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LOWER BOUNDS FOR THE SPREAD OF A NONNEGATIVE MATRIX

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Abstract. Given an integer $n \geqslant 2$ and a real number $a \geqslant 0$, let $\mathscr{C}_n(a)$ be the collection of all nonnegative $n \times n$ matrices $A = [a_{i,j}]_{i,j=1}^n$ such that $a = \min_{1 \leqslant i \leqslant n} a_{i,i}$ and $r(A) > a$, where $r(A)$ denotes the spectral radius of A. We prove some lower bounds for the spread $s(A)$ of $A \in \mathscr{C}_n(a)$ that is defined as the maximum distance between any two eigenvalues of A. In particular, we prove that

$$s(A) > \frac{2}{2+\sqrt{2n}}(r(A)-a)$$

for all $A \in \mathscr{C}_n(a)$.

1. Introduction

Let A be a complex $n \times n$ matrix with the spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The spectral radius and the trace of A are denoted by $r(A)$ and $\mathrm{tr}\,(A)$, respectively. The spread $s(A)$ of A is the maximum distance between any two eigenvalues, that is,

$$s(A) = \max_{i,j} |\lambda_i - \lambda_j|.$$

This quantity was introduced by Mirsky [5], and it has been studied by several authors; see e.g. [4] and the references therein. Note that $s(\lambda A) = |\lambda| s(A)$ for every complex number λ.

Given an integer $n \geqslant 2$ and a real number $a \geqslant 0$, let $\mathscr{C}_n(a)$ be the collection of all nonnegative $n \times n$ matrices $A = [a_{i,j}]_{i,j=1}^n$ such that $a = \min_{1 \leqslant i \leqslant n} a_{i,i}$ and $r(A) > a$. We are searching for lower bounds for the spread of $A \in \mathscr{C}_n(a)$. In [1] we have already proved some lower bounds for the spread of $A \in \mathscr{C}_n(0)$. The present paper improves and extends some results from [1]. We will also restrict our attention to a special subset of $\mathscr{C}_n(a)$. Given an integer $n \geqslant 2$ and a real number $a \geqslant 0$, let $\mathscr{D}_n(a)$ be the collection of all matrices in $\mathscr{C}_n(a)$ having exactly two distinct eigenvalues.

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2. The case of $A \in \mathscr{C}_n(0)$

For the convenience of the reader, we first recall relevant results from [1]. We begin with [1, Proposition 2.1].

PROPOSITION 2.1. *If $A \in \mathscr{C}_n(0)$, then*

$$s(A) \geqslant \frac{1}{n} r(A) .$$

Let A be a nonnegative $n \times n$ matrix and let $s_k := \mathrm{tr}(A^k)$ for $k \in \mathbb{N}$. The JLL-inequalities (discovered independently by Loewy and London [3], and Johnson [2]) state that

$$s_k^m \leqslant n^{m-1} s_{km}$$

for all positive integers k and m. These inequalities follow easily from Hölder's inequality. A slight modification of their proof gives the following inequalities; see [1, Proposition 2.2].

PROPOSITION 2.2. *If $A \in \mathscr{C}_n(0)$, then*

$$s_1^m \leqslant (n-1)^{m-1} s_m$$

for all $m \in \mathbb{N}$.

Applying Proposition 2.2 one can show the following theorem; see [1, Theorem 2.3].

THEOREM 2.3. *If $A \in \mathscr{C}_n(0)$, then*

$$s(A) > \frac{2}{4+\sqrt{2(n+3)}} r(A)$$

for $n \geqslant 6$,

$$s(A) \geqslant \frac{5}{8+\sqrt{74}} r(A)$$

for $n = 5$, and

$$s(A) \geqslant \frac{1}{3} r(A)$$

for $n = 4$.

For $n \in \{2,3\}$ one can show sharp bounds for the spread of a matrix in $\mathscr{C}_n(0)$; see [1, Proposition 2.4].

