Mathematical
Inequalities
& Applications
Volume 24, Number 3 (2021), 775–791 doi:10.7153/mia-2021-24-54

A TRUDINGER–MOSER INEQUALITY WITH MEAN VALUE ZERO ON A COMPACT RIEMANN SURFACE WITH BOUNDARY

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(*Communicated by M. Praljak*)

Abstract. In this paper, on a compact Riemann surface (Σ, g) with smooth boundary $\partial\Sigma$, we concern a Trudinger-Moser inequality with mean value zero. To be exact, let $\lambda_1(\Sigma)$ denotes the first eigenvalue of the Laplace-Beltrami operator with respect to the zero mean value condition and $\mathscr{S} = \left\{u \in W^{1,2}(\Sigma, g) : \|\nabla_g u\|_2^2 \leqslant 1 \text{ and } \int_\Sigma u \mathrm{d}v_g = 0\right\}$, where $W^{1,2}(\Sigma, g)$ is the usual Sobolev space, $\|\cdot\|_2$ denotes the standard L^2-norm and ∇_g represent the gradient. By the method of blow-up analysis, we obtain

$$\sup_{u\in\mathscr{S}} \int_\Sigma e^{2\pi u^2\left(1+\alpha\|u\|_2^2\right)} \mathrm{d}v_g < +\infty, \ \forall\, 0 \leqslant \alpha < \lambda_1(\Sigma);$$

when $\alpha \geqslant \lambda_1(\Sigma)$, the supremum is infinite. Moreover, we prove the supremum is attained by a function $u_\alpha \in C^\infty\left(\overline{\Sigma}\right) \cap \mathscr{S}$ for sufficiently small $\alpha > 0$. Based on the similar work in the Euclidean space, which was accomplished by Lu-Yang [19], we strengthen the result of Yang [29].

1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded domain and $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the Sobolev norm $\|\nabla_{\mathbb{R}^2} u\|_2^2 = \int_\Omega |\nabla_{\mathbb{R}^2} u|^2 \mathrm{d}x$, where $\nabla_{\mathbb{R}^2}$ is the gradient operator on $\mathbb{R}^2$ and $\|\cdot\|_2$ denotes the standard L^2-norm. The classical Trudinger-Moser inequality [37, 24, 23, 27, 20], as the limit case of the Sobolev embedding, says

$$\sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla_{\mathbb{R}^2} u\|_2 \leqslant 1} \int_\Omega e^{\beta u^2} \mathrm{d}x < +\infty, \ \forall\, \beta \leqslant 4\pi. \tag{1}$$

Moreover, 4π is called the best constant for this inequality in the sense that when $\beta > 4\pi$, all integrals in (1) are still finite, but the supremum is infinite. It is interesting to know whether or not the supremum in (1) can be attained. For this topic, we refer the reader to Carleson-Chang [4], Flucher [12], Lin [18], Struwe [25], Adimurthi-Struwe [2], Li [15], Yang [28], Zhu [38], Tintarev [26] and the references therein.

Mathematics subject classification (2020): 46E35, 58J05, 58J32.

Keywords and phrases: Trudinger-Moser inequality, Riemann surface, blow-up analysis, extremal function.

Supported by the Fundamental Research Funds for the Central Universities, and the Research Funds of Renmin University of China (21XNH150).

There are many extensions of (1). Adimurthi-Druet [1] generalized (1) to the following form

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla_{\mathbb{R}^2}u\|_2\leqslant 1}\int_\Omega e^{4\pi u^2\left(1+\alpha\|u\|_2^2\right)}\mathrm{d}x<+\infty,\ \forall\, 0\leqslant\alpha<\lambda_1(\Omega), \tag{2}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian with Dirichlet boundary condition in Ω. This inequality is sharp in the sense that if $\alpha\geqslant\lambda_1(\Omega)$, all integrals in (2) are still finite, but the supremum is infinite. Obviously, (2) is reduced to (1) when $\alpha=0$. Various extensions of the inequality (2) were obtained by Yang [28, 33], Tintarev [26] and Zhu [38] respectively. It was extended by Lu-Yang [19] to a version, namely

$$\sup_{u\in W^{1,2}(\Omega),\int_\Omega u\mathrm{d}x=0,\|\nabla_{\mathbb{R}^2}u\|_2\leqslant 1}\int_\Omega e^{2\pi u^2\left(1+\alpha\|u\|_2^2\right)}\mathrm{d}x<+\infty,\ \forall\, 0\leqslant\alpha<\overline{\lambda}_1(\Omega), \tag{3}$$

where $\overline{\lambda}_1(\Omega)$ denotes the first nonzero Neumann eigenvalue of the Laplacian operator. This inequality is sharp in the sense that all integrals in (3) are still finite when $\alpha\geqslant\overline{\lambda}_1(\Omega)$, but the supremum is infinite. Moreover, for sufficiently small $\alpha>0$, the supremum is attained.

Trudinger-Moser inequalities were introduced on Riemannian manifolds by Aubin [3], Cherrier [6] and Fontana [13]. In particular, let (Σ,g) be a 2-dimensional compact Riemann surface, $W^{1,2}(\Sigma,g)$ the completion of $C^\infty(\Sigma)$ under the norm $\|u\|^2_{W^{1,2}(\Sigma,g)}=\int_\Sigma(u^2+|\nabla_g u|^2)\mathrm{d}v_g$, where ∇_g stands for the gradient operator on (Σ,g). When (Σ,g) is closed Riemann surface, there holds

$$\sup_{u\in W^{1,2}(\Sigma,g),\int_\Sigma u\mathrm{d}v_g=0,\|\nabla_g u\|_2\leqslant 1}\int_\Sigma e^{\beta u^2}\mathrm{d}v_g<+\infty,\ \forall\,\beta\leqslant 4\pi. \tag{4}$$

Moreover, 4π is called the best constant for this inequality in the sense that when $\beta>4\pi$, all integrals in (4) are still finite, but the supremum is infinite. Based on the works of Ding-Jost-Li-Wang [9] and Adimurthi-Struwe [2], Li [14, 15] proved the existence of extremals for the supremum in (4). When (Σ,g) is a compact Riemann surface with smooth boundary $\partial\Sigma$, Yang [29] obtained the same inequality as (4), namely

$$\sup_{u\in W^{1,2}(\Sigma,g),\int_\Sigma u\mathrm{d}v_g=0,\|\nabla_g u\|_2\leqslant 1}\int_\Sigma e^{\beta u^2}\mathrm{d}v_g<+\infty,\ \forall\,\beta\leqslant 2\pi. \tag{5}$$

This inequality is sharp in the sense that if $\beta>2\pi$, all integrals in (5) are still finite, but the supremum is infinite. Furthermore, the supremum in (5) can be attained.

In view of the inequality (3) in the Euclidean space, we strengthen (5) on (Σ,g) with smooth boundary $\partial\Sigma$. Precisely we have the following:

THEOREM 1. *Let (Σ,g) be a compact Riemann surface with smooth boundary $\partial\Sigma$ and*

$$\lambda_1(\Sigma)=\inf_{u\in W^{1,2}(\Sigma,g),\int_\Sigma u\mathrm{d}v_g=0,u\not\equiv 0}\frac{\|\nabla_g u\|_2^2}{\|u\|_2^2} \tag{6}$$

be the first eigenvalue of the Laplace-Beltrami operator Δ_g with respect to the zero mean value condition. Denote a function space

$$\mathscr{S} = \left\{ u \in W^{1,2}(\Sigma, g) : \int_\Sigma u \mathrm{d}v_g = 0,\ \|\nabla_g u\|_2 \leqslant 1 \right\}$$

and

$$F_\alpha^\beta(u) = \int_\Sigma e^{\beta u^2\left(1+\alpha\|u\|_2^2\right)} \mathrm{d}v_g.$$

Then there hold
(i) for any $\alpha \geqslant \lambda_1(\Sigma)$, $\sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u) = +\infty$;
(ii) for any $0 \leqslant \alpha < \lambda_1(\Sigma)$, $\sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u) < +\infty$;
(iii) for sufficiently small $\alpha > 0$, $\sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u)$ can be attained by some function $u_\alpha \in C^\infty\left(\overline{\Sigma}\right) \cap \mathscr{S}$.

