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MATRIX VALUED CONJUGATE CONVOLUTION OPERATORS ON MATRIX VALUED L^p–SPACES

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Abstract. Let G be a locally compact group equipped with the left Haar measure m_G, M_n be an $n\times n$ matrix with entries in $\mathbb{C}$ and let $M(G,M_n)$ be the Banach algebra consisting all M_n-valued measures on G. We define the left and right conjugate convolution operators on $L^p(G,M_n)$ and characterize these operators. Moreover, we give some necessary and sufficient conditions, in terms of conjugate convolution, for a bounded operator on $L^p(G,M_n)$ to be translation invariant.

1. Introduction

Let G be a locally compact group, m_G be the left Haar measure on G, $1<p,q<\infty$ such that $1/p+1/q=1$ and Δ be the modular function on G. For any $f\in L^1(G)$ and $g\in L^p(G)$, the conjugate convolution $f\circledast g$ was introduced by Yuan [12] as follows:

$$f\circledast g(x)=\int_G f(y)g(y^{-1}xy)\Delta^{\frac{1}{p}}(y)\,\mathrm{d}m_G(y). \tag{1}$$

The above defined product on $L^p(G)$ spaces studied widely by Ghaffari, see [8, 9]. Let M_n be an $n\times n$, $n\in\mathbb{N}$, matrix with entries in $\mathbb{C}$. We equip M_n with the C^*-norm and consider the trace map $\mathrm{Tr}:M_n\longrightarrow\mathbb{C}$ is a positive linear functional of norm n. Suppose that $\mathscr{B}$ is a σ-algebra of Borel sets in G, $\mu:G\longrightarrow M_n$ is a countably additive function that we call it an M_n-valued measure on G and denote by an $n\times n$ matrix $\mu=(\mu_{ij})$ of complex valued measures μ_{ij} on G. The variation of μ is $|\mu|$ that is a positive real finite measure on G defined by

$$|\mu|(E)=\sup_{\mathscr{P}}\left\{\sum_{E_i\in\mathscr{P}}\|\mu(E_i)\|:E\in\mathscr{B}\right\},$$

where $\mathscr{P}$ is a partition of E into a finite number of pairwise disjoint Borel sets. Define the norm of μ as $\|\mu\|=|\mu|(G)$. Following [1, 2], μ has a polar representation $\mathrm{d}\mu=\omega\cdot\mathrm{d}|\mu|$ where $\omega:G\longrightarrow M_n$ is a Bochner integrable function with $\|\omega(\cdot)\|=1$. A

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function $f=(f_{ij}):G\longrightarrow M_n$ is called μ-integrable if each f_{ij} is a Borel function and the integral $\int_G f_{ij}\,\mathrm{d}\mu_{k\ell}$ exist in which case. For any $E\in\mathscr{B}$, the integral $\int_E f\,\mathrm{d}\mu$ is an $n\times n$ matrix with ij-th entry

$$\sum_k \int_E f_{ik}\,\mathrm{d}\mu_{kj}.$$

Then, by [1, Lemma 4], we have the norm of f, as follows

$$\left\|\int_G f\,\mathrm{d}\mu\right\| = \left\|\int_G f(x)\omega(x)\,\mathrm{d}|\mu|(x)\right\| \leqslant \int_G \|f(x)\|\ \mathrm{d}|\mu|(x) \leqslant \|f\|\|\mu\|. \tag{2}$$

The trace-norm $\|\cdot\|_{tr}$ is equivalent to the C^*-norm on M_n and $M_n^*=(M_n,\|\cdot\|_{tr})$, by this, we can regard an M_n^*-valued measure on G as an M_n-valued measure on G, and vice versa. We denote the space of all M_n^*-valued measures on G by $M(G,M_n^*)$ with the total variation norm $\|\cdot\|_{tr}$. This space is linearly isomorphic to the space $(M(G,M_n),\|\cdot\|)$. By $C_0(G,M_n)$, we mean the Banach space of continuous M_n-valued functions on G vanishing at infinity with the supremum norm and $C_C(G,M_n)$ denotes the subspace of $C_0(G,M_n)$ consists all M_n-valued continuous functions with compact supports. By [1, Lemma 5], $M(G,M_n^*)$ is linearly isomorphic order-isomorphic to the dual of $C_0(G,M_n)$, with the following duality formula:

$$\langle\cdot,\cdot\rangle : C_0(G,M_n)\times M(G,M_n^*) \longrightarrow \mathbb{C}$$
$$\langle f,\mu\rangle = \mathrm{Tr}\left(\int_G f\mathrm{d}\mu\right) = \sum_{i,k}\int_G f_{ik}\mathrm{d}\mu_{k,i}, \tag{3}$$

for any $f=(f_{ij})\in C_0(G,M_n)$ and $\mu=(\mu_{ij})\in M(G,M_n^*)$. By [3, Proposition 2.4], $(M(G,M_n^*),\|\cdot\|_{tr})$ is a Banach algebra with the following convolution product:

$$\langle f,\mu*\nu\rangle = \mathrm{Tr}\left(\int_G\int_G f(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)\right), \tag{4}$$

for all $f\in C_0(G,M_n)$ and $\mu,\nu\in M(G,M_n^*)$. Also, $(M(G,M_n),\|\cdot\|)$ becomes a Banach algebra with the convolution product and is algebraically isomorphic to $(M(G,M_n^*),\|\cdot\|_{tr})$. Let $f=(f_{ij})$ be a Borel M_n-valued function on G and $\mu=(\mu_{ij})$ be a M_n-valued measure on G. An M_n-valued convolution $f*\mu$, if exists at $x\in G$, is defined by

$$(f*\mu)(x) = \int_G f(xy^{-1})\,\mathrm{d}\mu(y). \tag{5}$$

The left convolution $\mu*_\ell f$ is the following integral if it exists:

$$(\mu*_\ell f)(x) = \int_G \mathrm{d}\mu(y)f(y^{-1}x) \qquad (x\in G). \tag{6}$$

The transposed integral $\int_G \mathrm{d}\mu(x)f(x)$ which is defined to have ij-entry

$$\left(\int_G \mathrm{d}\mu(x)f(x)\right)_{ij} = \sum_k\int_G f_{kj}(x)\,\mathrm{d}\mu_{ik}(x).$$

Moreover, similar to (2), we have

$$\left\| \int_G \mathrm{d}\mu(x) f(x) \right\| \leqslant \int_G \|f(x)\| \,\mathrm{d}|\mu|(x) \leqslant \|f\| \|\mu\|. \tag{7}$$

For a given $\mu \in M(G,M_n)$, following [2, Page 24], we consider $\widetilde{\mu} \in (G,M_n)$ by $\mathrm{d}\widetilde{\mu}(x) = \mathrm{d}\mu(x^{-1})$, for all $x \in G$. Consider the complex vector space $L^p(G,M_n)$. Then by [5], the dual of $L^p(G,M_n)$ is identified by $L^q(G,M_n^*)$ with the following duality formula:

$$\begin{aligned} \langle \cdot,\cdot \rangle : L^p(G,M_n) \times L^q(G,M_n^*) &\longrightarrow \mathbb{C} \\ \langle f,g \rangle = \mathrm{Tr}\left(\int_G f(x)g(x)\mathrm{d}m_G(x) \right). \end{aligned} \tag{8}$$

For any $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$, we have

$$\langle f*\mu, g \rangle = \mathrm{Tr}\left(\int_G \int_G g(xy) f(x) \,\mathrm{d}\mu(y) \,\mathrm{d}m_G(x) \right) = \langle f, \widetilde{\mu} * g \rangle \tag{9}$$

and

$$\|f\|_p = \left(\int_G \|f(x)\|_{tr}^p \,\mathrm{d}m_G(x) \right)^{\frac{1}{p}}. \tag{10}$$

Let G be a locally compact group, $L^p(G,M_n)$ and $L^q(G,M_n)$ be as before. For all $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$, $\int_G f(x)g(x)\,\mathrm{d}m_G(x)$ is in M_n. We denote this M_n-valued integral by

$$\int_G f(x)g(x)\,\mathrm{d}m_G(x) = \langle f,g \rangle_{M_n},$$

indeed it is M_n-valued duality formula and $\mathrm{Tr}(\langle f,g \rangle_{M_n}) = \mathrm{Tr}\langle f,g \rangle$.