PROPOSITION 2.4. *If $A \in \mathscr{C}_2(0)$, then $s(A) \geqslant r(A)$; if $A \in \mathscr{C}_3(0)$, then $s(A) \geqslant \frac{3}{4} r(A)$. Both bounds are sharp.*

The following sharp lower bound for the spread of a matrix in $\mathscr{D}_n(0)$ is proved in [1, Theorem 2.5].

THEOREM 2.5. *If $A \in \mathscr{D}_n(0)$, then*

$$s(A) \geqslant \frac{n}{2(n-1)} r(A).$$

Moreover, this bound is sharp, i.e., there is a (necessarily irreducible) matrix $A \in \mathscr{D}_n(0)$ such that $s(A) = \frac{n}{2(n-1)} r(A)$.

Here we recall that a nonnegative $n \times n$ matrix is irreducible, if there exists no permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices.

Using Proposition 2.2 we now improve Theorem 2.3.

THEOREM 2.6. *If $n \geqslant 3$ and $A \in \mathscr{C}_n(0)$, then*

$$s(A) > \frac{2}{2+\sqrt{2n}} r(A) \ .$$

Proof. With no loss of generality we can assume that $r(A) = 1$. Since the result is true if $s(A) \geqslant 1$, we may also assume that $s := s(A) \in [0,1)$. Let $\lambda_1 = r(A) = 1$, λ_2, λ_3, …, λ_n be the spectrum of A. By Proposition 2.2, we have

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 = s_1^2 \leqslant (n-1)s_2 = (n-1)\sum_{i=1}^{n} \lambda_i^2$$

or

$$\left(1+\sum_{i=2}^{n} \lambda_i\right)^2 \leqslant (n-1)\left(1+\sum_{i=2}^{n} \lambda_i^2\right)$$

or

$$1+2\sum_{i=2}^{n} \lambda_i + \left(\sum_{i=2}^{n} \lambda_i\right)^2 \leqslant (n-1)+(n-1)\sum_{i=2}^{n} \lambda_i^2.$$

Since

$$\sum_{i=2}^{n-1}\sum_{j=i+1}^{n} (\lambda_i-\lambda_j)^2 + \left(\sum_{i=2}^{n} \lambda_i\right)^2 = (n-1)\sum_{i=2}^{n} \lambda_i^2,$$

this inequality can be rewritten to the form

$$1+2\sum_{i=2}^{n} \lambda_i \leqslant (n-1)+\sum_{i=2}^{n-1}\sum_{j=i+1}^{n} (\lambda_i-\lambda_j)^2. \qquad (1)$$

The right-hand side of (1) is clearly at most $(n-1)+(n-1)(n-2)s^2/2$. To obtain a lower bound for the left-hand side of (1), we observe that

$$\sum_{i=2}^{n} \lambda_i = \mathrm{Re} \sum_{i=2}^{n} \lambda_i = \sum_{i=2}^{n} \mathrm{Re}\, \lambda_i \geqslant (n-1)(1-s),$$

since $\mathrm{Re}\,(1-\lambda_i) \leqslant s$, and so $\mathrm{Re}\,\lambda_i \geqslant 1-s$. Therefore, the inequality (1) gives that

$$(n-1) + \frac{(n-1)(n-2)}{2} s^2 \geqslant 1 + 2(n-1)(1-s).$$

This implies the inequality

$$2(n-1) + (n-1)(n-2)s^2 > 4(n-1)(1-s)$$

or

$$(n-2)s^2 + 4s - 2 > 0.$$

It follows that

$$s > \frac{-4+\sqrt{8n}}{2(n-2)} = \frac{2}{2+\sqrt{2n}}.$$

This completes the proof. □

The following proposition shows the lower bound for the spread of a matrix in $\mathscr{C}_4(0)$ that is better than the bound in Theorem 2.6 for $n=4$.

