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A PICONE'S IDENTITY FOR THE $p(x)$–LAPLACE EQUATIONS AND ITS APPLICATIONS

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Abstract. In this paper, we present a Picone identity for the $p(x)$-Laplace equations and establish some applications of this formula, such as Caccioppoli inequalities, nonexistence of positive supersolutions, domain monotonicity property, uniqueness and simplicity of the first eigenvalue, Hardy type inequality, Barta-type inequality, a nonlinear system with singular nonlinearity and Sturmian comparison theorem for the $p(x)$-Laplace equations.

1. Introduction

Over the past 20 years, the differential equations and variational problems with non-standard growth conditions and the corresponding function spaces with variable exponents have been a very attractive field. We can refer to the book [1] and the survey papers [2] for a quite comprehensive bibliography on this topics. These investigations are stimulated mainly by the development of the study of electrorheological fluids [3], image restoration [4] and the theory of nonlinear elasticity [5]–[7]. In this paper, our work is closed with this subject.

If u and v are differentiable functions such that $u \geqslant 0$ and $v > 0$, then

$$|\nabla u|^2 - \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geqslant 0. \tag{1}$$

This formula is known as Picone's identity which was first established by Picone (see [8, 9]). Since the classical Picone's identity was introduced, its extensions and applications have been extensively investigated. For instance, we refer to Kreith [10, 11], Swanson [12, 13] for linear differential equations, and Berestycki, Capuzzo-Dolcetta and Nirenberg [14] who proved a generalized Picone's identity in order to study indefinite superlinear elliptic problems. Tyagi [15] also proved a nonlinear analogue of classical Picone's identity (1) and gave its applications. In order to apply Picone's

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identity to p-Laplace equations, the classical Picone's identity (1) is extended to the following formula by Allegretto and Huang [16] (see also [17]).

$$\begin{aligned} &|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla u|^{p-2}\nabla u \nabla v \\ &= |\nabla u|^p - |\nabla v|^{p-2}\nabla\left(\frac{u^p}{v^{p-1}}\right)\nabla v, \end{aligned} \tag{2}$$

where $u \geqslant 0$ and $v > 0$. Bal [19] also extended Tyagi's in [15] to deal with p-Laplace equations. Dwivedi [23] gave a generalized nonlinear Picone's identity for the p-Laplacian. Jaroš established Picone's identity for Finsler p-Laplacian [20] and A-harmonic operators [21]. Niu, Zhang and Luo obtained Picone's identity for p-Laplace equations on Heisenberg group in [27]. Lately, Dwivedi and Tyagi [24] proved a nonlinear a nonlinear analogue of classical Picone's identity for biharmonic operator on Heisenberg group and dealt with some applications. Picone's identities have played an important role in the study of qualitative properties of solutions of differential equations. Its applications are mainly focused on follows: Caccioppoli inequality, Morse index and Hardy type inequality, nonexistence of positive supersolutions, uniqueness and simplicity of the first eigenvalue, domain monotonicity property of the first eigenvalue, Barta-type inequality, Sturmian comparison and oscillation etc. For these applications, we can also refer the reader to [18, 22, 25, 26, 28, 29, 30, 31, 32, 33].

In Picone's identity (2), p is a positive constant. If p is a variable exponent, that is, p is a positive function of variable x. whether is there a formula similar to (2)? In other words, can one establish Picone's identity for the $p(x)$-Laplacian? The related questions had already been discussed in a few articals (see [34, 35, 36, 38, 39, 40, 41, 49]). It is noted that $p(x)$-Laplace equations haven't the homogeneity, that is, any constant multiple of any non-zero solution of $p(x)$-Laplace equation isn't its solution. However, p-Laplace equations possess this property, and this property plays an important role in the constant exponents case. In order to remove this "drawback" and establish Picone's identity, Yoshida [34] (see also [35, 37, 39]) introduced an extra term "$a(x)(\ln|u|)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla p(x)$" in quasilinear elliptic equations with $p(x)$-Laplacians (or in (3.1a)). It is the purpose of this paper to establish Picone's identity for the $p(x)$-Laplace equations without this extra term, which is an extension of the identity (2) for the p-Laplace equations.

The paper is organized as follows. In Section 2, we state some preliminaries about the variable exponent Lebesgue and Sobolev spaces. In Section 3, the more general Picone's identity for the $p(x)$-Laplace equations is proved. Section 4 we give some applications of this identity.

2. Notations and some preliminary results

In order to deal with the variable exponent problems, we need recall the theory of variable exponent Lebesgue and Sobolev spaces (see [1, 43, 44]).

Throughout this paper, let $\Omega \subset \mathbb{R}^N$, with $N \geqslant 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For any continuous function $p:\overline{\Omega}\to(1,\infty)$ we denote

$$1<p^{-}:=\inf_{x\in\Omega}p(x)\leqslant p(x)\leqslant \sup_{x\in\Omega}p(x):=p^{+}<N \text{ for all } x\in\Omega. \tag{3}$$

In the following we introduce the most important condition on the exponent in the study of variable exponent spaces, the log-Hölder continuity condition.

DEFINITION 1. A function $\alpha:\Omega\to\mathbb{R}$ is locally log-Hölder continuous in Ω if there exists $c_1>0$ such that

$$|\alpha(x)-\alpha(y)|\leqslant\frac{c_1}{\log(e+1/|x-y|)}$$

for all $x,y\in\Omega$, where the constant c_1 is called the local log-Hölder constant.

DEFINITION 2. α satisfies the log-Hölder decay condition if there exist $\alpha_\infty\in\mathbb{R}$ and a constant $c_2>0$ such that

$$|\alpha(x)-\alpha_\infty|\leqslant\frac{c_2}{\log(e+|x|)}$$

for all $x\in\Omega$, where the constant c_2 is called the log-Hölder decay constant.

A function α is globally log-Hölder continuous in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

From then on, assume that

$$p\in\left\{p\,\Big|\,\frac{1}{p}\text{ is globally log-Hölder continuous in}\,\Omega\right\}, \tag{4}$$

and denote

$$\mathscr{P}(\Omega)=\left\{p:p\in C(\bar{\Omega})\text{ such that (3) and (4) hold}\right\}.$$

Define the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega)=\left\{u|u\,\text{is a measurable function in}\,\Omega\text{ such that}\int_\Omega|u(x)|^{p(x)}dx<+\infty\right\}.$$

This space is equipped with the Luxemburg norm

$$|u|_{p(x)}=\inf\left\{\lambda>0;\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\frac{dx}{p(x)}\leqslant 1\right\}.$$

Let the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ defined by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\right\}$$

with the norm

$$\|u\|_{p(x)}=|u|_{p(x)}+|\nabla u|_{p(x)},$$

which is equivalent to

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0; \int_{\Omega} \left[\left|\frac{u}{\lambda}\right|^{p(x)} + \left|\frac{\nabla u}{\lambda}\right|^{p(x)}\right] \frac{dx}{p(x)} \leqslant 1\right\}.$$

PROPOSITION 1. *Both* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *and* $(W^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ *are separable, reflexive and uniformly convex Banach spaces. Moreover, the space* $W_0^{1,p(x)}(\Omega)$ *is defined as the closure of* $C_0^{\infty}(\Omega)$ *in* $W_0^{1,p(x)}(\Omega)$.

PROPOSITION 2. *Assume that* $1 < p(x) < +\infty$ *for all* $x \in \Omega$, $a, b \geqslant 0$ *and* $\tau > 0$, *then the following Young's inequality holds*

$$ab^{p(x)-1} \leqslant \frac{1}{p(x)\tau^{p(x)-1}} a^{p(x)} + \frac{p(x)-1}{p(x)} \tau b^{p(x)}. \tag{5}$$

Moreover, the equality holds if and only if $a = b\tau$.