Let G be a locally compact group and $1 < p < \infty$. Following [4], a bounded operator $T : L^p(G) \longrightarrow L^p(G)$ is called a p-convolution operator of G if $T({}_af) = {}_a T(f)$, for all $a \in G$ and $f \in L^p(G)$, where ${}_af(\cdot) = f(a\cdot)$ denotes the left translation of f. These operators are also called translation invariant operators, see [7], where Hörmander's studied these operators on $\mathbb{R}^n$. Following [4], we denote the set of all p-convolution operators of G by $CV_p(G)$. Suppose that $\mathscr{B}(L^p(G))$ denotes the space of all maps from $L^p(G)$ into itself and $B(L^p(G))$ denotes the Banach algebra consists all linear bounded operators from $L^p(G)$ into itself. Then clearly, $CV_p(G)$ is a subalgebra of $B(L^p(G))$. The notion of p-convolution operators on $L^p(G)$ has been generalized in [6] to left and right matrix valued p-convolution operators on $L^p(G,M_n)$. Moreover, positive type and positive definite functions on on $L^p(G,M_n)$ are characterized in [10].

Throughout this paper, we suppose that $1 < p,q < \infty$ and $1/p + 1/q = 1$. In the next section, we introduce the notions of left and right conjugate convolution operators on $L^p(G,M_n)$ Banach spaces, where G is a locally compact group equipped with the left Haar measure m_G. We give some results and properties of these operators. Moreover, we characterize the left and right conjugate convolution operators on $L^p(G,M_n)$. In section 3, we show the relationships between the matrix valued p-convolution operators and the conjugate convolution products.

2. Matrix valued conjugate convolution

In this section, we introduce the left and right conjugate convolution products on $L^p(G,M_n)$, where $1 \leqslant p < \infty$. Some properties of these operators are given and we characterize these operators. Following [2], for any $f \in L^p(G,M_n)$ and the scalar valued map Δ (the modular function of G), the product $f(x) \otimes \Delta(y)$ is given by

$$f(x) \otimes \Delta(y) = \begin{pmatrix} f_{11}(x)\Delta(y) & \cdots & f_{1n}(x)\Delta(y) \\ \vdots & \ddots & \vdots \\ f_{n1}(x)\Delta(y) & \cdots & f_{nn}(x)\Delta(y) \end{pmatrix} \tag{11}$$

Let $f \in L^p(G,M_n)$, $\mu \in M(G,M_n)$ and $1/p+1/q=1$. We now define two right and left conjugate convolution of f and μ as follows:

$$f \circledast \mu(x) = \int_G \left(f(y^{-1}xy) \otimes \Delta^{\frac{1}{p}}(y) \right) \mathrm{d}\mu(y) \tag{12}$$

and

$$\mu \circledast_\ell f(x) = \int_G \mathrm{d}\mu(y) \left(f(y^{-1}xy) \otimes \Delta^{\frac{1}{q}}(y) \right). \tag{13}$$

DEFINITION 1. Let G be a locally compact group with the left Haar measure m_G and $0 \neq \mu \in M(G,M_n)$. We say an operator $T_\mu : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is a right conjugate convolution operator if satisfies the following condition:

$$T_\mu(f) = f \circledast \mu \qquad (f \in L^p(G,M_n)).$$

Similarly, we define the left conjugate convolution operator $S_\mu : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ as follows

$$S_\mu(f) = \mu \circledast_\ell f \qquad (f \in L^p(G,M_n)).$$

LEMMA 1. *Let G be a locally compact group with the left Haar measure m_G. Then, for any $f \in L^p(G,M_n)$, $g \in L^q(G,M_n^*)$ and $\mu \in M(G,M_n)$, the following statements hold:*

(i) $\langle f \circledast \mu, g \rangle = \langle f, \widetilde{\mu} \circledast_\ell g \rangle$.

(ii) $\|f \circledast \mu\|_p \leqslant \|f\|_p \|\mu\|$ *and* $\|\mu \circledast_\ell f\|_p \leqslant \|f\|_p \|\mu\|$.