For the proof, we employ the method of blow-up analysis, which was originally used by Carleson-Chang[4], Ding-Jost-Li-Wang [9], Adimurthi-Struwe [2], Li [14], and Yang [31, 33]. For related works, we refer the reader to Adimurthi-Druet [1], do Ó-de Souza [8, 10], Nguyen [21, 22], Li-Yang [16], Zhu [39], Fang-Zhang [11], Yang-Zhu [35, 36] and Csató-Nguyen-Roy [7]. We should point out that the blow-up occurs on the boundary $\partial\Sigma$ in our case. The key ingredient in the proof of our theorem is the isothermal coordinate system on $\partial\Sigma$. Though such coordinates have been used by many authors (see for example Li-Liu [17] and Yang [29, 30, 32]), the proof of its existence around has just been provided by Yang-Zhou [34] via Riemann mapping theorems involving the boundary.

The remaining part of this paper will be organized as follows: In Section 2, we prove (Theorem 1, (i)) by constructing test functions; in Section 3, we prove (Theorem 1, (ii)) by using blow-up analysis; in Section 4, we construct a sequence of functions to show (Theorem 1, (iii)) holds. Hereafter we do not distinguish the sequence and the subsequence; moreover, we often denote various constants by the same C.

2. The case of $\alpha \geqslant \lambda_1(\Sigma)$

In this section, we select test functions to prove Theorem 1 (i). Let $\lambda_1(\Sigma)$ be defined by (6) and $\alpha \geqslant \lambda_1(\Sigma)$. From a direct method of variation, one obtains that there exists some function $u_0 \in \mathscr{S}$, such that

$$\lambda_1(\Sigma) = \left\|\nabla_g u_0\right\|_2^2. \tag{7}$$

By a direct calculation, we derive that u_0 satisfies the Euler-Lagrange equation

$$\begin{cases} \Delta_g u_0 = \lambda_1(\Sigma)\, u_0 \text{ in } \Sigma, \\ \dfrac{\partial u_0}{\partial \mathbf{n}} = 0 \text{ on } \partial\Sigma, \\ \displaystyle\int_\Sigma u_0 \mathrm{d}v_g = 0,\ \int_\Sigma u_0^2 \mathrm{d}v_g = 1, \end{cases} \tag{8}$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Sigma$. Applying elliptic estimates to (8), we obtain $u_0 \in \mathscr{S} \cap C^\infty\left(\overline{\Sigma}\right)$. Consequently, there exist a point $x_0 \in \partial\Sigma$ with $u_0(x_0) > 0$ and a neighborhood U of x_0 with $u_0(x) \geqslant u_0(x_0)/2$ in U. Let $\delta = \left(t_\varepsilon\sqrt{-\ln\varepsilon}\right)^{-1}$, where $t_\varepsilon > 0$ such that $-t_\varepsilon^2\ln\varepsilon \to +\infty$ and $t_\varepsilon^2\sqrt{-\ln\varepsilon} \to 0$ as $\varepsilon \to 0$. Following ([34], Lemma 4), we can take an isothermal coordinate system $\left(\phi^{-1}\left(\mathbb{B}_\delta^+\right), \phi\right)$ such that $\phi(x_0) = 0$ and $\phi^{-1}\left(\mathbb{B}_\delta^+\right) \subset U$, where $\mathbb{B}_\delta^+ = \{(x_1,\ x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant \delta^2, x_2 > 0\}$. In such coordinates, the metric g has the representation $g = e^{2f}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right)$ and f is a smooth function with $f(0) = 0$.

On $\overline{\mathbb{B}_\delta^+}$, we define a sequence of functions

$$\tilde{u}_\varepsilon(x) = \begin{cases} \sqrt{\dfrac{-\ln\varepsilon}{2\pi}}, & |x| \leqslant \delta\sqrt{\varepsilon}, \\ \sqrt{\dfrac{-2}{\pi\ln\varepsilon}}\ln\dfrac{\delta}{|x|}, & \delta\sqrt{\varepsilon} < |x| \leqslant \delta. \end{cases}$$

Moreover, we set

$$u_\varepsilon = \begin{cases} \tilde{u}_\varepsilon \circ \phi & \text{in} \quad \phi^{-1}\left(\overline{\mathbb{B}_\delta^+}\right), \\ s_\varepsilon\,\varphi & \text{in} \quad \Sigma \setminus \phi^{-1}\left(\overline{\mathbb{B}_\delta^+}\right), \end{cases} \tag{9}$$

where $\varphi \in C_0^\infty\left(\Sigma \setminus \phi^{-1}(\mathbb{B}_\delta)\right)$ and s_ε is a real number such that $\int_\Sigma u_\varepsilon \mathrm{d}v_g = 0$. Set $v_\varepsilon = u_\varepsilon + t_\varepsilon u_0$. According to (7)–(9), we have

$$\|v_\varepsilon\|_2^2 = \|u_\varepsilon\|_2^2 + t_\varepsilon^2\|u_0\|_2^2 + 2t_\varepsilon\int_\Sigma u_\varepsilon u_0 \mathrm{d}v_g = t_\varepsilon^2 + 2t_\varepsilon\int_\Sigma u_\varepsilon u_0\mathrm{d}v_g + O\left(\frac{-1}{\ln\varepsilon}\right) \tag{10}$$

and

$$\left\|\nabla_g v_\varepsilon\right\|_2^2 = 1 + \lambda_1(\Sigma)t_\varepsilon^2 + 2\lambda_1(\Sigma)t_\varepsilon\int_\Sigma u_\varepsilon u_0\mathrm{d}v_g + O\left(\frac{-1}{\ln\varepsilon}\right). \tag{11}$$

Take $v_\varepsilon^* = v_\varepsilon/\|\nabla_g v_\varepsilon\|_2^2 \in \mathscr{S}$. From $\alpha \geqslant \lambda_1(\Sigma)$ and (9)–(11), we have that on $\phi^{-1}\left(\mathbb{B}_{\delta\sqrt{\varepsilon}}^+\right)$

$$\begin{aligned} 2\pi v_\varepsilon^{*2}\left(1 + \alpha\|v_\varepsilon^*\|_2^2\right) =& 2\pi v_\varepsilon^2\frac{1}{\left\|\nabla_g v_\varepsilon\right\|_2^2}\left(1 + \alpha\frac{\|v_\varepsilon\|_2^2}{\left\|\nabla_g v_\varepsilon\right\|_2^2}\right) \\ \geqslant& \left(2\pi t_\varepsilon^2 u_0^2 - \ln\varepsilon + 4\pi t_\varepsilon\sqrt{\frac{-\ln\varepsilon}{2\pi}}u_0\right)\left(1 + o\left(\frac{t_\varepsilon}{\sqrt{-\ln\varepsilon}}\right)\right) \\ \geqslant& -\ln\varepsilon + t_\varepsilon\sqrt{-\ln\varepsilon}\left(\sqrt{8\pi}u_0 + o(1)\right). \end{aligned}$$