PROPOSITION 3. *Let* $\rho_{p(x)}(u) = \int_{\Omega}[|u|^{p(x)} + |\nabla u|^{p(x)}]\frac{dx}{p(x)}$. *For any* $u, u_k \in W^{1,p(x)}(\Omega)$ $(k = 1, 2, \ldots)$, *we have*

1. $\|u\|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (= 1; > 1)$;
2. $\|u\|_{p(x)} \leqslant 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leqslant \rho_{p(x)}(u) \leqslant \|u\|_{p(x)}^{p^-}$;
3. $\|u\|_{p(x)} \geqslant 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leqslant \rho_{p(x)}(u) \leqslant \|u\|_{p(x)}^{p^+}$;
4. $\|u_k - u\|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_k - u) \to 0$.

3. Picone's identity for the $p(x)$-Laplace equations

In this section, we will establish Picone's identity for the following $p(x)$-Laplace equations

$$\begin{cases} (3.1a)\ q[u] := \operatorname{div}\left(a(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + c(x)\,|u|^{p(x)-2}\,u = 0, \\ (3.1b)\ Q[u] := \operatorname{div}\left(A(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + C(x)\,|u|^{p(x)-2}\,u = 0, \end{cases} \tag{6}$$

where $a(x)$, $A(x) \in C^1(\Omega; (0, +\infty)) \cap C(\overline{\Omega}; (0, +\infty))$ and $c(x)$, $C(x) \in C(\overline{\Omega}; \mathbb{R})$.

The main result of this section is as follows.

THEOREM 1. (Picone's identity) *Assume that u and v are C^2 differentiable functions defined in $\Omega \subset \mathbb{R}^N$ with $v \neq 0$ in Ω, then we have*

$$\begin{aligned} &\operatorname{div}\left(u\, a(x)\,|\nabla u|^{p(x)-2}\nabla u - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}A(x)\,|\nabla v|^{p(x)-2}\nabla v\right) \\ &= A(x)\,\Phi(u,v) - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\left(\ln\left|\frac{u}{v}\right|\right)A(x)\,|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p \\ &\quad + (a(x)-A(x))\,|\nabla u|^{p(x)} - (c(x)-C(x))\,|u|^{p(x)} + uq[u] - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}Q[v], \end{aligned} \tag{7}$$

where

$$\begin{aligned} \Phi(u,v) = |\nabla u|^{p(x)} - p(x)\frac{|u|^{p(x)-2}u}{|v|^{p(x)-2}v}\,|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u \\ + (p(x)-1)\left|\frac{u\nabla v}{v}\right|^{p(x)}, \end{aligned} \tag{8}$$

and $\Phi(u,v) \geqslant 0$ in Ω. Moreover, $\Phi(u,v)=0$ a.e. in Ω if and only if $u = kv,\ k \in \mathbb{R}$ in Ω.

Proof. A direct calculation yields

$$\nabla\left(\frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\right) = p(x)\frac{|u|^{p(x)-2}u}{|v|^{p(x)-2}v}\nabla u - (p(x)-1)\left|\frac{u}{v}\right|^{p(x)}\nabla v + \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\left(\ln\left|\frac{u}{v}\right|\right)\nabla p.$$

It follows from (3.1a) and (3.2a) that

$$\begin{aligned} &\operatorname{div}\left(u\, a(x)\,|\nabla u|^{p(x)-2}\nabla u - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}A(x)\,|\nabla v|^{p(x)-2}\nabla v\right) \\ &= a(x)\,|\nabla u|^{p(x)} + u\operatorname{div}\left(a(x)\,|\nabla u|^{p(x)-2}\nabla u\right) \\ &\quad - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\operatorname{div}\left(A(x)\,|\nabla v|^{p(x)-2}\nabla v\right) - \nabla\left(\frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\right)\left(A(x)\,|\nabla v|^{p(x)-2}\nabla v\right) \\ &= a(x)\,|\nabla u|^{p(x)} + uq[u] - c(x)\,|u|^{p(x)} - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}Q[v] + C(x)\,|u|^{p(x)} \\ &\quad + (p(x)-1)\left|\frac{u\nabla v}{v}\right|^{p(x)} - p(x)\frac{|u|^{p(x)-2}u}{|v|^{p(x)-2}v}A(x)\,|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u \\ &\quad - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\left(\ln\left|\frac{u}{v}\right|\right)A(x)\,|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p, \end{aligned}$$

which yields the desired identity (7).

Then, using Young's inequality (5), here replacing a with $|\nabla u|$, b with $\left|\frac{u\nabla v}{v}\right|$ and τ with 1, it follows

$$|\nabla u|\left|\frac{u\nabla v}{v}\right|^{p(x)-1} \leqslant \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{p(x)-1}{p(x)}\left|\frac{u\nabla v}{v}\right|^{p(x)}. \tag{9}$$

Therefore,

$$\begin{aligned}
\Phi(u,v) &= |\nabla u|^{p(x)} - p(x)\frac{|u|^{p(x)-2}u}{|v|^{p(x)-2}v}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u + (p(x)-1)\left|\frac{u\nabla v}{v}\right|^{p(x)} \\
&= |\nabla u|^{p(x)} - p(x)|\nabla u|\left|\frac{u\nabla v}{v}\right|^{p(x)-1} + (p(x)-1)\left|\frac{u\nabla v}{v}\right|^{p(x)} \\
&\quad + p(x)\frac{|u\nabla v|^{p(x)-2}}{|v|^{p(x)-2}}\left(\left|\frac{u}{v}\nabla v\right||\nabla u| - \frac{u}{v}\nabla v\cdot\nabla u\right) \\
&\geqslant 0
\end{aligned}$$

Now, on one hand, if $\Phi(u,v)(x_0)=0$ and $u(x_0)\neq 0$, we must have $|\nabla u| = \left|\frac{u\nabla v}{v}\right|$ and $\left|\frac{u\nabla v}{v}\right||\nabla u| = \frac{u\nabla v}{v}\cdot\nabla u$, i.e. $\nabla\left(\frac{u}{v}\right)=0$. On the other hand, if $S=\{x\in\Omega : u(x)=0\}$, then $\nabla u = 0$ a.e. in S (see [48]), and thus $\nabla\left(\frac{u}{v}\right)=0$ a.e. in S. We conclude that $\nabla\left(\frac{u}{v}\right)=0$ a.e. in Ω and consequently $u=kv$ for some constant k. $\square$

COROLLARY 1. *If the assumptions of Theorem 1 hold, $c(x)=C(x)=0$ and $a(x)=A(x)=1$, then Picone's identity (7) becomes*

$$\begin{aligned}
&|\nabla u|^{p(x)} - |\nabla v|^{p(x)-2}\nabla v\cdot\nabla\left(\frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\right) \\
&= \Phi(u,v) - \frac{|u|^{p(x)}}{|v|^{p(x)-2}v}\left(\ln\left|\frac{u}{v}\right|\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p,
\end{aligned} \tag{10}$$

where $\Phi(u,v)$ is the same as in (8).

REMARK 1. As $p(x)\equiv p$, and $u\geqslant 0$, $v>0$, (10) reduces to (2).

4. Applications

In this section, we give some applications for the $p(x)$-Laplace equations by using Picone's identity (10).

4.1. Caccioppoli inequalities

Consider the problem

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = g(x)|u|^{p(x)-2}u, \tag{11}$$

where $0 \leqslant g(x) \in L^{\infty}_{loc}(\Omega)$.

$u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ is called a weak solution of problem (11) if it satisfies

$$\int_{\Omega} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \eta \, dx = \int_{\Omega} g(x) |v|^{p(x)-2} v\eta \, dx, \forall \eta \in W^{1,p(x)}_{0}(\Omega) \cap C(\Omega). \tag{12}$$

Notice that if v is a solution, so is $|v|$ and $|v| \geqslant 0$. In view of Theorem 4.1 in [45] and Theorem 5.3 in [2], we obtain $v > 0$ in Ω.

Similarly, weak supersolutions and subsolutions of problem (11) can also be defined.

