (2.7). Since $g(t) = \left(\frac{h^t + h^{-t}}{2}\right)^{-1/t}$ is decreasing on $[-1,0]$ and $g(0) = 1$, and $g(-1) = \frac{h+h^{-1}}{2} \geqslant \frac{(h+1)^2}{4h}$, there exists $t_0 \in (-1,0)$ such that

$$G_K(\omega;\mathbb{A}) \leqslant \left(\frac{h^t + h^{-t}}{2}\right)^{-1/t} P_t(\omega;\mathbb{A}) \leqslant \frac{(h+1)^2}{4h} P_t(\omega;\mathbb{A})$$

for all $-1 < t_0 < t < 0$. Hence we have the desired inequality (2.8). $\square$

COROLLARY 1. *Let* $\mathbb{A} = (A_1, \ldots, A_n)$ *be a* n*-tuple of positive invertible operators such that* $mI \leqslant A_j \leqslant MI$ *for all* $j = 1, \ldots, n$, *and* $\omega = (\omega_1, \ldots, \omega_n)$ *a weight vector. Put* $h = M/m$. *Then there exists* $t_0 \in (0,1)$ *such that*

$$\left(\frac{h^t + h^{-t}}{2}\right)^{-1/t} P_t(\omega;\mathbb{A}) \leqslant G_K(\omega;\mathbb{A}) \leqslant \left(\frac{h^s + h^{-s}}{2}\right)^{1/s} P_s(\omega;\mathbb{A}) \qquad (2.9)$$

for all $-t_0 < s < 0 < t < t_0$, *and the left-hand side of* (2.9) *converges to the middle term as* $t \downarrow 0$. *Similarly, the right-hand side of* (2.9) *converges to the middle term as* $s \uparrow 0$.

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