PROPOSITION 2.7. *If $A \in \mathscr{C}_4(0)$, then*

$$s(A) \geqslant \frac{4}{3+\sqrt{17}}\, r(A)\,.$$

Proof. As in the proof of Theorem 2.6, we can assume that $r(A)=1$ and $s := s(A) \in [0,1)$. Let $\lambda_1 = r(A) = 1$, λ_2, λ_3, and λ_4 be the spectrum of A. Then the inequality (1) gives that

$$2(\lambda_2+\lambda_3+\lambda_4) \leqslant 2 + (\lambda_2-\lambda_3)^2 + (\lambda_2-\lambda_4)^2 + (\lambda_3-\lambda_4)^2\,. \tag{2}$$

We claim that

$$(\lambda_2-\lambda_3)^2 + (\lambda_2-\lambda_4)^2 + (\lambda_3-\lambda_4)^2 \leqslant 2s^2\,.$$

Suppose first that all eigenvalues of A are real, so that we can assume that $1 = \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$. Then

$$(\lambda_2-\lambda_3)^2 + (\lambda_3-\lambda_4)^2 \leqslant ((\lambda_2-\lambda_3)+(\lambda_3-\lambda_4))^2 = (\lambda_2-\lambda_4)^2 \leqslant s^2,$$

and so the claim follows. Suppose now that two eigenvalues of A are complex, so that we can assume that $\lambda_3 = \overline{\lambda_2}$ and $\lambda_4 \in \mathbb{R}$. Then $(\lambda_2-\lambda_3)^2 = (\lambda_2-\overline{\lambda}_2)^2 < 0$ and $(\lambda_2-\lambda_4)^2 + (\lambda_3-\lambda_4)^2 \leqslant 2s^2$, and so the claim follows also in this case.

Therefore, the right-hand side of (2) is at most $2+2s^2$, and the left-hand side of (2) is at least $6(1-s)$. Consequently, we have

$$1+s^2 \geqslant 3(1-s)$$

or

$$s^2+3s-2 \geqslant 0.$$

It follows that

$$s > \frac{-3+\sqrt{17}}{2} = \frac{4}{3+\sqrt{17}},$$

completing the proof. □

3. The case of $A \in \mathscr{C}_n(a)$

We start with an easy extension of Proposition 2.1.

PROPOSITION 3.1. *Given an integer $n \geqslant 2$ and a real number $a \geqslant 0$, let $A \in \mathscr{C}_n(a)$. Then*

$$s(A) \geqslant \frac{1}{n}(r(A)-a)\,.$$

Proof. Let $B := A - aI$, where I denotes the identity matrix. Then $B \in \mathscr{C}_n(0)$. Since $r(B)$ is the Perron eigenvalue of B, $r(B)+a$ is the Perron eigenvalue of $A = B+aI$, and so $r(A) = r(B)+a$. By Proposition 2.1,

$$s(A) = s(B) \geqslant \frac{1}{n}r(B) = \frac{1}{n}(r(A)-a)\,,$$

completing the proof. □

In a similar manner we can extend Theorem 2.6, Proposition 2.4, Proposition 2.7 and Theorem 2.5.

THEOREM 3.2. *Given an integer $n \geqslant 3$ and a real number $a \geqslant 0$, let $A \in \mathscr{C}_n(a)$. Then*

$$s(A) > \frac{2}{2+\sqrt{2n}}(r(A)-a)\,.$$

Proof. It is clear that $B := A - aI \in \mathscr{C}_n(0)$, $s(A) = s(B)$ and $r(A) = r(B)+a$. By Theorem 2.6, we have

$$s(B) > \frac{2}{2+\sqrt{2n}}r(B)\,,$$

and so

$$s(A) > \frac{2}{2+\sqrt{2n}}(r(A)-a)\,. \quad \square$$

PROPOSITION 3.3. *Let a be a nonnegative number. If $A \in \mathscr{C}_2(a)$, then $s(A) \geqslant r(A) - a$. If $A \in \mathscr{C}_3(a)$, then $s(A) \geqslant \frac{3}{4}(r(A) - a)$. Both bounds are sharp.*