The following two theorems establish Caccioppoli-type estimates for positive sub- and supersolutions of nonlinear equations involving $p(x)$-Laplace operators.

THEOREM 2. *Assume that $0 < u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ is a weak subsolution of (11), $p(x), q(x)$ are differentiable functions and $q(x) > p(x) - 1$ pointwise for all $x \in \Omega$. Then the inequality*

$$\begin{aligned} &\int_{\Omega} \frac{q(x)-p(x)+1}{p(x)} \left| \varphi u^{\frac{q(x)}{p(x)}-1} \nabla u \right|^{p(x)} dx \\ &\leqslant \int_{\Omega} g(x) u^{q(x)} \varphi^{p(x)} dx + \int_{\Omega} \left(\frac{p(x)}{q(x)-p(x)+1} \right)^{p(x)-1} u^{q(x)} |\nabla \varphi|^{p(x)} dx \\ &\quad + \int_{\Omega} \varphi^{p(x)} u^{q(x)-p(x)+1} \left[p(x) |\ln u| |\nabla u|^{p(x)-1} \left| \nabla \left(\frac{q(x)}{p(x)} \right) \right| \right. \\ &\qquad \left. + \left| \ln \left(\varphi u^{\frac{q(x)}{p(x)}-1} \right) \right| |\nabla u|^{p(x)-1} |\nabla p(x)| \right] dx, \end{aligned} \tag{13}$$

holds for all $0 \leqslant \varphi \in C^{\infty}_{0}(\Omega)$.

Proof. Let $\varepsilon > 0$. Then, integrating Picone's identity (10), here replacing u with $(u+\varepsilon)^{\frac{q(x)}{p(x)}} \varphi$ and v with $u+\varepsilon$ in Ω, it follows

$$I_1 = I_2, \tag{14}$$

where

$$\begin{aligned} I_1 &= \int_{\Omega} \left| \nabla ((u+\varepsilon)^{\frac{q(x)}{p(x)}} \varphi) \right|^{p(x)} dx \\ &\qquad - \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \left((u+\varepsilon)^{q(x)-p(x)+1} \varphi^{p(x)} \right) dx, \\ I_2 &= \int_{\Omega} \left| \nabla ((u+\varepsilon)^{\frac{q(x)}{p(x)}} \varphi) \right|^{p(x)} dx \\ &\quad - \int_{\Omega} p(x) \, \varphi^{p(x)-1} (u+\varepsilon)^{(p(x)-1)(\frac{q(x)}{p(x)}-1)} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla ((u+\varepsilon)^{\frac{q(x)}{p(x)}} \varphi) dx \end{aligned}$$

$$+\int_\Omega (p(x)-1)\left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)}dx$$
$$-\int_\Omega \varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla p(x)\,dx.$$

Since $0\leqslant\varphi\in C_0^\infty(\Omega)$, $0<u\in W^{1,p(x)}_{loc}(\Omega)\cap C(\Omega)$ is a weak subsolution of (11) and

$$\begin{aligned}&\nabla\left((u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}\right)\\&=(u+\varepsilon)^{q(x)-p(x)+1}\nabla(\varphi^{p(x)})+(q(x)-p(x)+1)\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)}\nabla u\\&\quad+\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}\ln(u+\varepsilon)\nabla(q(x)-p(x)+1),\end{aligned}$$

we can choose $(u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}$ as a test function and obtain that

$$\begin{aligned}&\int_\Omega g(x)u^{p(x)-1}(u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}dx\\&\quad\geqslant\int_\Omega|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\left((u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}\right)dx\\&\quad=\int_\Omega\left|\nabla((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi)\right|^{p(x)}dx-I_1.\end{aligned}\tag{15}$$

Since

$$\begin{aligned}&\nabla((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi)\\&=(u+\varepsilon)^{\frac{q(x)}{p(x)}}\nabla\varphi+\frac{q(x)}{p(x)}\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u+\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}}\ln(u+\varepsilon)\nabla\left(\frac{q(x)}{p(x)}\right),\end{aligned}$$

we have

$$\begin{aligned}I_2=&\int_\Omega\left|\nabla((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi)\right|^{p(x)}dx\\&-\int_\Omega p(x)\,\varphi^{p(x)-1}(u+\varepsilon)^{q(x)-p(x)+1}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi dx\\&-\int_\Omega q(x)\,\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)}|\nabla u|^{p(x)}dx\\&-\int_\Omega p(x)\,\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}\ln(u+\varepsilon)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\left(\frac{q(x)}{p(x)}\right)dx\\&+\int_\Omega(p(x)-1)\left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)}dx\\&-\int_\Omega\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla p(x)\,dx,\end{aligned}\tag{16}$$

then an easy calculation shows that

$$
\begin{aligned}
I_2 \leqslant & \int_\Omega \left|\nabla\left((u+\varepsilon)^{\frac{q(x)}{p(x)}} \varphi\right)\right|^{p(x)} dx \\
& + \int_\Omega p(x) \left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1} \nabla u\right|^{p(x)-1} (u+\varepsilon)^{\frac{q(x)}{p(x)}} |\nabla\varphi| \, dx \\
& + \int_\Omega (p(x)-1-q(x)) \left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1} \nabla u\right|^{p(x)} dx \\
& + \int_\Omega p(x)\ \varphi^{p(x)} (u+\varepsilon)^{q(x)-p(x)+1} \left|\ln(u+\varepsilon)\right| |\nabla u|^{p(x)-1} \left|\nabla\left(\frac{q(x)}{p(x)}\right)\right| dx \\
& + \int_\Omega \varphi^{p(x)} (u+\varepsilon)^{q(x)-p(x)+1} \left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right| |\nabla u|^{p(x)-1} |\nabla p(x)| \, dx.
\end{aligned} \tag{17}
$$

By using Young's inequality (5) with

$$
\begin{cases}
a = (u+\varepsilon)^{\frac{q(x)}{p(x)}} |\nabla\varphi|, \\
b = \varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1} |\nabla u|, \\
0 < \tau < \frac{q(x)-p(x)+1}{p(x)-1},
\end{cases}
$$

we have

$$
\begin{aligned}
I_2 \leqslant & \int_\Omega \left|\nabla\left((u+\varepsilon)^{\frac{q(x)}{p(x)}} \varphi\right)\right|^{p(x)} dx + \int_\Omega \frac{1}{\tau^{p(x)-1}} (u+\varepsilon)^{q(x)} |\nabla\varphi|^{p(x)} \, dx \\
& + \int_\Omega (p(x)-1-q(x)+(p(x)-1)\tau) \left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1} \nabla u\right|^{p(x)} dx \\
& + \int_\Omega p(x)\ \varphi^{p(x)} (u+\varepsilon)^{q(x)-p(x)+1} \left|\ln(u+\varepsilon)\right| |\nabla u|^{p(x)-1} \left|\nabla\left(\frac{q(x)}{p(x)}\right)\right| dx \\
& + \int_\Omega \varphi^{p(x)} (u+\varepsilon)^{q(x)-p(x)+1} \left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right| |\nabla u|^{p(x)-1} |\nabla p(x)| \, dx.
\end{aligned} \tag{18, 19}
$$

Therefore, (14)–(17) imply that

$$
\int_\Omega (q(x)-p(x)+1-(p(x)-1)\tau) \left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1} \nabla u\right|^{p(x)} dx \tag{20}
$$

$$
\begin{aligned}
\leqslant & \int_\Omega g(x) u^{p(x)-1} (u+\varepsilon)^{q(x)-p(x)+1} \varphi^{p(x)} dx \\
& + \int_\Omega \frac{1}{\tau^{p(x)-1}} (u+\varepsilon)^{q(x)} |\nabla\varphi|^{p(x)} \, dx \\
& + \int_\Omega p(x)\ \varphi^{p(x)} (u+\varepsilon)^{q(x)-p(x)+1} \left|\ln(u+\varepsilon)\right| |\nabla u|^{p(x)-1} \left|\nabla\left(\frac{q(x)}{p(x)}\right)\right| dx
\end{aligned} \tag{21}
$$

$$+\int_{\Omega}\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}\left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla u|^{p(x)-1}|\nabla p(x)|\,dx.$$