Proof. (i) Similar to (9), for any $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$, we have

$$\begin{aligned} \langle f \circledast \mu, g \rangle &= \mathrm{Tr}\left(\int_G \int_G f \circledast \mu(x) g(x) \mathrm{d}m_G(x) \right) \\ &= \mathrm{Tr}\left(\int_G \int_G f(y^{-1}xy) \otimes \Delta^{\frac{1}{p}}(y) \mathrm{d}\mu(y) g(x) \mathrm{d}m_G(x) \right) \end{aligned}$$

$$= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}\mu(y)f(x)\left(g(yxy^{-1})\otimes\Delta^{\frac{-1}{q}}(y)\right)\mathrm{d}m_G(x)\right)$$
$$= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}\mu(y)f(x)\left(g(y^{-1}xy)\otimes\Delta^{\frac{1}{q}}(y)\right)\mathrm{d}m_G(x)\right)$$
$$= \langle f, \widetilde{\mu}\circledast_\ell g\rangle. \tag{14}$$

(ii) For a fixed $y \in G$, we set $\Gamma_y f(x) = f(y^{-1}xy)\Delta^{\frac{1}{p}}(y)$, for all $x \in G$ and $f \in L^p(G,M_n)$. Then by (10),

$$\|f\|_p^p = \int_G \|f(x)\|_{tr}^p \mathrm{d}m_G(x)$$
$$= \int_G \left\|f(y^{-1}xy)\otimes\Delta^{\frac{1}{p}}(y)\right\|_{tr}^p \mathrm{d}m_G(x)$$
$$= \int_G \left\|\Gamma_y f(x)\right\|_{tr}^p \mathrm{d}m_G(x)$$
$$= \|\Gamma_y f(x)\|_p^p. \tag{15}$$

Now for a fixed $x \in G$, set $F_x(y) = f(y^{-1}xy)\otimes\Delta^{\frac{1}{p}}(y)$. By (2) and (15),

$$\|f\circledast\mu\|_p^p = \int_G \|f\circledast\mu(x)\|_{tr}^p \mathrm{d}m_G(x)$$
$$= \int_G \left\|\int_G f(y^{-1}xy)\otimes\Delta^{\frac{1}{p}}(y)\mathrm{d}\mu(y)\right\|_{tr}^p \mathrm{d}m_G(x)$$
$$= \int_G \left\|\int_G F_x(y)\mathrm{d}\mu(y)\right\|_{tr}^p \mathrm{d}m_G(x)$$
$$\leqslant \int_G \left(\int_G \|F_x(y)\|_{tr}\mathrm{d}|\mu|(y)\right)^p \mathrm{d}m_G(x)$$
$$\leqslant \int_G \|F_x(y)\|_{tr}^p \|\mu\|^p \mathrm{d}m_G(x)$$
$$\leqslant \int_G \left\|f(y^{-1}xy)\otimes\Delta^{\frac{1}{p}}(y)\right\|_{tr}^p \|\mu\|^p \mathrm{d}m_G(x)$$
$$= \|\Gamma_y f(x)\|_p^p \|\mu\|^p$$
$$= \|f\|_p^p\|\mu\|^p.$$

Similarly, by (7), we can show that $\|\mu\circledast_\ell f\|_p \leqslant \|f\|_p\|\mu\|$. □

The operators in Definition 1 are different from the right and left convolution products defined in [1, 2].

EXAMPLE 1. Let $G = \{e,a\}$. Define $f \in L^1(G)$ and $\mu \in M(G)$ similar to [2, Example 3.1.3] as follows

$$f(x) = \begin{cases} (b_{ij}), & x = e; \\ (a_{ij}), & x = a. \end{cases} \qquad \text{and} \qquad \mu(x) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, & x = e; \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & x = a. \end{cases}$$