Hence there holds

$$\begin{aligned} \int_\Sigma e^{2\pi v_\varepsilon^{*2}\left(1+\alpha\|v_\varepsilon^*\|_2^2\right)}\mathrm{d}v_g \geqslant& \int_{\phi^{-1}\left(\mathbb{B}_{\delta\sqrt{\varepsilon}}^+\right)} \frac{1}{\varepsilon}e^{t_\varepsilon\sqrt{-\ln\varepsilon}\left(\sqrt{8\pi}u_0+o(1)\right)}\mathrm{d}v_g \\ \geqslant& C(\delta)e^{t_\varepsilon\sqrt{-\ln\varepsilon}\left(\sqrt{2\pi}u_0(x_0)+o(1)\right)} \end{aligned}$$

for some positive constant $C(\delta)$. In view of $u_0(x_0) > 0$, we get $\sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u) \geqslant \lim_{\varepsilon\to 0} F_\alpha^{2\pi}(v_\varepsilon^*) = +\infty$. This completes the proof of Theorem 1 (i).

3. The case of $0 \leqslant \alpha < \lambda_1(\Sigma)$

In this section, we will prove Theorem 1 (ii) in three steps: firstly, we consider the existence of maximizers for subcritical functionals and give the corresponding Euler-Lagrange equation; secondly, we deal with the asymptotic behavior of the maximizers through blow-up analysis; finally, we deduce an upper bound of the supremum $\sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u)$ under the assumption that blow-up occurs.

Step 1. Existence of maximizers for subcritical functionals

Using the similar proof of ([19], Step 1), we have the following

LEMMA 1. *For any $\varepsilon > 0$, there exists some function $u_\varepsilon \in \mathscr{S} \cap C^\infty\left(\overline{\Sigma}\right)$ with $\|\nabla_g u_\varepsilon\|_2^2 = 1$, such that*

$$\sup_{u\in\mathscr{S}} F_\alpha^{2\pi-\varepsilon}(u) = F_\alpha^{2\pi-\varepsilon}(u_\varepsilon).$$

Moreover, u_ε satisfies the Euler-Lagrange equation

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 \text{ on } \partial\Sigma, \\ \Delta_g u_\varepsilon = \dfrac{\beta_\varepsilon}{\lambda_\varepsilon} u_\varepsilon e^{\alpha_\varepsilon u_\varepsilon^2} + \gamma_\varepsilon u_\varepsilon - \dfrac{\mu_\varepsilon}{\lambda_\varepsilon} \text{ in } \Sigma, \\ \alpha_\varepsilon = (2\pi-\varepsilon)\left(1+\alpha\|u_\varepsilon\|_2^2\right), \ \ \beta_\varepsilon = \dfrac{1+\alpha\|u_\varepsilon\|_2^2}{1+2\alpha\|u_\varepsilon\|_2^2}, \\ \gamma_\varepsilon = \dfrac{\alpha}{1+2\alpha\|u_\varepsilon\|_2^2}, \ \ \lambda_\varepsilon = \displaystyle\int_\Sigma u_\varepsilon^2 e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g, \ \ \mu_\varepsilon = \dfrac{\beta_\varepsilon}{\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma u_\varepsilon e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g, \end{cases} \tag{12}$$

where $\mathrm{Area}(\Sigma) = \int_\Sigma 1 \mathrm{d}v_g$.

It follows from Lebesgue's dominated convergence theorem that

$$\lim_{\varepsilon\to 0} F_\alpha^{2\pi-\varepsilon}(u_\varepsilon) = \sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u). \tag{13}$$

Seeing the fact of $1 + te^t \geqslant e^t$ for any $t \geqslant 0$, we get

$$\lambda_\varepsilon = \int_\Sigma u_\varepsilon^2 e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g \geqslant \frac{1}{\alpha_\varepsilon} \int_\Sigma \left(e^{\alpha_\varepsilon u_\varepsilon^2} - 1\right) \mathrm{d}v_g,$$

which together with (13) leads to

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon > 0. \tag{14}$$

In view of (12), (14) and $\beta_\varepsilon \leqslant 1$, we obtain

$$\begin{aligned}\left|\frac{\mu_\varepsilon}{\lambda_\varepsilon}\right| &\leqslant \frac{1}{\lambda_\varepsilon \mathrm{Area}(\Sigma)}\left(\int_{\{u\in\Sigma:|u_\varepsilon|\geqslant 1\}}|u_\varepsilon|e^{\alpha_\varepsilon u_\varepsilon^2}\mathrm{d}v_g+\int_{\{u\in\Sigma:|u_\varepsilon|<1\}}|u_\varepsilon|e^{\alpha_\varepsilon u_\varepsilon^2}\mathrm{d}v_g\right)\\ &\leqslant \frac{1}{\lambda_\varepsilon \mathrm{Area}(\Sigma)}\left(\int_{\{u\in\Sigma:|u_\varepsilon|\geqslant 1\}}u_\varepsilon^2 e^{\alpha_\varepsilon u_\varepsilon^2}\mathrm{d}v_g+\int_{\{u\in\Sigma:|u_\varepsilon|<1\}}e^{\alpha_\varepsilon u_\varepsilon^2}\mathrm{d}v_g\right)\\ &\leqslant \frac{1}{\mathrm{Area}(\Sigma)}+\frac{e^{\alpha_\varepsilon}}{\lambda_\varepsilon}\\ &\leqslant C. \end{aligned} \tag{15}$$

Step 2. Blow-up analysis

Since u_ε is bounded in $W^{1,2}(\Sigma,g)$, there exists some function $u_0 \in W^{1,2}(\Sigma,g)$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \text{ weakly in } W^{1,2}(\Sigma,g),\\ u_\varepsilon \to u_0 \text{ strongly in } L^p(\Sigma,g), \forall p>1,\\ u_\varepsilon \to u_0 \text{ a.e. in } \Sigma. \end{cases} \tag{16}$$

Then we have $\int_\Sigma u_0 dv_g = 0$ and $\|\nabla_g u_0\|_2^2 \leqslant 1$.

We set $c_\varepsilon = |u_\varepsilon(x_\varepsilon)| = \max_{\overline{\Sigma}}|u_\varepsilon|$. We first assume that c_ε is bounded, which together with elliptic estimates completes the proof of Theorem 1 (ii). Without loss of generality, we assume

$$c_\varepsilon = u_\varepsilon(x_\varepsilon) \to +\infty \tag{17}$$

and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. Applying maximum principle to (12), we have $x_0 \in \partial\Sigma$.

Following ([34], Lemma 4), we can take an isothermal coordinate system (U,ϕ) near x_0, such that $\phi(x_0)=0$, $\phi(U)=\mathbb{B}_r^+$ and $\phi(U\cap\partial\Sigma)=\partial\mathbb{R}_+^2\cap\mathbb{B}_r$ for some fixed $r>0$, where $\mathbb{R}_+^2=\{x=(x_1,x_2)\in\mathbb{R}^2 : x_2>0\}$. In such coordinates, the metric g has the representation $g=e^{2f}\left(\mathrm{d}x_1^2+\mathrm{d}x_2^2\right)$ and f is a smooth function with $f(0)=0$. Denote $\tilde{x}_\varepsilon=\phi(x_\varepsilon)$ and $\tilde{u}_\varepsilon=u_\varepsilon\circ\phi^{-1}$. To proceed, we observe an energy concentration phenomenon of u_ε.