Choosing $\tau=\frac{q(x)-p(x)+1}{p(x)}$ in (18) leads to

$$\begin{aligned}
&\int_{\Omega}\frac{q(x)-p(x)+1}{p(x)}\left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)}dx\\
&\leqslant\int_{\Omega}g(x)u^{p(x)-1}(u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}dx\\
&\quad+\int_{\Omega}\left(\frac{p(x)}{q(x)-p(x)+1}\right)^{p(x)-1}(u+\varepsilon)^{q(x)}|\nabla\varphi|^{p(x)}dx\\
&\quad+\int_{\Omega}p(x)\;\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}|\ln(u+\varepsilon)||\nabla u|^{p(x)-1}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|dx\\
&\quad+\int_{\Omega}\varphi^{p(x)}(u+\varepsilon)^{q(x)-p(x)+1}\left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla u|^{p(x)-1}|\nabla p(x)|\,dx.
\end{aligned}$$

Passing to the limit as $\varepsilon\to 0$ and using the Lebesgue dominated convergence theorem on the right hand side and Fatou's lemma on the left hand side, we obtain

$$\begin{aligned}
&\int_{\Omega}\frac{q(x)-p(x)+1}{p(x)}\left|\varphi u^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)}dx\\
&\leqslant\int_{\Omega}g(x)u^{q(x)}\varphi^{p(x)}dx+\int_{\Omega}\left(\frac{p(x)}{q(x)-p(x)+1}\right)^{p(x)-1}u^{q(x)}|\nabla\varphi|^{p(x)}dx\\
&\quad+\int_{\Omega}\varphi^{p(x)}u^{q(x)-p(x)+1}\Bigg[p(x)|\ln u||\nabla u|^{p(x)-1}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|\\
&\quad+\left|\ln\left(\varphi u^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla u|^{p(x)-1}|\nabla p(x)|\Bigg]dx,
\end{aligned}$$

as claimed. □

COROLLARY 2. *If the assumptions of Theorem 2 hold. Then we have*

$$\begin{aligned}
&\int_{\Omega}\frac{q(x)-p(x)+1}{p(x)}\left|\varphi u^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)}dx\\
&\leqslant\int_{\Omega}g(x)u^{q(x)}\varphi^{p(x)}dx+\int_{\Omega}\left(\frac{p(x)+2}{q(x)-p(x)+1}\right)^{p(x)-1}u^{q(x)}|\nabla\varphi|^{p(x)}dx\\
&\quad+\int_{\Omega}\left(\frac{p(x)+2}{q(x)-p(x)+1}\right)^{p(x)-1}\varphi^{p(x)}u^{q(x)}\\
&\qquad\Bigg[p(x)^{p(x)-1}|\ln u|^{p(x)}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|^{p(x)}\\
&\qquad+\frac{1}{p(x)}\left|\ln\left(\varphi u^{\frac{q(x)}{p(x)}-1}\right)\right|^{p(x)}|\nabla p(x)|^{p(x)}\Bigg]dx.
\end{aligned}\tag{22}$$

Proof. (16) yields

$$
\begin{aligned}
I_2 \leqslant & \int_\Omega \left|\nabla\left((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi\right)\right|^{p(x)} dx \\
&+ \int_\Omega p(x)\left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)-1}(u+\varepsilon)^{\frac{q(x)}{p(x)}}|\nabla\varphi|\,dx \\
&+ \int_\Omega (p(x)-1-q(x))\left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx \\
&+ \int_\Omega p(x)\left(\frac{1}{p(x)}\right)^{\frac{p(x)-1}{p(x)}}\left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)-1} \\
&\qquad\qquad p(x)^{\frac{p(x)-1}{p(x)}}\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}}|\ln(u+\varepsilon)|\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right| dx \\
&+ \int_\Omega \left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)-1}\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}} \\
&\qquad\qquad \left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla p(x)|\,dx.
\end{aligned}
\tag{23}
$$

Using Young's inequality (5) in the case of $0<\tau<\frac{p(x)(q(x)-p(x)+1)}{(p(x)-1)(p(x)+2)}$ yields

$$
\begin{aligned}
I_2 \leqslant & \int_\Omega \left|\nabla\left((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi\right)\right|^{p(x)} dx \\
&+ \int_\Omega \left(p(x)-1-q(x)+\tau(p(x)-1)\frac{p(x)+2}{p(x)}\right) \\
&\qquad\qquad \left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx \\
&+ \int_\Omega \frac{1}{\tau^{p(x)-1}}(u+\varepsilon)^{q(x)}|\nabla\varphi|^{p(x)}\,dx \\
&+ \int_\Omega \frac{1}{\tau^{p(x)-1}}p(x)^{p(x)-1}\varphi^{p(x)}(u+\varepsilon)^{q(x)}|\ln(u+\varepsilon)|^{p(x)}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|^{p(x)} dx \\
&+ \int_\Omega \frac{1}{p(x)\tau^{p(x)-1}}\varphi^{p(x)}(u+\varepsilon)^{q(x)}\left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right|^{p(x)}|\nabla p(x)|^{p(x)}\,dx.
\end{aligned}
\tag{24}
$$

Choosing $\tau=\frac{q(x)-p(x)+1}{p(x)+2}$ in (24) and taking into account (14) and (15) lead to

$$
\begin{aligned}
&\int_\Omega \frac{q(x)-p(x)+1}{p(x)}\left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx \\
&\leqslant \int_\Omega g(x)u^{p(x)-1}(u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}dx \\
&\quad + \int_\Omega \left(\frac{p(x)+2}{q(x)-p(x)+1}\right)^{p(x)-1}(u+\varepsilon)^{q(x)}|\nabla\varphi|^{p(x)}\,dx
\end{aligned}
$$

$$+\int_\Omega \left(\frac{p(x)+2}{q(x)-p(x)+1}\right)^{p(x)-1} \varphi^{p(x)}(u+\varepsilon)^{q(x)}$$
$$\left[p(x)^{p(x)-1}|\ln(u+\varepsilon)|^{p(x)}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|^{p(x)}\right.$$
$$\left.+\frac{1}{p(x)}\left|\ln\left(\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right|^{p(x)}|\nabla p(x)|^{p(x)}\right]dx.$$

Letting $\varepsilon \to 0$ and using the Lebesgue dominated convergence theorem and Fatou's lemma, we conclude that

$$\begin{aligned}
&\int_\Omega \frac{q(x)-p(x)+1}{p(x)}\left|\varphi u^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx\\
&\leqslant \int_\Omega g(x)u^{q(x)}\varphi^{p(x)}dx + \int_\Omega \left(\frac{p(x)+2}{q(x)-p(x)+1}\right)^{p(x)-1} u^{q(x)}|\nabla\varphi|^{p(x)}dx\\
&\quad+\int_\Omega \left(\frac{p(x)+2}{q(x)-p(x)+1}\right)^{p(x)-1}\varphi^{p(x)}u^{q(x)}\left[p(x)^{p(x)-1}|\ln u|^{p(x)}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|^{p(x)}\right.\\
&\quad\left.+\frac{1}{p(x)}\left|\ln\left(\varphi u^{\frac{q(x)}{p(x)}-1}\right)\right|^{p(x)}|\nabla p(x)|^{p(x)}\right]dx. \quad \square
\end{aligned}$$

REMARK 2. As $p(x) \equiv p$ and $q(x) \equiv q$, then Theorem 2 reduces to Caccioppoli-type estimate for positive subsolution of p-Laplace equation (see [22]).

An analogous result as Theorem 2 above holds for positive supersolutions of problem (11).