We now compare the right and left conjugate convolution operators with the right and left convolution operators:

$$f \circledast \mu(e) = \begin{pmatrix} b_{11} & 2b_{12} \\ b_{21} & 2b_{22} \end{pmatrix}, \qquad f * \mu(e) = \begin{pmatrix} a_{11} & 2b_{12} \\ a_{21} & 2b_{22} \end{pmatrix},$$

$$f \circledast \mu(a) = \begin{pmatrix} a_{11} & 2a_{12} \\ a_{21} & 2a_{22} \end{pmatrix}, \qquad f * \mu(a) = \begin{pmatrix} b_{11} & 2a_{12} \\ b_{21} & 2a_{22} \end{pmatrix},$$

$$\mu \circledast_\ell f(e) = \begin{pmatrix} b_{11} & b_{12} \\ 2b_{21} & 2b_{22} \end{pmatrix}, \qquad \mu *_\ell f(e) = \begin{pmatrix} a_{11} & a_{12} \\ 2b_{21} & 2b_{22} \end{pmatrix},$$

and

$$\mu \circledast_\ell f(a) = \begin{pmatrix} a_{11} & a_{12} \\ 2a_{21} & 2a_{22} \end{pmatrix}, \qquad \mu *_\ell f(a) = \begin{pmatrix} b_{11} & b_{12} \\ 2a_{21} & 2a_{22} \end{pmatrix}.$$

PROPOSITION 1. Let G be a locally compact group with the left Haar measure m_G and $\mu \in M(G, M_n)$. Let $T_\mu : L^p(G, M_n) \longrightarrow L^p(G, M_n)$ be a right conjugate convolution operator, then $S_{\tilde{\mu}} : L^q(G, M_n^*) \longrightarrow L^q(G, M_n^*)$ is a right conjugate convolution operator.

Proof. Apply Lemma 1(i). $\square$

Let G be a locally compact group and $\mu, \nu \in M(G, M_n)$. Following [1], we define the convolution $\mu * \nu$ by

$$(\mu * \nu)(E) = (\mu \times \nu)\{(x, y) \in G \times G : xy \in E\}.$$

The above definition follows the following formula

$$\int_G f(x)\,\mathrm{d}(\mu * \nu)(x) = \int_G \int_G f(x, y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y),$$

for all $f \in C_0(G, M_n)$ (see [1, p. 26]). For any $f \in L^p(G, M_n)$ and $y \in G$, we set $\rho_y(f)(x) = f(y^{-1}xy)$ and for a fixed $z \in G$, set $F_z f(y) = f(y^{-1}zy) \otimes \Delta^{\frac{1}{p}}(y)$.

PROPOSITION 2. Let G be a locally compact group with the left Haar measure m_G and $\mu, \nu \in M(G, M_n)$. Let $T_\mu, T_\nu : L^p(G, M_n) \longrightarrow L^p(G, M_n)$ be right conjugate convolution operators, then $T_\mu \circ T_\nu = T_{\mu * \nu}$.

Proof. For any $f \in L^p(G, M_n)$ and $g \in L^q(G, M_n^*)$, we have

$$\begin{aligned}
\langle T_{\mu*\nu}(f), g\rangle &= \operatorname{Tr}\left(\int_G T_{\mu*\nu}(f)(x) g(x) \mathrm{d}m_G(x)\right) \\
&= \operatorname{Tr}\left(\int_G (f \circledast \mu * \nu)(x) g(x) \mathrm{d}m_G\right) \\
&= \operatorname{Tr}\left(\int_G \left(\int_G \left(f(y^{-1}xy) \otimes \Delta^{\frac{1}{p}}(y)\right) \mathrm{d}\mu * \nu(y)\right) g(x) \mathrm{d}m_G(x)\right) \\
&= \operatorname{Tr}\left(\int_G \int_G \int_G \left(f(z^{-1}y^{-1}xyz) \otimes \Delta^{\frac{1}{p}}(y) \Delta^{\frac{1}{p}}(z)\right) \mathrm{d}\mu(y) \mathrm{d}\nu(z) g(x) \mathrm{d}m_G(x)\right).
\end{aligned}$$