Proof. Let $A \in \mathscr{C}_2(a)$. Then $B := A - aI \in \mathscr{C}_2(0)$, $s(A) = s(B)$ and $r(A) = r(B) + a$. By Proposition 2.4, we have $s(B) \geqslant r(B)$, and so $s(A) \geqslant r(A) - a$. The diagonal matrix $\operatorname{diag}(a, a+1) \in \mathscr{C}_2(a)$ shows that this lower bound can be achieved.

Similarly, we can prove the second assertion of the proposition. To prove that the bound is sharp, we define a matrix

$$A = \begin{bmatrix} a & 2 & 0 \\ 0 & a+3 & 1 \\ 2 & 0 & a+3 \end{bmatrix} \in \mathscr{C}_3(a).$$

Its spectrum is equal to $\{a+4, a+1, a+1\}$, so that $s(A) = 3$ and $r(A) = a+4$. □

PROPOSITION 3.4. *Let a be a nonnegative number. If $A \in \mathscr{C}_4(a)$, then*

$$s(A) \geqslant \frac{4}{3+\sqrt{17}}(r(A) - a).$$

THEOREM 3.5. *Let a be a nonnegative number and $n \geqslant 2$ an integer. If $A \in \mathscr{D}_n(a)$, then*

$$s(A) \geqslant \frac{n}{2(n-1)}(r(A) - a).$$

Moreover, this bound can be achieved, i.e., there is a (necessarily irreducible) matrix $A_0 \in \mathscr{D}_n(a)$ such that $s(A_0) = \frac{n}{2(n-1)}(r(A_0) - a)$.

Proof. Let $A \in \mathscr{D}_n(a)$. Then $B := A - aI \in \mathscr{D}_n(0)$, $s(A) = s(B)$ and $r(A) = r(B) + a$. Now, the desired lower bound for the spread follows from Theorem 2.5.

To show that the lower bound can be achieved, as in [1] we define the matrix $A = [a_{i,j}]_{i,j=1}^n$ with nonzero elements: $a_{i,i+1} = n - i$ for $i = 1, 2, \ldots, n-1$, $a_{i,i} = n$ for $i = 2, 3, \ldots, n$, and $a_{i,j} = 2$ if $i - j$ is an even positive integer. We also introduce the upper triangular matrix $U = [u_{i,j}]_{i,j=1}^n$ with nonzero elements: $u_{i,i+1} = n - i$ for $i = 1, 2, \ldots, n-1$, $u_{1,1} = 2(n-1)$ and $u_{i,i} = n-2$ for $i = 2, 3, \ldots, n$. It is shown in [1] that A and U are similar matrices. Put $A_0 := A + aI \in \mathscr{D}_n(a)$. Then $r(A_0) = r(A) + a = r(U) + a = 2(n-1) + a$ and $s(A_0) = s(A) = s(U) = n$. □

Using the matrix A from the last proof we can show the following result.

PROPOSITION 3.6. *Given an integer $n \geqslant 2$ and real numbers $a \geqslant 0$ and $d > 0$, there exists a matrix in $\mathscr{D}_n(a)$ the spectrum of which is contained in the interval $[a + (n-2)d, a + 2(n-1)d]$.*

Proof. Let us keep the notation of the last proof. Then $B := dA + aI \in \mathscr{D}_n(a)$ and the spectrum of B is equal to $\{a + (n-2)d, a + 2(n-1)d\}$. □

It would be interesting to know the following infimum

$$k_n := \inf\left\{ \frac{s(A)}{r(A)} : A \in \mathscr{C}_n(0) \right\}.$$

By Theorem 2.6, we have $k_n \geqslant \frac{2}{2+\sqrt{2n}}$.

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