THEOREM 3. *Assume that $0 < u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ is a weak supersolution of (11), $p(x), q(x)$ are differentiable functions and $q(x) < p(x)-1$ pointwise for all $x \in \Omega$. Then*

$$\begin{aligned}
&\int_\Omega \frac{p(x)-1-q(x)}{p(x)}\left|\varphi u^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx\\
&\leqslant -\int_\Omega g(x)u^{q(x)}\varphi^{p(x)}dx\\
&\quad+\int_\Omega \left(\frac{p(x)}{p(x)-1-q(x)}\right)^{p(x)-1} u^{q(x)}|\nabla\varphi|^{p(x)}dx\\
&\quad+\int_\Omega \varphi^{p(x)}u^{q(x)-p(x)+1}\left[p(x)|\ln u||\nabla u|^{p(x)-1}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|\right.\\
&\quad\left.+\left|\ln\left(\varphi u^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla u|^{p(x)-1}|\nabla p(x)|\right]dx,
\end{aligned} \tag{25}$$

holds for all $0 \leqslant \varphi \in C_0^\infty(\Omega)$.

Proof. (16) yields

$$I_2 \geqslant \int_\Omega \left|\nabla((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi)\right|^{p(x)} dx \tag{26}$$

$$\begin{aligned}
&- \int_\Omega p(x)\left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)-1}(u+\varepsilon)^{\frac{q(x)}{p(x)}}|\nabla\varphi|\,dx \\
&+ \int_\Omega (p(x)-1-q(x))\left|\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx \\
&- \int_\Omega p(x)\left(\frac{1}{p(x)}\right)^{\frac{p(x)-1}{p(x)}}\left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)-1} \\
&\qquad p(x)^{\frac{p(x)-1}{p(x)}}\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}}|\ln(u+\varepsilon)|\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right| dx \\
&- \int_\Omega \left|\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)-1}\varphi(u+\varepsilon)^{\frac{q(x)}{p(x)}} \\
&\qquad \left|\ln\left(\varphi\,(u+\varepsilon)^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla p(x)|\,dx.
\end{aligned} \tag{27}$$

Since $0 < u \in W^{1,p(x)}_{loc}(\Omega)\cap C(\Omega)$ is a weak supersolution of (11), then

$$\begin{aligned}
&\int_\Omega g(x)u^{p(x)-1}(u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}dx \\
&\qquad \leqslant \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\left((u+\varepsilon)^{q(x)-p(x)+1}\varphi^{p(x)}\right)dx \\
&\qquad = \int_\Omega \left|\nabla((u+\varepsilon)^{\frac{q(x)}{p(x)}}\varphi)\right|^{p(x)} dx - I_1.
\end{aligned} \tag{28}$$

Using the same argument as in the proof of Theorem 2 and taking into account (14), (27) and (28), we have

$$\begin{aligned}
&\int_\Omega \frac{p(x)-1-q(x)}{p(x)}\left|\varphi u^{\frac{q(x)}{p(x)}-1}\nabla u\right|^{p(x)} dx \\
&\leqslant -\int_\Omega g(x)u^{q(x)}\varphi^{p(x)}dx + \int_\Omega\left(\frac{p(x)}{p(x)-1-q(x)}\right)^{p(x)-1}u^{q(x)}|\nabla\varphi|^{p(x)}dx \\
&\quad + \int_\Omega \varphi^{p(x)}u^{q(x)-p(x)+1}\left[p(x)\,|\ln u|\,|\nabla u|^{p(x)-1}\left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|\right. \\
&\quad \left. + \left|\ln\left(\varphi u^{\frac{q(x)}{p(x)}-1}\right)\right||\nabla u|^{p(x)-1}|\nabla p(x)|\right]dx.
\end{aligned}$$

Thus the proof is completed. □

Using the same argument as in the proof of Corollary 2, we obtain

COROLLARY 3. *If the assumptions of Theorem 3 hold. Then we have*

$$\int_\Omega \frac{p(x)-1-q(x)}{p(x)} \left|\varphi u^{\frac{q(x)}{p(x)}-1} \nabla u\right|^{p(x)} dx \tag{29}$$

$$\leqslant -\int_\Omega g(x) u^{q(x)} \varphi^{p(x)} dx$$

$$+\int_\Omega \left(\frac{p(x)+2}{p(x)-1-q(x)}\right)^{p(x)-1} u^{q(x)} |\nabla\varphi|^{p(x)} dx \tag{30}$$

$$+\int_\Omega \left(\frac{p(x)+2}{p(x)-1-q(x)}\right)^{p(x)-1} \varphi^{p(x)} u^{q(x)}$$

$$\left[p(x)^{p(x)-1} |\ln u|^{p(x)} \left|\nabla\left(\frac{q(x)}{p(x)}\right)\right|^{p(x)}\right.$$

$$\left. +\frac{1}{p(x)} \left|\ln\left(\varphi u^{\frac{q(x)}{p(x)}-1}\right)\right|^{p(x)} |\nabla p(x)|^{p(x)}\right] dx.$$

REMARK 3. As $p(x) \equiv p$ and $q(x) \equiv q$, then Theorem 3 reduces to Caccioppoli-type estimate for positive supersolution of p-Laplace equations (see [22]).

4.2. Nonexistence of positive supersolutions

Consider the equation

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) - g(x) |u|^{p(x)-2} u = f(x), x \in \Omega \tag{31}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 \leqslant g(x) \in L^\infty(\Omega)$ and $0 \leqslant f(x) \in L^{\frac{p(x)}{p(x)-1}}(\Omega)$.

$u \in W^{1,p(x)}(\Omega)$ is called a weak supersolution of equation (31) if it satisfies

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla\eta \, dx - \int_\Omega g(x) |u|^{p(x)-2} u\eta \, dx \geqslant \int_\Omega f(x)\eta \, dx, \tag{32}$$

for any nonnegative function $\eta \in W_0^{1,p(x)}(\Omega)$. A weak subsolution $u \in W^{1,p(x)}(\Omega)$ is defined analogously with the inequality in (32) reversed.

The main result of this subsection is as follows.

THEOREM 4. *If (31) has a strictly postive supersolution $v \in W^{1,p(x)}(\Omega)$. Then,*

$$\int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega g(x)|u|^{p(x)} dx \geqslant \int_\Omega \Phi(u,v) dx + \int_\Omega f(x) \frac{|u|^{p(x)}}{v^{p(x)-1}} dx \tag{33}$$

$$-\int_\Omega \frac{|u|^{p(x)}}{v^{p(x)-1}} \left(\ln\left(\frac{|u|}{v}\right)\right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx$$

holds for any $u \in W_0^{1,p(x)}(\Omega)$.

Proof. For any given $u \in W_0^{1,p(x)}(\Omega)$, then there exits a sequence $\{\varphi_n\} \subset C_0^\infty(\Omega)$ such that $\|\varphi_n - u\|_{p(x)} \to 0$ and $\varphi_n \to u$ a.e. in Ω as $n \to \infty$.