LEMMA 2. *There hold $u_0=0$ and $|\nabla_g u_\varepsilon|^2\mathrm{d}v_g \rightharpoonup \delta_{x_0}$ in sense of measure, where δ_{x_0} stands for the Dirac measure centered at x_0.*

Proof. We first prove $u_0\equiv 0$. Suppose not, we can see that $0<\|\nabla_g u_0\|_2^2\leqslant 1$. Letting $\eta=\|\nabla_g u_0\|_2^2$, one has $\|\nabla_g(u_\varepsilon-u_0)\|_2^2\to 1-\eta<1$ and $1+\alpha\|u_\varepsilon\|_2^2\to 1+\alpha\|u_0\|_2^2\leqslant 1+\eta$ as $\varepsilon\to 0$. For sufficiently small ε, we obtain

$$\left(1+\alpha\|u_\varepsilon\|_2^2\right)\|\nabla_g(u_\varepsilon-u_0)\|_2^2\leqslant\frac{2-\eta^2}{2}<1.$$

From the Hölder inequality, there holds

$$\int_\Sigma e^{q\alpha_\varepsilon u_\varepsilon^2} dv_g \leqslant \int_\Sigma e^{q\left(1+\frac{1}{\delta}\right)\alpha_\varepsilon u_0^2 + q(1+\delta)\alpha_\varepsilon (u_\varepsilon - u_0)^2} dv_g$$
$$\leqslant C \left(\int_\Sigma e^{sq(1+\delta)(2\pi-\varepsilon)\frac{2-\eta^2}{2} \frac{(u_\varepsilon - u_0)^2}{\|\nabla_g (u_\varepsilon - u_0)\|_2^2}} dv_g \right)^{\frac{1}{s}}$$

for sufficiently small δ, some $r,\ s,\ q > 1$ satisfying $sq(1+\delta)(2-\eta^2)/2 < 1$ and $1/r + 1/s = 1$. In view of the Trudinger-Moser inequality (5), we get $e^{\alpha_\varepsilon u_\varepsilon^2}$ is bounded in $L^q(\Sigma, g)$. Hence $\Delta_g u_\varepsilon$ is bounded in some $L^q(\Sigma, g)$ from (12) and (15). Applying the elliptic estimate to (12), one gets u_ε is uniformly bounded, which contradicts our assumption $c_\varepsilon \to +\infty$. That is to say $u_0 \equiv 0$.

Next we prove $|\nabla_g u_\varepsilon|^2 dv_g \rightharpoonup \delta_{x_0}$ in sense of measure. Suppose not. There exists some $r > 0$ such that

$$\lim_{\varepsilon \to 0} \int_{B_r(x_0)} \left|\nabla_g u_\varepsilon\right|^2 dv_g := \eta < 1,$$

where $B_r(x_0)$ is a geodesic ball centered at x_0 with radius r. For sufficiently small ε, we can see that $\int_{B_r(x_0)} |\nabla_g u_\varepsilon|^2 dv_g \leqslant (\eta+1)/2 < 1$. Then we choose a cut-off function ρ in $C_0^1\left(\phi\left(B_{r_0}(x_0)\right)\right)$, which is equal to 1 in $\overline{\phi\left(B_{r_0/2}(x_0)\right)}$ such that

$$\int_{\phi(B_r(x_0))} |\nabla_g (\rho \tilde{u}_\varepsilon)|^2 dx \leqslant \frac{\eta+3}{4} < 1.$$

Hence we obtain

$$\int_{B_{r/2}(x_0)} e^{\alpha_\varepsilon q u_\varepsilon^2} dv_g = \int_{\phi\left(B_{r/2}(x_0)\right)} e^{\alpha_\varepsilon q \tilde{u}_\varepsilon^2} e^{2f} dx$$
$$\leqslant C \int_{\phi(B_r(x_0))} e^{\alpha_\varepsilon q (\rho \tilde{u}_\varepsilon)^2} dx$$
$$\leqslant C \int_{\phi(B_r(x_0))} e^{\alpha_\varepsilon q \frac{\eta+3}{4} \frac{(\rho \tilde{u}_\varepsilon)^2}{\|\nabla_g (\rho \tilde{u}_\varepsilon)\|_2^2}} dx.$$

From the Trudinger-Moser inequality (5), we get $e^{\alpha_\varepsilon u_\varepsilon^2}$ is bounded in $L^q\left(B_{r/2}(x_0), g\right)$ for any $q > 1$ satisfying $q(\eta+3)/4 \leqslant 1$. Applying the elliptic estimate to (12), one gets u_ε is uniformly bounded in $B_{r/4}(x_0)$. This contradicts (17) and ends the proof of the lemma. □

Denote

$$r_\varepsilon = \sqrt{\frac{\lambda_\varepsilon}{\beta_\varepsilon c_\varepsilon^2 e^{\alpha_\varepsilon c_\varepsilon^2}}}. \tag{18}$$

Using $c_\varepsilon = \max_\Sigma |u_\varepsilon|$, the inequality (5), (12) and Lemma 2, we have

$$
\begin{aligned}
r_\varepsilon^2 c_\varepsilon^k &= \frac{c_\varepsilon^k}{\beta_\varepsilon c_\varepsilon^2 e^{\alpha_\varepsilon c_\varepsilon^2}} \int_\Sigma u_\varepsilon^2 e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g \\
&\leqslant \frac{c_\varepsilon^k}{(1+o_\varepsilon(1)) e^{2\pi(1-\delta)c_\varepsilon^2}} \int_\Sigma e^{(2\pi+o_\varepsilon(1))(1-\delta)u_\varepsilon^2} \mathrm{d}v_g \\
&\leqslant C \frac{c_\varepsilon^k}{e^{2\pi(1-\delta)c_\varepsilon^2}},
\end{aligned}
$$

where k is an integer and $0 < \delta < 1$. It follows from (17) that

$$\lim_{\varepsilon\to 0} r_\varepsilon^2 c_\varepsilon^k = 0. \tag{19}$$

Define

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon \circ \phi^{-1}(x_1, x_2), & x_2 \geqslant 0, \\ u_\varepsilon \circ \phi^{-1}(x_1, -x_2), & x_2 < 0, \end{cases}$$

and

$$\tilde{f}(x) = \begin{cases} f(x_1, x_2), & x_2 \geqslant 0, \\ f(x_1, -x_2), & x_2 < 0, \end{cases}$$

on $\mathbb{B}_r$. Let $U_\varepsilon = \left\{x \in \mathbb{R}^2 : \tilde{x}_\varepsilon + r_\varepsilon x \in \mathbb{B}_r\right\}$. Then one has $U_\varepsilon \to \mathbb{R}^2$ as $\varepsilon \to 0$ from (19). Define two blowing up functions on U_ε,

$$\psi_\varepsilon(x) = \frac{\tilde{u}(\tilde{x}_\varepsilon + r_\varepsilon x)}{c_\varepsilon}, \tag{20}$$

$$\varphi_\varepsilon(x) = c_\varepsilon(\tilde{u}(\tilde{x}_\varepsilon + r_\varepsilon x) - c_\varepsilon). \tag{21}$$

Now we study the convergence behavior of ψ_ε and φ_ε.