On the other hand, for any $f \in L^p(G,M_n)$ and $g \in L^q(G,M_n^*)$,

$$\begin{aligned}
\langle T_\mu \circ T_\nu(f), g\rangle &= \langle T_\nu(f), T_\mu^*(g)\rangle \\
&= \mathrm{Tr}\left(\int_G T_\nu(f)(x)T_\mu^*(g)(x)\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G (f \circledast \nu)(x)T_\mu^*(g)(x)\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G \left(f(z^{-1}xz)\otimes\Delta^{\frac{1}{p}}(y)\right)\mathrm{d}\nu(z)T_\mu^*(g)(x)\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G \left(f(z^{-1}xz)\otimes\Delta^{\frac{1}{p}}(y)\right)T_\mu^*(g)(x)\mathrm{d}m_G(x)\mathrm{d}\nu(z)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G F_z(x)T_\mu^*(g)(x)\mathrm{d}m_G(x)\mathrm{d}\nu(z)\right) \\
&= \mathrm{Tr}\left(\int_G \langle F_z, T_\mu^*(g)\rangle_{M_n}\mathrm{d}\nu(z)\right) \\
&= \mathrm{Tr}\left(\int_G \langle T_\mu(F_z), g\rangle_{M_n}\mathrm{d}\nu(z)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G T_\mu(F_z)(x)g(x)\mathrm{d}m_G(x)\mathrm{d}\nu(z)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G\int_G \left(F_z(y^{-1}xy)\otimes\Delta^{\frac{1}{p}}(y)\right)\mathrm{d}\mu(y)g(x)\mathrm{d}m_G(x)\mathrm{d}\nu(z)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G\int_G \left(f(z^{-1}y^{-1}xyz)\otimes\Delta^{\frac{1}{p}}(y)\Delta^{\frac{1}{p}}(z)\right)\mathrm{d}\mu(y)\mathrm{d}\nu(z)g(x)\mathrm{d}m_G(x)\right).
\end{aligned}$$

Thus, the above equalities imply that $T_\mu \circ T_\nu = T_{\mu*\nu}$. $\square$

By a similar argument in the proof of Proposition 2, we have the following result for the left conjugate operators.

PROPOSITION 3. Let G be a locally compact group with the left Haar measure m_G and $\mu,\nu \in M(G,M_n)$. Let $S_\mu, S_\nu : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ be left conjugate convolution operators, then $S_\mu \circ S_\nu = S_{\mu*\nu}$.

We again recall that for any $f \in L^p(G,M_n)$ and $y \in G$, we set $\rho_y(f)(x) = f(y^{-1}xy)$ and for a fixed $z \in G$, set $F_z f(y) = f(y^{-1}zy)\otimes\Delta^{\frac{1}{p}}(y)$.

LEMMA 2. *Let G be a locally compact group with the left Haar measure m_G and $\mu \in M(G,M_n)$. Then*

(i) T_μ, S_μ *are bounded and* $\|T_\mu\| \leqslant \|\mu\|$, $\|S_\mu\| \leqslant \|\mu\|$.

(ii) $\rho_x T_\mu = T_\mu \rho_x$ *and* $\rho_x S_\mu = S_\mu \rho_x$, *for any* $x \in G$.

(iii) $F_x T_\mu = T_\mu F_x$ *and* $F_x S_\mu = S_\mu F_x$*, for any* $x \in G$.

(iv) $T_{\delta_a}(f) = {}_{a^{-1}}f_a \otimes \Delta^{\frac{1}{p}}(a)$ *and* $S_{\delta_a}(f) = {}_{a^{-1}}f_a \otimes \Delta^{\frac{1}{q}}(a)$*, for all* $a \in G$.

(v) $T_\mu(\cdot) = \int_G T_{\delta_y}(\cdot)\,\mathrm{d}\mu(y)$ *and* $S_\mu(\cdot) = \int_G \mathrm{d}\mu(y) S_{\delta_y}(\cdot)$.

Proof. (i) By Lemma 1(ii), we have

$$\|T_\mu\| = \sup_{\|f\|_p \leqslant 1} \|T_\mu(f)\|_p = \sup_{\|f\|_p \leqslant 1} \|f \circledast \mu\|_p \leqslant \|\mu\|.$$