Let $\varepsilon > 0$. By using

$$\eta = \frac{|\varphi_n|^{p(x)}}{(v+\varepsilon)^{p(x)-1}}$$

as a test function in (32) and Picone's identity (10) for the pair φ_n and $v+\varepsilon$, we have

$$\begin{aligned}
&\int_\Omega g(x) v^{p(x)-1} \frac{|\varphi_n|^{p(x)}}{(v+\varepsilon)^{p(x)-1}} dx \qquad (34)\\
&\leqslant \int_\Omega |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \left(\frac{|\varphi_n|^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \right) dx - \int_\Omega f(x) \frac{|\varphi_n|^{p(x)}}{(v+\varepsilon)^{p(x)-1}} dx \\
&= \int_\Omega |\nabla(\varphi_n)|^{p(x)} dx - \int_\Omega \Phi(\varphi_n, v+\varepsilon) dx - \int_\Omega f(x) \frac{|\varphi_n|^{p(x)}}{(v+\varepsilon)^{p(x)-1}} dx \\
&\quad + \int_\Omega \frac{|\varphi_n|^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \left(\ln \left(\frac{|\varphi_n|}{v+\varepsilon} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx.
\end{aligned}$$

Letting $n \to \infty$ and $\varepsilon \to 0$ respectively, we can obtain (33). Thus the proof of Theorem 4 is completed. $\square$

As an immediate consequence of Theorem 4, we have

COROLLARY 4. *Assume that there exist two functions $u \in W_0^{1,p(x)}(\Omega)$ and $0 < v \in W^{1,p(x)}(\Omega)$ such that*

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega g(x)|u|^{p(x)} dx < \int_\Omega \Phi(u,v) dx + \int_\Omega f(x) \frac{|u|^{p(x)}}{v^{p(x)-1}} dx \qquad (35)\\
- \int_\Omega \frac{|u|^{p(x)}}{v^{p(x)-1}} \left(\ln \left(\frac{|u|}{v} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx.
\end{aligned}$$

Then v is not the positive supersolution of (31).

4.3. Some properties of the first Dirichlet eigenvalue of $p(x)$-Laplace equations

The purpose of this subsection is to prove the properties of the first Dirichlet eigenvalue of $p(x)$-Laplace equations by using Picone's identity (10).

In [42], Fan, Zhang and Zhao studied the eigenvalues of the $p(x)$-Laplacian Dirichlet problems

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p(x)-2} \nabla u \right) = \lambda \, |u|^{p(x)-2} u, \text{ in } \Omega, \\ u = 0, \qquad \text{on } \partial\Omega \end{cases} \qquad (36)$$

and showed that the first eigenvalue $\lambda_{p(x)}$ defined by the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{p(x)}(\Omega), u \not\equiv 0} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}, \qquad (37)$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ holds. More specifically,

LEMMA 1. *Let $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be a bounded domain, if there is $x_0 \notin \bar{\Omega}$ such that for any $\omega \in \mathbb{R}^N \setminus \{0\}$ with $\|\omega\| = 1$ the function $f(t) = p(x_0 + t\omega)$ is monotone for $t \in \{t \in \mathbb{R} \,|\, x_0 + t\omega \in \Omega\}$, then $\lambda_{p(x)} > 0$.*

In what follows, we assume that the first eigenvalue $\lambda_{p(x)} > 0$. Notice that if u is a nontrivial eigenfunction corresponding to $\lambda_{p(x)}$ in the problem (36), so is $|u|$, and $|u| \geqslant 0$. In view of Theorem 4.1 in [45] and Theorem 5.3 in [2], we obtain $u > 0$ in Ω. Following the arguments of [46, Theorem 4.1], it is easy to see that $u \in L^\infty(\Omega)$. By using regularity result (see [47, Theorem 1.2]), we conclude that $u \in C^{1,\alpha}(\overline{\Omega})$.

The main result of this subsection is as follows.

4.3.1. Domain monotonicity property

THEOREM 5. *Let $\Omega_1 \subset \Omega_2$ and u_1, u_2 be positive eigenfunctions associated with $\lambda_{p(x)}(\Omega_1)$, $\lambda_{p(x)}(\Omega_2)$ respectively, and*

$$\int_{\Omega_1} \frac{u_1^{p(x)}}{u_2^{p(x)-1}} \left(\ln \left(\frac{u_1}{u_2} \right) \right) |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla p \, dx \leqslant 0, \tag{38}$$

then

$$\lambda_{p(x)}(\Omega_1) \geqslant \lambda_{p(x)}(\Omega_2). \tag{39}$$

Moreover, if $\Omega_1 \subset \Omega_2$ and $\Omega_1 \not\equiv \Omega_2$, then the inequality in (39) is strict.

Proof. Let $\varepsilon > 0$. Integrating identity (10) where $u = u_1$ and $v_1 = u_2 + \varepsilon$ over Ω_1, we get

$$\begin{aligned} &\int_{\Omega_1} |\nabla u_1|^{p(x)} dx - \int_{\Omega_1} |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla \left(\frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}} \right) dx \\ &= \int_{\Omega_1} \Phi(u_1, u_2+\varepsilon) dx \\ &\quad - \int_{\Omega_1} \frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}} \left(\ln \left(\frac{u_1}{u_2+\varepsilon} \right) \right) |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla p \, dx. \end{aligned} \tag{40}$$

Since

$$\begin{aligned} &\int_{\Omega_1} |\nabla u_1|^{p(x)} dx - \int_{\Omega_1} |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla \left(\frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}} \right) dx \\ &= \lambda_{p(x)}(\Omega_1) \int_{\Omega_1} u_1^{p(x)} dx - \lambda_{p(x)}(\Omega_2) \int_{\Omega_1} u_2^{p(x)-1} \frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}} dx, \end{aligned} \tag{41}$$

then, taking into account (40) and (41), letting $\varepsilon \to 0$ and using the Lebesgue dominated convergence theorem and Fatou's lemma, we obtain

$$\lambda_{p(x)}(\Omega_1) \geqslant \lambda_{p(x)}(\Omega_2) + \frac{\int_{\Omega_1} \Phi(u_1,u_2)dx - \int_{\Omega_1} \frac{u_1^{p(x)}}{u_2^{p(x)-1}} \left(\ln\left(\frac{u_1}{u_2}\right)\right) |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla p\, dx}{\int_{\Omega_1} u_1^{p(x)} dx}. \tag{42}$$

$\Phi(u_1,u_2) \geqslant 0$ and (38) yield

$$\lambda_{p(x)}(\Omega_1) \geqslant \lambda_{p(x)}(\Omega_2).$$

If $\Omega_1 \subset \Omega_2$, $\Omega_1 \not\equiv \Omega_2$ and

$$\lambda_{p(x)}(\Omega_1) = \lambda_{p(x)}(\Omega_2),$$

it follows that $\Phi(u_1,u_2) = 0$ a.e. in Ω_1 and

$$\int_{\Omega_1} \frac{u_1^{p(x)}}{u_2^{p(x)-1}} \left(\ln\left(\frac{u_1}{u_2}\right)\right) |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla p\, dx = 0,$$

thus $u_1 = ku_2$ and

$$\int_{\Omega_1} u_2 k^{p(x)} |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla p\, dx = 0,$$

for some constant $k > 0$. But $u_1 = ku_2$ contradicts the assumption that $\Omega_1 \subset \Omega_2$, $\Omega_1 \not\equiv \Omega_2$. The proof is completed. □

REMARK 4. Taking $\frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}}$ as a test function, which is valid since by using regularity results, $\frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}} \in W_0^{1,p(x)}(\Omega_1)$. Hence,

$$\int_{\Omega_1} |\nabla u_2|^{p(x)-2} \nabla u_2 \cdot \nabla \left(\frac{u_1^{p(x)}}{(u_2+\varepsilon)^{p(x)-1}}\right) dx < +\infty.$$

In a recent paper [38], Feng and Han take $\frac{u_1^{p(x)}}{u_2^{p(x)-1}}$ as a test function, however, to the best of our knowledge, $\frac{u_1^{p(x)}}{u_2^{p(x)-1}} \in W_0^{1,p(x)}(\Omega_1)$ is not already known. As a consequence, the test functions used in the third Section of [38] are unreasonable.