LEMMA 3. *Up to a subsequence, there hold*

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(x) = 1 \quad \text{in} \quad C^1_{loc}(\mathbb{R}^2), \tag{22}$$

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(x) = \varphi(x) \quad \text{in} \quad C^1_{loc}(\mathbb{R}^2), \tag{23}$$

where

$$\varphi(x) = -\frac{1}{2\pi} \ln\left(1 + \frac{\pi}{2}|x|^2\right) \tag{24}$$

and

$$\int_{\mathbb{R}^2_+} e^{4\pi\varphi(x)} \mathrm{d}x = 1. \tag{25}$$

Proof. By (12) and (18)–(21), a direct computation shows

$$-\Delta_{\mathbb{R}^2}\psi_\varepsilon = \left(c_\varepsilon^{-2}\psi_\varepsilon e^{\alpha_\varepsilon(\psi_\varepsilon+1)\varphi_\varepsilon} + r_\varepsilon^2\gamma_\varepsilon\psi_\varepsilon - \frac{r_\varepsilon^2\mu_\varepsilon}{c_\varepsilon\lambda_\varepsilon}\right)e^{2\tilde{f}(\tilde{x}_\varepsilon+r_\varepsilon x)}, \tag{26}$$

$$-\Delta_{\mathbb{R}^2}\varphi_\varepsilon = \left(\psi_\varepsilon e^{\alpha_\varepsilon(\psi_\varepsilon+1)\varphi_\varepsilon} + c_\varepsilon^2 r_\varepsilon^2\gamma_\varepsilon\psi_\varepsilon - \frac{c_\varepsilon r_\varepsilon^2\mu_\varepsilon}{\lambda_\varepsilon}\right)e^{2\tilde{f}(\tilde{x}_\varepsilon+r_\varepsilon x)}. \tag{27}$$

Since $|\psi_\varepsilon| \leqslant 1$ and $\lim_{\varepsilon\to 0} -\Delta_{\mathbb{R}^2}\psi_\varepsilon = 0$, we have by the elliptic estimate to (26) that $\lim_{\varepsilon\to 0}\psi_\varepsilon = \psi$ in $C^1_{loc}(\mathbb{R}^2)$, where ψ is a bounded harmonic function in $\mathbb{R}^2$. Note that $\psi(0) = \lim_{\varepsilon\to 0}\psi_\varepsilon(0) = 1$. It follows from the Liouville theorem that $\psi \equiv 1$ in $\mathbb{R}^2$. That is to say (22) holds.

Note that $\varphi_\varepsilon(x) \leqslant \varphi_\varepsilon(0) = 0$ for any $x \in U_\varepsilon$. Applying (19) and the elliptic estimate to (27), we obtain (23), where φ satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2}\varphi = e^{4\pi\varphi} \text{ in } \mathbb{R}^2, \\ \varphi(0) = 0 = \sup\limits_{\mathbb{R}^2}\varphi, \\ \displaystyle\int_{\mathbb{R}^2} e^{4\pi\varphi}\mathrm{d}x \leqslant 2. \end{cases}$$

By the uniqueness theorem in Chen-Li [5], we have (24). Moreover, a simple calculation gives

$$\int_{\mathbb{R}^2} e^{4\pi\varphi}\mathrm{d}x = 2. \tag{28}$$

For any fixed $R > 0$, let $\mathbb{B}'_R = \{x \in \mathbb{B}_R : \tilde{x}_\varepsilon + r_\varepsilon x \in \mathbb{B}_r^+\}$ and $\mathbb{B}''_R = \{x \in \mathbb{B}_R : \tilde{x}_\varepsilon + r_\varepsilon x \in \mathbb{B}_r^-\}$, we have

$$\begin{aligned} \int_{\mathbb{B}_R} e^{4\pi\varphi}\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{\mathbb{B}_R}\frac{1}{\beta_\varepsilon}\psi_\varepsilon^2 e^{\alpha_\varepsilon(1+\psi_\varepsilon)\varphi_\varepsilon}\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{B}_{Rr_\varepsilon}(\tilde{x}_\varepsilon)}\frac{1}{\lambda_\varepsilon}\tilde{u}_\varepsilon^2 e^{\alpha_\varepsilon\tilde{u}_\varepsilon^2}\mathrm{d}x \\ &\leqslant \lim_{\varepsilon\to 0}\int_{\mathbb{B}^+_{Rr_\varepsilon}(\tilde{x}_\varepsilon)}\frac{1}{\lambda_\varepsilon}\tilde{u}_\varepsilon^2 e^{\alpha_\varepsilon\tilde{u}_\varepsilon^2}\mathrm{d}x + \lim_{\varepsilon\to 0}\int_{\mathbb{B}^-_{Rr_\varepsilon}(\tilde{x}_\varepsilon)}\frac{1}{\lambda_\varepsilon}\tilde{u}_\varepsilon^2 e^{\alpha_\varepsilon\tilde{u}_\varepsilon^2}\mathrm{d}x. \end{aligned}$$

This inequality together with $\int_{U_\varepsilon}\tilde{u}_\varepsilon^2 e^{\alpha_\varepsilon\tilde{u}_\varepsilon^2}\mathrm{d}x \leqslant \lambda_\varepsilon$ and (28) gives

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\mathbb{B}^+_{Rr_\varepsilon}(\tilde{x}_\varepsilon)}\frac{1}{\lambda_\varepsilon}\tilde{u}_\varepsilon^2 e^{\alpha_\varepsilon\tilde{u}_\varepsilon^2}\mathrm{d}x = 1,$$

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\mathbb{B}^-_{Rr_\varepsilon}(\tilde{x}_\varepsilon)}\frac{1}{\lambda_\varepsilon}\tilde{u}_\varepsilon^2 e^{\alpha_\varepsilon\tilde{u}_\varepsilon^2}\mathrm{d}x = 1.$$

That is to say (25) holds. Then we have the lemma. □

Next we discuss the convergence behavior of u_ε away from x_0. Denote $u_{\varepsilon,\beta} = \min\{\beta c_\varepsilon, u_\varepsilon\} \in W^{1,2}(\Sigma, g)$ for any real number $0<\beta<1$. Following ([29], Lemma 3.6), we get

$$\lim_{\varepsilon\to 0} \left\|\nabla_g u_{\varepsilon,\beta}\right\|_2^2 = \beta. \tag{29}$$

LEMMA 4. *Letting λ_ε be defined by (12), we obtain*

$$\limsup_{\varepsilon\to 0} F_\alpha^{2\pi-\varepsilon}(u_\varepsilon) = \mathrm{Area}(\Sigma) + \lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{c_\varepsilon^2} \tag{30}$$

and

$$\lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{c_\varepsilon^2} = \lim_{R\to+\infty} \lim_{\varepsilon\to 0} \int_{\phi^{-1}\left(\mathbb{B}^+_{Rr_\varepsilon}(\tilde{x}_\varepsilon)\right)} e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g. \tag{31}$$