4.3.2. Simplicity

THEOREM 6. *Let u be the first eigenfunction of problem (36) with $u > 0$ in Ω, and $u_1 \in W_0^{1,p(x)}(\Omega)$ be any eigenfunctions associated with $\lambda_{p(x)}$ such that*

$$\int_{\Omega} \frac{u_1^{p(x)}}{u^{p(x)-1}} \left(\ln\left(\frac{|u_1|}{u}\right)\right) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla p\, dx \leqslant 0, \tag{43}$$

then $u_1 = ku$ *and*

$$\int_\Omega u\,|k|^{p(x)}\,|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla p\,dx = 0, \tag{44}$$

for some constant $k \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$. Integrating identity (10), here taking $u = u_1$ and $v_1 = u + \varepsilon$ over Ω, we get

$$\begin{aligned}
\int_\Omega \Phi(u_1, u+\varepsilon)dx &= \int_\Omega |\nabla u_1|^{p(x)}\,dx - \int_\Omega |\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla\left(\frac{|u_1|^{p(x)}}{(u+\varepsilon)^{p(x)-1}}\right)dx \\
&\quad + \int_\Omega \frac{|u_1|^{p(x)}}{(u+\varepsilon)^{p(x)-1}}\left(\ln\left(\frac{|u_1|}{u+\varepsilon}\right)\right)|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla p\,dx \\
&= \lambda_{p(x)}\int_\Omega |u_1|^{p(x)}dx - \lambda_{p(x)}\int_\Omega u^{p(x)-1}\frac{|u_1|^{p(x)}}{(u+\varepsilon)^{p(x)-1}}dx \\
&\quad + \int_\Omega \frac{|u_1|^{p(x)}}{(u+\varepsilon)^{p(x)-1}}\left(\ln\left(\frac{|u_1|}{u+\varepsilon}\right)\right)|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla p\,dx.
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$ and using the Lebesgue dominated convergence theorem and Fatou's lemma, we obtain

$$\int_\Omega \Phi(u_1,u)dx \leqslant \int_\Omega \frac{|u_1|^{p(x)}}{u^{p(x)-1}}\left(\ln\left(\frac{|u_1|}{u}\right)\right)|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla p\,dx.$$

$\Phi(u_1,u) \geqslant 0$ and (43) yield

$$\int_\Omega \Phi(u_1,u)dx = \int_\Omega \frac{|u_1|^{p(x)}}{u^{p(x)-1}}\left(\ln\frac{|u_1|}{u}\right)|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla p\,dx = 0,$$

which implies that $u_1 = ku$ and

$$\int_\Omega u\,|k|^{p(x)}\,|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla p\,dx = 0.$$

for some constant $k \in \mathbb{R}$, this completes the proof. □

4.3.3. Uniqueness

THEOREM 7. *Let* (λ, u_1) *be an eigenpair for problem* (36) *with* $u_1 > 0$ *in* Ω, *and* $0 < u \in W_0^{1,p(x)}(\Omega)$ *be the first eigenfunction such that*

$$\int_\Omega \frac{u^{p(x)}}{u_1^{p(x)-1}}\left(\ln\left(\frac{u}{u_1}\right)\right)|\nabla u_1|^{p(x)-2}\,\nabla u_1\cdot\nabla p\,dx \leqslant 0, \tag{45}$$

then

$$\lambda = \lambda_{p(x)}. \tag{46}$$

Moreover, we obtain $u = ku_1$ *and*

$$\int_\Omega k^{p(x)} u_1 |\nabla u_1|^{p(x)-2} \nabla u_1 \cdot \nabla p \, dx = 0,$$

for some constant $k \in \mathbb{R}^+$.

Proof. Let $\varepsilon > 0$. Integrating identity (10), here taking $v_1 = u_1 + \varepsilon$ over Ω and using the same procedures as those in the proof of Theorem 6, we get

$$\int_\Omega \Phi(u, u_1) dx \leqslant (\lambda_{p(x)} - \lambda) \int_\Omega u^{p(x)} dx \\ + \int_\Omega \frac{u^{p(x)}}{u_1^{p(x)-1}} \left(\ln \left(\frac{u}{u_1} \right) \right) |\nabla u_1|^{p(x)-2} \nabla u_1 \cdot \nabla p \, dx. \tag{47}$$

$\Phi(u, u_1) \geqslant 0$ and (45) yield

$$\lambda \leqslant \lambda_{p(x)}.$$

Taking into account $\lambda \geqslant \lambda_{p(x)}$, it follows that $\lambda = \lambda_{p(x)}$. Therefore, (45)–(47) imply $\Phi(u, u_1) = 0$ and

$$\int_\Omega \frac{u^{p(x)}}{u_1^{p(x)-1}} \left(\ln \left(\frac{u}{u_1} \right) \right) |\nabla u_1|^{p(x)-2} \nabla u_1 \cdot \nabla p \, dx = 0,$$

which mean $u = ku_1$ and

$$\int_\Omega k^{p(x)} u_1 |\nabla u_1|^{p(x)-2} \nabla u_1 \cdot \nabla p \, dx = 0,$$

for some constant $k \in \mathbb{R}^+$. This completes the proof. $\square$

REMARK 5. If $p(x) \equiv p$, then Theorem 5, Theorem 6 and Theorem 7 reduce to the case of p-Laplace equations (see [16, 21, 26]).

4.4. Hardy type inequality

THEOREM 8. *Suppose* $0 < v \in W^{1,p(x)}(\Omega)$ *that*

$$-\operatorname{div}(|\nabla v|^{p(x)-2} \nabla v) \geqslant \lambda g(x) v^{p(x)-1}, \text{ in } \Omega, \tag{48}$$

for some $\lambda > 0$ *and* $0 \leqslant g(x) \in L^\infty(\Omega)$. *Then for any* $0 \leqslant u \in W_0^{1,p(x)}(\Omega)$ *such that*

$$\int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \left(\ln \left(\frac{u}{v} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx \leqslant 0, \tag{49}$$

the following inequality holds

$$\int_\Omega \lambda g(x) u^{p(x)} dx \leqslant \int_\Omega |\nabla u|^{p(x)} dx. \tag{50}$$

Moreover, the equality holds if and only if $u = kv$ and

$$\int_\Omega v k^{p(x)} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx = 0,$$

for some constant $k \in \mathbb{R}^+$.

Proof. Since $0 \leqslant u \in W_0^{1,p(x)}(\Omega)$, then there exits a positive sequence $\{\varphi_n\} \subset C_0^\infty(\Omega)$ such that $\|\varphi_n - u\|_{p(x)} \to 0$ and $\varphi_n \to u$ a.e. in Ω as $n \to \infty$.

Let $\varepsilon > 0$. Since

$$\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \in W_0^{1,p(x)}(\Omega),$$

applying identity (10) to the pair φ_n and $v+\varepsilon$ and taking into account (48) yield

$$\begin{aligned} &\int_\Omega \lambda g(x) v^{p(x)-1} \frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}} dx \\ &\quad \leqslant \int_\Omega |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \left(\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \right) dx \\ &\quad = \int_\Omega |\nabla (\varphi_n)|^{p(x)} \, dx - \int_\Omega \Phi(\varphi_n, v+\varepsilon) dx \\ &\qquad + \int_\Omega \frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \left(\ln \left(\frac{\varphi_n}{v+\varepsilon} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx. \end{aligned} \tag{51}$$

Letting $n \to \infty$, and making use of Lebesgue dominated convergence theorem on the right hand side and Fatou's lemma on the left-hand side, we obtain

$$\begin{aligned} \int_\Omega \lambda g(x) v^{p(x)-1} \frac{u^{p(x)}}{(v+\varepsilon)^{p(x)-1}} dx &\leqslant \int_\Omega |\nabla u|^{p(x)} \, dx - \int_\Omega \Phi(u, v+\varepsilon) dx \\ &\quad + \int_\Omega \frac{u^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \left(\ln \left(\frac{u}{v+\varepsilon} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx. \end{aligned} \tag{52}$$

Once again we repeat the above procedures by letting $\varepsilon \to 0$ obtain

$$\begin{aligned} \int_\Omega \lambda g(x) u^{p(x)} dx &\leqslant \int_\Omega |\nabla u|^{p(x)} \, dx - \int_\Omega \Phi(u, v) dx \\ &\quad + \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \left(\ln \left(\frac{u}{v} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx. \end{aligned} \tag{53}$$

Therefore, $\Phi(u, v) \geqslant 0$ and (49) imply (50) holds. If

$$\int_\Omega \lambda g(x) u^{p(x)} dx = \int_\Omega |\nabla u|^{p(x)} \, dx,$$

it follows that $\Phi(u, v) = 0$ and

$$\int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \left(\ln \left(\frac{u}{v} \right) \right) |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx = 0,$$

which yield $u = kv$ and

$$\int_\Omega v k^{p(x)} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx = 0,$$

for some constant $k \in \mathbb{R}^+$. Thus this theorem is proved. $\square$

REMARK 6. Theorem 8 extends the reult as that of [16] (see also [26]) to $p(x)$-Laplacian.

4.5. Barta-type inequality

THEOREM 9. *If the assumptions of Theorem 8 hold. Then the following Barta's inequality holds*

$$\inf_{0 \leqslant u \in W_0^{1,p(x)}(\Omega), u \not\equiv 0} \frac{\int_\Omega |\nabla u|^{p(x)} \, dx - \int_\Omega \lambda g(x) u^{p(x)} dx}{\int_\Omega u^{p(x)} dx} \geqslant \inf_{x \in \Omega} \frac{-\mathrm{div}(|\nabla v|^{p(x)-2} \nabla v) - \lambda g(x) v^{p(x)-1}}{v^{p(x)-1}}. \tag{54}$$

Moreover, the equality holds if and only if $u = kv$ and

$$\int_\Omega v k^{p(x)} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx = 0,$$

for some constant $k \in \mathbb{R}^+$.

Proof. Denote the right hand side of (54) by $\beta(\lambda)$. Then we have

$$-\mathrm{div}(|\nabla v|^{p(x)-2} \nabla v) - \lambda g(x) v^{p(x)-1} \geqslant \beta(\lambda) v^{p(x)-1}.$$

Applying (50) (with obvious extension) we obtain

$$\int_\Omega |\nabla u|^{p(x)} \, dx - \int_\Omega \lambda g(x) u^{p(x)} dx \geqslant \beta(\lambda) \int_\Omega u^{p(x)} dx, \tag{55}$$

the equality holds if and only if $u = kv$ and

$$\int_\Omega v k^{p(x)} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx = 0,$$

for some constant $k \in \mathbb{R}^+$. If $u \not\equiv 0$ in Ω, we can divide by $\int_\Omega u^{p(x)} dx$ and the result (54) follows. $\square$

COROLLARY 5. *If the assumptions of Theorem 8 hold and $g(x) \equiv 0$. Then*

$$\lambda_{p(x)} \geqslant \inf_{x \in \Omega} \frac{-\mathrm{div}(|\nabla v|^{p(x)-2} \nabla v)}{v^{p(x)-1}}, \tag{56}$$

with the equality if and only if v is an eigenfunction corresponding to $\lambda_{p(x)}$ and

$$\int_\Omega v k^{p(x)} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla p \, dx = 0.$$

REMARK 7. If $p(x) \equiv p$, Theorem 9 and Corollary 5 reduce to the case of p-Laplace equations (see [21]).

4.6. A nonlinear system with singular nonlinearity

In this subsection, we consider the following system with singular nonlinearity.

THEOREM 10. *Consider*

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = v^{p(x)-1}, \text{ in } \Omega, \\ -\operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) = \frac{v^{2p(x)-2}}{u^{p(x)-1}}, \text{ in } \Omega, \\ u > 0, v > 0, \qquad \text{in } \Omega, \\ u = v = 0, \qquad \text{on } \partial\Omega. \end{cases} \tag{57}$$

If $(u,v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,p(x)}(\Omega)$ *is a weak solution of (57) and*

$$\int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}}\left(\ln\left(\frac{u}{v}\right)\right)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla p\, dx \leqslant 0, \tag{58}$$

then $u = kv$ *and*

$$\int_\Omega v k^{p(x)}|\nabla v|^{p(x)-2}\nabla v \cdot \nabla p\, dx = 0,$$

for some constant $k \in \mathbb{R}^+$.

Proof. Since $0 \leqslant u \in W_0^{1,p(x)}(\Omega)$, then there exits a positive sequence $\{\varphi_n\} \subset C_0^\infty(\Omega)$ such that $\|\varphi_n - u\|_{p(x)} \to 0$ and $\varphi_n \to u$ a.e. in Ω as $n \to \infty$. Let $\varepsilon > 0$.

Since
$$\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}} \in W_0^{1,p(x)}(\Omega),$$
we have

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\left(\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}}\right)dx \tag{59}$$
$$= \int_\Omega \frac{(|v|^{p(x)-2}v)^2}{u^{p(x)-1}}\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}}dx.$$

Since
$$u \in W_0^{1,p(x)}(\Omega),$$
we have
$$\int_\Omega |\nabla u|^{p(x)}\, dx = \int_\Omega v^{p(x)-1}u\, dx. \tag{60}$$

By using Picone's identity (10) for the pair φ_n and $v+\varepsilon$, we have

$$\begin{aligned}\int_\Omega \Phi(\varphi_n, v+\varepsilon)dx &= \int_\Omega |\nabla(\varphi_n)|^{p(x)}\,dx - \int_\Omega |\nabla v|^{p(x)-2}\nabla v\cdot\nabla\left(\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}}\right)dx \\ &\quad + \int_\Omega \frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}}\left(\ln\left(\frac{\varphi_n}{v+\varepsilon}\right)\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx \\ &= \int_\Omega |\nabla(\varphi_n)|^{p(x)}\,dx - \int_\Omega \frac{v^{2p(x)-2}}{u^{p(x)-1}}\frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}}dx \\ &\quad + \int_\Omega \frac{(\varphi_n)^{p(x)}}{(v+\varepsilon)^{p(x)-1}}\left(\ln\left(\frac{\varphi_n}{v+\varepsilon}\right)\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx. \quad (61)\end{aligned}$$

Letting $n\to\infty$ and $\varepsilon\to 0$ respectively, we have

$$\int_\Omega \Phi(u,v)dx \leqslant \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}}\left(\ln\left(\frac{u}{v}\right)\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx. \quad (62)$$

$\Phi(u,v)\geqslant 0$ and (58) imply

$$\Phi(u,v) = \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}}\left(\ln\left(\frac{u}{v}\right)\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx = 0,$$

namely, $u = kv$ and

$$\int_\Omega v\,k^{p(x)}\,|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx = 0,$$

for some constant $k\in\mathbb{R}^+$. The proof is completed. □

REMARK 8. As $p(x)\equiv p$, this result is discussed in [26].

4.7. Sturmian comparison theorem

In this subsection, we discuss Sturmian comparison theorem because it plays an important role in the qualitative theory of elliptic partial differential equations.

THEOREM 11. *Let $g_1, g_2 \in L^\infty(\Omega)$, $g_1 \leqslant g_2$ and $g_1 \not\equiv g_2$ in Ω. If $0 < u \in W_0^{1,p(x)}(\Omega)$ is a solution of*

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = g_1(x)\,|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (63)$$

then any solution of the equation

$$\begin{cases} -\mathrm{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) = g_2(x)\,|v|^{p(x)-2}v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega \end{cases} \quad (64)$$

satisfying

$$\int_\Omega \frac{u^{p(x)}}{|v|^{p(x)-2}v}\left(\ln\left(\frac{u}{|v|}\right)\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx \leqslant 0, \tag{65}$$

must change sign in Ω.

Proof. Suppose the contrary. Let $v>0$. Then an easy calculation shows that

$$\begin{aligned} 0\leqslant \int_\Omega \Phi(u,v)dx &= \int_\Omega (g_1(x)-g_2(x))u^{p(x)}dx \\ &\quad + \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}}\left(\ln\left(\frac{u}{v}\right)\right)|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p\,dx \\ &< 0. \end{aligned} \tag{66}$$

This is a contradiction. Hence v must change sign in Ω. □

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