Proof. Recalling (12) and (29), for any real number $0<\beta<1$, one gets

$$\begin{aligned}
F_\alpha^{2\pi-\varepsilon}(u_\varepsilon) - \mathrm{Area}(\Sigma) &= \int_{\{x\in\Sigma:\, u_\varepsilon\leqslant\beta c_\varepsilon\}} (e^{\alpha_\varepsilon u_\varepsilon^2}-1)\,\mathrm{d}v_g + \int_{\{x\in\Sigma:\, u_\varepsilon>\beta c_\varepsilon\}} (e^{\alpha_\varepsilon u_\varepsilon^2}-1)\,\mathrm{d}v_g \\
&\leqslant \int_\Sigma (e^{\alpha_\varepsilon u_{\varepsilon,\beta}^2}-1)\,\mathrm{d}v_g + \frac{u_\varepsilon^2}{\beta^2 c_\varepsilon^2} \int_{\{x\in\Sigma:\, u_\varepsilon>\beta c_\varepsilon\}} e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g \\
&\leqslant \int_\Sigma e^{\alpha_\varepsilon u_{\varepsilon,\beta}^2} \alpha_\varepsilon u_\varepsilon^2 \mathrm{d}v_g + \frac{\lambda_\varepsilon}{\beta^2 c_\varepsilon^2} \\
&\leqslant \left(\int_\Sigma e^{r\alpha_\varepsilon u_{\varepsilon,\beta}^2} \mathrm{d}v_g\right)^{1/r} \left(\int_\Sigma \alpha_\varepsilon^s u_\varepsilon^{2s} \mathrm{d}v_g\right)^{1/s} + \frac{\lambda_\varepsilon}{\beta^2 c_\varepsilon^2}.
\end{aligned}$$

By (5) and (29), $e^{\alpha_\varepsilon u_{\varepsilon,\beta}^2}$ is bounded in $L^r(\Sigma, g)$ for some $r>1$. Then letting $\varepsilon\to 0$ first, and then $\beta\to 1$, we obtain

$$\limsup_{\varepsilon\to 0} F_\alpha^{2\pi-\varepsilon}(u_\varepsilon) - \mathrm{Area}(\Sigma) \leqslant \limsup_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{32}$$

According to $c_\varepsilon = \max_{\overline{\Sigma}} u_\varepsilon$, (12) and Lemma 2, we have

$$F_\alpha^{2\pi-\varepsilon}(u_\varepsilon) - \mathrm{Area}(\Sigma) \geqslant \frac{\lambda_\varepsilon}{c_\varepsilon^2} - \frac{1}{c_\varepsilon^2}\int_\Sigma u_\varepsilon^2 \mathrm{d}v_g,$$

that is to say

$$\limsup_{\varepsilon\to 0} F_\alpha^{2\pi-\varepsilon}(u_\varepsilon) - \mathrm{Area}(\Sigma) \geqslant \liminf_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{33}$$

Combining (32) and (33), one gets (30).

Applying (12) and (18)–(21), we obtain

$$\begin{aligned}\int_{\phi^{-1}\left(\mathbb{B}^+_{Rr_\varepsilon}(\tilde{x}_\varepsilon)\right)} e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g &= \int_{\mathbb{B}^+_{Rr_\varepsilon}(\tilde{x}_\varepsilon)} e^{\alpha_\varepsilon u_\varepsilon^2(x)} e^{2f(x)} \mathrm{d}x \\ &= \int_{\mathbb{B}^+_R(0)} r_\varepsilon^2 e^{\alpha_\varepsilon c_\varepsilon^2(x)} e^{\alpha_\varepsilon(\psi_\varepsilon(x)+1)\varphi_\varepsilon(x)} e^{2f(\tilde{x}_\varepsilon+r_\varepsilon x)} \mathrm{d}x \\ &= \frac{\lambda_\varepsilon}{c_\varepsilon^2} \int_{\mathbb{B}^+_R(0)} \frac{1}{\beta_\varepsilon} e^{\alpha_\varepsilon(\psi_\varepsilon(x)+1)\,\varphi_\varepsilon(x)} e^{2f(\tilde{x}_\varepsilon+r_\varepsilon x)} \mathrm{d}x.\end{aligned}$$

Letting $\varepsilon \to 0$ first and then $R \to +\infty$, we have (31) by (23)–(25). □

Next we consider the properties of $c_\varepsilon u_\varepsilon$. Using the similar idea of ([29], Lemma 3.9), one gets

$$\frac{\beta_\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g \rightharpoonup \delta_{x_0}. \tag{34}$$

After a slight modification of ([31], Lemma 4.8), we have

LEMMA 5. *Assume $u \in C^\infty\left(\overline{\Sigma}\right)$ is a solution of $\Delta_g u = f(x)$ in (Σ, g) and satisfies $\|u\|_1 \leqslant c_0 \|f\|_1$. Then for any $1 < q < 2$, there holds $\|\nabla_g u\|_q \leqslant C(q, c_0, \Sigma, g)\|f\|_1$.*

LEMMA 6. *For any $1 < q < 2$, $c_\varepsilon u_\varepsilon$ is bounded in $W^{1,q}(\Sigma, g)$. Moreover, there holds*

$$\begin{cases} c_\varepsilon u_\varepsilon \rightharpoonup G \text{ weakly in } W^{1,q}(\Sigma, g), \forall 1 < q < 2, \\ c_\varepsilon u_\varepsilon \to G \text{ strongly in } L^s(\Sigma, g), \forall 1 < s < \dfrac{2q}{2-q}, \\ c_\varepsilon u_\varepsilon \to G \text{ in } C^1_{loc}(\Sigma \setminus \{x_0\}), \end{cases}$$

where G is a Green function satisfying

$$\begin{cases} \Delta_g G = \delta_{x_0} + \alpha G - \dfrac{1}{\mathrm{Area}(\Sigma)} \text{ in } \Sigma, \\ \dfrac{\partial G}{\partial \mathbf{n}} = 0 \text{ on } \partial\Sigma \setminus \{x_0\}, \\ \displaystyle\int_\Sigma G \mathrm{d}v_g = 0. \end{cases} \tag{35}$$

Proof. It follows from (12) that

$$\Delta_g(c_\varepsilon u_\varepsilon) = \frac{\beta_\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{\alpha_\varepsilon u_\varepsilon^2} + \gamma_\varepsilon c_\varepsilon u_\varepsilon - c_\varepsilon \frac{\mu_\varepsilon}{\lambda_\varepsilon}. \tag{36}$$

According to (12) and (34), we have

$$\left|\frac{c_\varepsilon \mu_\varepsilon}{\lambda_\varepsilon}\right| = \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\beta_\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{\alpha_\varepsilon u_\varepsilon^2} \mathrm{d}v_g = \frac{1}{\mathrm{Area}(\Sigma)}(1 + o_\varepsilon(1)). \tag{37}$$

In view of Lemma 2, (12), (15), (34), (36) and (37), we have $\Delta_g(c_\varepsilon u_\varepsilon)$ is bounded in $L^1(\Sigma,g)$. From Lemma 5, there holds $c_\varepsilon u_\varepsilon$ is bounded in $W^{1,q}(\Sigma,g)$ for any $1<q<2$. Then $c_\varepsilon u_\varepsilon \rightharpoonup G$ weakly in $W^{1,q}(\Sigma,g)$ for any $1<q<2$ and $c_\varepsilon u_\varepsilon \to G$ strongly in $L^s(\Sigma,g)$ for any $1<s<2q/(2-q)$.

We choose a cut-off function ρ in $C^\infty\left(\overline{\Sigma}\right)$, which is equal to 0 in $\overline{B_\delta(x_0)}$ and equal to 1 in $\Sigma\setminus B_{2\delta}(x_0)$ such that $\lim_{\varepsilon\to 0}\|\nabla_g(\rho u_\varepsilon)\|_2^2=0$. Hence there holds

$$\int_{\Sigma\setminus B_{2\delta}(x_0)} e^{s\alpha_\varepsilon u_\varepsilon^2}\mathrm{d}x \leqslant \int_{\Sigma\setminus B_{2\delta}(x_0)} e^{s\alpha_\varepsilon\|\nabla_g(\rho u_\varepsilon)\|_2^2\frac{\rho^2u_\varepsilon^2}{\|\nabla_g(\rho u_\varepsilon)\|_2^2}}\mathrm{d}x.$$

From the Trudinger-Moser inequality (5), $e^{\alpha_\varepsilon u_\varepsilon^2}$ is bounded in $L^s(\Sigma,g)$ for some $s>1$. Applying the elliptic estimate and the compact embedding theorem to (36), we obtain $c_\varepsilon u_\varepsilon \to G$ in $C^1_{loc}(\Sigma\setminus\{x_0\})$. Testing (36) by $\phi\in C^1(\Sigma)$, we obtain (35). $\square$

Applying the elliptic estimate, we can decompose G as the form

$$G=-\frac{1}{\pi}\ln|x-x_0|+A_{x_0}+\sigma(x), \tag{38}$$

where A_{x_0} is a constant only on x_0 and $\sigma(x)\in C^\infty\left(\overline{\Sigma}\right)$ with $\sigma(x_0)=0$.

Step 3. Upper bound estimate

To derive an upper bound of $\sup_{u\in\mathscr{S}}F_\alpha^{2\pi}(u)$, we use the capacity estimate, which was first used by Li [14] in this topic.

LEMMA 7. *There holds*

$$\sup_{u\in\mathscr{S}}F_\alpha^{2\pi}(u)\leqslant \mathrm{Area}(\Sigma)+\frac{\pi}{2}e^{1+2\pi A_{x_0}}.$$

Proof. We take an isothermal coordinate system (U,ϕ) near x_0 such that $\phi(x_0)=0$, $\phi(U)\subset\mathbb{R}^2_+$ and $\phi(U\cap\partial\Sigma)\subset\partial\mathbb{R}^2_+$. In such coordinates, the metric g has the representation $g=e^{2f}\left(\mathrm{d}x_1^2+\mathrm{d}x_2^2\right)$ and f is a smooth function with $f(0)=0$. Denote $\tilde{u}_\varepsilon=u_\varepsilon\circ\phi^{-1}$. We claim that

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}\leqslant\frac{\pi}{2}e^{1+2\pi A_{x_0}}. \tag{39}$$

To confirm this claim, we set $a=\sup_{\partial\mathbb{B}_\delta\cap\mathbb{R}^2_+}\tilde{u}_\varepsilon$ and $b=\inf_{\partial\mathbb{B}_{R r_\varepsilon}\cap\mathbb{R}^2_+}\tilde{u}_\varepsilon$ for sufficiently small $\delta>0$ and some fixed $R>0$. According to (23), (24), (38) and Lemma 6, one gets

$$a=\frac{1}{c_\varepsilon}\left(\frac{1}{\pi}\ln\frac{1}{\delta}+A_{x_0}+o_\delta(1)+o_\varepsilon(1)\right),$$
$$b=c_\varepsilon+\frac{1}{c_\varepsilon}\left(-\frac{1}{2\pi}\ln\left(1+\frac{\pi}{2}R^2\right)+o_\varepsilon(1)\right),$$

where $o_\delta(1) \to 0$ as $\delta \to 0$ and $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. It follows from a direct computation that

$$\pi(a-b)^2 = \pi c_\varepsilon^2 + 2\ln\delta - 2\pi A_{x_0} - \ln\left(1+\frac{\pi}{2}R^2\right) + o_\delta(1) + o_\varepsilon(1). \tag{40}$$

Define a function space

$$W_{a,b} = \left\{ \tilde{u} \in W^{1,2}\left(\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+\right) : \tilde{u}|_{\partial\mathbb{B}_\delta \cap \mathbb{R}_+^2} = a,\, \tilde{u}|_{\partial\mathbb{B}_{Rr_\varepsilon} \cap \mathbb{R}_+^2} = b,\, \frac{\partial \tilde{u}}{\partial \mathbf{v}}\bigg|_{\partial\mathbb{R}_+^2 \cap \left(\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\varepsilon}\right)} = 0 \right\},$$

where $\mathbf{v}$ denotes the outward unit normal vector on $\partial\mathbb{R}_+^2$. Applying the direct method of variation, we obtain $\inf_{u\in W_{a,b}} \int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} u|^2 \mathrm{d}x$ can be attained by some function $m(x) \in W_{a,b}$ with $\Delta_{\mathbb{R}^2} m(x) = 0$. We can check that

$$m(x) = \frac{a(\ln|x| - \ln(Rr_\varepsilon)) + b(\ln\delta - \ln|x|)}{\ln\delta - \ln(Rr_\varepsilon)}$$

and

$$\int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} m(x)|^2 dx = \frac{\pi(a-b)^2}{\ln\delta - \ln(Rr_\varepsilon)}. \tag{41}$$

Recalling (12) and (18), we have

$$\ln\delta - \ln(Rr_\varepsilon) = \ln\delta - \ln R - \frac{1}{2}\ln\frac{\lambda_\varepsilon}{\beta_\varepsilon c_\varepsilon^2} + \frac{1}{2}\alpha_\varepsilon c_\varepsilon^2. \tag{42}$$

Letting $u_\varepsilon^* = \max\{a,\, \min\{b,\, \tilde{u}_\varepsilon\}\} \in W_{a,b}$, one gets $|\nabla_{\mathbb{R}^2} u_\varepsilon^*| \leqslant |\nabla_{\mathbb{R}^2} \tilde{u}_\varepsilon|$ in $\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+$ for sufficiently small ε. According to this and $\|\nabla_g u_\varepsilon\|_2^2 = 1$, we obtain

$$\begin{aligned} \int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} m(x)|^2 \mathrm{d}x &\leqslant \int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} u_\varepsilon^*(x)|^2 \mathrm{d}x \\ &\leqslant 1 - \int_{\Sigma \setminus \phi^{-1}\left(\mathbb{B}_\delta^+\right)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g - \int_{\phi^{-1}\left(\mathbb{B}_{Rr_\varepsilon}^+\right)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g. \end{aligned} \tag{43}$$

Now we compute $\int_{\Sigma \setminus \phi^{-1}\left(\mathbb{B}_\delta^+\right)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g$ and $\int_{\phi^{-1}\left(\mathbb{B}_{Rr_\varepsilon}^+\right)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g$. In view of (35) and (38), we obtain

$$\int_{\Sigma \setminus \phi^{-1}\left(\mathbb{B}_\delta^+\right)} |\nabla_g G|^2 \mathrm{d}v_g = \frac{1}{\pi}\ln\frac{1}{\delta} + A_{x_0} + \alpha\|G\|_2^2 + o_\varepsilon(1) + o_\delta(1).$$

Hence we have by Lemma 6

$$\int_{\Sigma \setminus \phi^{-1}\left(\mathbb{B}_\delta^+\right)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g = \frac{1}{c_\varepsilon^2}\left(\frac{1}{\pi}\ln\frac{1}{\delta} + A_{x_0} + \alpha\|G\|_2^2 + o_\varepsilon(1) + o_\delta(1)\right). \tag{44}$$

It follows from (21), (23) and (24) that

$$\int_{\phi^{-1}\left(\mathbb{B}^+_{R r_\varepsilon}\right)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g = \frac{1}{c_\varepsilon^2}\left(\frac{1}{2\pi}\ln\left(1+\frac{\pi}{2}R^2\right) - \frac{1}{2\pi} + o_\varepsilon(1) + o_R(1)\right), \tag{45}$$

where $o_R(1) \to 0$ as $R \to +\infty$. Recalling (40)–(45), we obtain

$$\ln\frac{\lambda_\varepsilon}{c_\varepsilon^2} \leqslant \ln\frac{\pi}{2} + 1 + 2\pi A_{x_0} + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$ first, then $R \to +\infty$ and $\delta \to 0$. Hence (39) is followed. Combining (13), (39) and Lemma 4, we finish the proof of the lemma. □

From Lemma 7, the proof of Theorem 1 (ii) follows immediately under the hypothesis of $c_\varepsilon \to +\infty$.

4. Existence of the extremal functions

The content in this section is carried out under the condition $0 \leqslant \alpha < \lambda_1(\overline{\Sigma})$ and $c_\varepsilon \to +\infty$. Set a cut-off function $\xi \in C_0^\infty(B_{2R\varepsilon}(x_0))$ with $\xi = 1$ on $\overline{B_{R\varepsilon}(x_0)}$ and $\|\nabla_g \xi\|_{L^\infty} = O(1/(R\varepsilon))$. Denote $\tau = G + \pi^{-1}\ln|x-x_0| - A_{x_0}$, where G is defined as in (38). Let $R = -\ln\varepsilon$, then $R \to +\infty$ and $R\varepsilon \to 0$ as $\varepsilon \to 0$. We construct a blow-up sequence

$$v_\varepsilon = \begin{cases} \dfrac{c^2 - \frac{1}{2\pi}\ln\left(1+\frac{\pi}{2}\frac{|x-x_0|^2}{\varepsilon^2}\right) + b}{\sqrt{c^2+\alpha\|G\|_2^2}}, & x \in B_{R\varepsilon}(x_0), \\ \dfrac{G - \xi\tau}{\sqrt{c^2+\alpha\|G\|_2^2}}, & x \in B_{2R\varepsilon}(x_0)\setminus B_{R\varepsilon}(x_0), \\ \dfrac{G}{\sqrt{c^2+\alpha\|G\|_2^2}}, & x \in \Sigma \setminus B_{2R\varepsilon}(x_0), \end{cases} \tag{46}$$

where b and c are constants to be determined later. In order to assure that $v_\varepsilon \in C^\infty\left(\overline{\Sigma}\right)$, we obtain

$$c^2 - \frac{1}{2\pi}\ln\left(1+\frac{\pi}{2}R^2\right) + b = -\frac{1}{\pi}\ln(R\varepsilon) + A_{x_0}. \tag{47}$$

It follows from $\|\nabla_g v_\varepsilon\|_2 = 1$ that

$$c^2 = A_{x_0} - \frac{1}{\pi}\ln\varepsilon + \frac{1}{2\pi}\ln\frac{\pi}{2} - \frac{1}{2\pi} + O\left(\frac{1}{R^2}\right) + O(R\varepsilon\ln(R\varepsilon)) + o_\varepsilon(1). \tag{48}$$

In view of (47) and (48), we have

$$b = \frac{1}{2\pi} + O\left(\frac{1}{R^2}\right) + O(R\varepsilon\ln(R\varepsilon)) + o_\varepsilon(1). \tag{49}$$

A delicate and simple calculation shows

$$\|v_\varepsilon\|_2^2 = \frac{\|G\|_2^2 + O(R\varepsilon\ln(R\varepsilon))}{c^2+\alpha\|G\|_2^2} \geqslant \frac{\|G\|_2^2 + O(R\varepsilon\ln(R\varepsilon))}{c^2}\left(1-\frac{\alpha\|G\|_2^2}{c^2}\right), \quad (50)$$

which gives on $(B_{R\varepsilon}(x_0), g)$

$$2\pi v_\varepsilon^2\left(1+\alpha\|v_\varepsilon\|_2^2\right) \geqslant 2\pi c^2 + 4\pi b - 2\ln\left(1+\frac{\pi}{2}\frac{|x-x_0|^2}{\varepsilon^2}\right) - \frac{4\pi\alpha^2\|G\|_2^4}{c^2} + O\left(\frac{\ln R}{c^4}\right).$$

Denote $v_\varepsilon^* = \int_\Sigma v_\varepsilon dv_g/\mathrm{Area}(\Sigma)$. It is easy to know that $v_\varepsilon^* = O\left((R\varepsilon)^2\ln\varepsilon\right)$ and $v_\varepsilon - v_\varepsilon^* \in \mathscr{S}$. On the one hand, by (47)–(50), there holds

$$\begin{aligned}&\int_{B_{R\varepsilon}(x_0)} e^{2\pi(v_\varepsilon-v_\varepsilon^*)^2\left(1+\alpha\|v_\varepsilon-v_\varepsilon^*\|_2^2\right)}dv_g\\ \geqslant&\frac{\pi}{2}e^{1+2\pi A_{x_0}} - \frac{2\pi^2\alpha^2\|G\|_2^4}{c^2}e^{1+2\pi A_{x_0}} + O\left(\frac{\ln R}{c^4}\right) + O\left(\frac{\ln\ln\varepsilon}{R^2}\right).\end{aligned} \quad (51)$$

On the other hand, from the fact of $e^t \geqslant t+1$ for any $t \geqslant 0$ and (46), one gets

$$\begin{aligned}&\int_{\Sigma\setminus B_{R\varepsilon}(x_0)} e^{2\pi(v_\varepsilon-v_\varepsilon^*)^2\left(1+\alpha\|v_\varepsilon-v_\varepsilon^*\|_2^2\right)}dv_g\\ \geqslant&\int_{\Sigma\setminus B_{2R\varepsilon}(x_0)}\left(1+2\pi(v_\varepsilon-v_\varepsilon^*)^2\right)dv_g\\ \geqslant&\mathrm{Area}(\Sigma) + 2\pi\frac{\|G\|_2^2}{c^2} + O\left(\frac{\ln R}{c^4}\right) + O\left(R^2\varepsilon^2\right).\end{aligned} \quad (52)$$

It follows from (51) and (52) that

$$\begin{aligned}&\int_\Sigma e^{2\pi(v_\varepsilon-v_\varepsilon^*)^2\left(1+\alpha\|v_\varepsilon-v_\varepsilon^*\|_2^2\right)}dv_g\\ \geqslant&\mathrm{Area}(\Sigma) + \frac{\pi}{2}e^{1+2\pi A_{x_0}} + \frac{2\pi\|G\|_2^2}{c^2}\left(1-\pi\alpha^2\|G\|_2^2e^{1+2\pi A_{x_0}}\right)\\ &+O\left(\frac{\ln\ln\varepsilon}{R^2}\right) + O\left(\frac{\ln R}{c^4}\right) + O\left(R^2\varepsilon^2\right).\end{aligned}$$

According to $R=-\ln\varepsilon$ and (47), we obtain

$$F_\alpha^{2\pi}(v_\varepsilon - v_\varepsilon^*) > \mathrm{Area}(\Sigma) + \frac{\pi}{2}e^{1+2\pi A_{x_0}}. \quad (53)$$

for sufficiently small α and ε. The contradiction between (13) and (53) indicates that c_ε must be bounded when α is sufficiently small. When $|c_\varepsilon| \leqslant C$, using Lebesgue's dominated convergence, we have

$$F_\alpha^{2\pi}(u_0) = \sup_{u\in\mathscr{S}} F_\alpha^{2\pi}(u).$$

Moreover, it is easy to see $u_0 \in C^\infty\left(\overline{\Sigma}\right)\cap\mathscr{S}$ from Lemma 1 and (16). Therefore, we obtain Theorem 1 (iii).

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(Received December 2, 2020)

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