Similarly, one can show that $\|S_\mu\| \leqslant \|\mu\|$.
(ii) For any $f \in L^p(G, M_n)$,

$$\begin{aligned}
\rho_x T_\mu(f)(y) = \rho_x(f \circledast \mu)(y) &= f \circledast \mu(x^{-1}yx) \\
&= \int_G \left(f(z^{-1}x^{-1}yxz) \otimes \Delta^{\frac{1}{p}}(z) \right) \mathrm{d}\mu(z) \\
&= \int_G \left(\rho_z f(x^{-1}yx) \otimes \Delta^{\frac{1}{p}}(z) \right) \mathrm{d}\mu(z) \\
&= \int_G \left(\rho_z \rho_x f(y) \otimes \Delta^{\frac{1}{p}}(z) \right) \mathrm{d}\mu(z) \\
&= T_\mu \rho_x(f)(y),
\end{aligned}$$

for all $y \in G$. Similarly, we have $\rho_x S_\mu = S_\mu \rho_x$, for any $x \in G$.
(iii) By a similar argument as in (ii), the results hold.
(iv) Straightforward.
(v) For any $f \in L^p(G, M_n)$ and $g \in L^q(G, M_n^*)$, by (iv), we have

$$\begin{aligned}
\langle T_\mu(f), g \rangle &= \mathrm{Tr}\left(\int_G T_\mu(f)(x) g(x) \mathrm{d}m_G(x) \right) \\
&= \mathrm{Tr}\left(\int_G T_\mu(f)(x) g(x) \mathrm{d}m_G(x) \right) \\
&= \mathrm{Tr}\left(\int_G \int_G (f(y^{-1}xy) \otimes \Delta^{\frac{1}{p}}(y)) \mathrm{d}\mu(y) g(x) \mathrm{d}m_G(x) \right) \\
&= \mathrm{Tr}\left(\int_G \int_G {}_{y^{-1}}f_y(x) \otimes \Delta^{\frac{1}{p}}(y) \mathrm{d}\mu(y) g(x) \mathrm{d}m_G(x) \right) \\
&= \mathrm{Tr}\left(\int_G \left(\int_G \left(T_{\delta_y}(f) \right)(x) \mathrm{d}\mu(y) \right) g(x) \mathrm{d}m_G(x) \right) \\
&= \left\langle \int_G T_{\delta_y}(f) \mathrm{d}\mu(y), g \right\rangle.
\end{aligned}$$

This shows that $T_\mu(\cdot) = \int_G T_{\delta_y}(\cdot)\,\mathrm{d}\mu(y)$. By a similar argument we have $S_\mu(\cdot) = \int_G \mathrm{d}\mu(y) S_{\delta_y}(\cdot)$. □

Now, we consider conjugate convolution operators on $L^p(G, M_n)$.

THEOREM 1. *Let G be a locally compact group with the left Haar measure m_G and $T : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ be a bounded M_n-linear map. Then for some $\mu \in M(G,M_n)$, $T = T_\mu$ if and only if $F_xT = TF_x$, for all $x \in G$ and T maps $C_C(G,M_n)$ into $C_b(G,M_n)$ continuously in the spectrum norm.*

Proof. Suppose that $F_xT = TF_x$, for all $x \in G$ and T maps $C_C(G,M_n)$ into $C_b(G,M_n)$ continuously in the spectrum norm. Define an M_n-linear map $\Psi : C_C(G,M_n) \longrightarrow M_n$ by $\Psi(f) = Tf(e)$, for all $f \in C_C(G,M_n)$. From that T maps $C_C(G,M_n)$ into $C_b(G,M_n)$ continuously in the spectrum norm, Ψ is continuous. By [2, Lemma 3.1.6], there exists $\mu \in M(G,M_n)$ such that

$$\Psi(f) = \int_G f \,\mathrm{d}\mu \qquad (f \in C_C(G,M_n).$$

Then, by Lemma 2(iii),

$$\begin{aligned} T(f)(x) &= F_xT(f)(e) \\ &= T(F_xf)(e) \\ &= \Psi(F_xf) \\ &= \int_G F_x(y)\mathrm{d}\mu(y) \\ &= \int_G \left(f(y^{-1}xy) \otimes \Delta^{\frac{1}{p}}(y)\right) \mathrm{d}\mu(y) \\ &= f \circledast \mu(x), \end{aligned}$$

for all $f \in C_C(G,M_n)$. For any $f \in L^p(G,M_n)$ there is a net $(f_\alpha)_\alpha \subseteq C_C(G,M_n)$ such that f_α converges to f. Thus,

$$T(f) = \lim_\alpha T(f_\alpha) = \lim_\alpha f_\alpha \circledast \mu = f \circledast \mu.$$

This show that $T = T_\mu$, for some $\mu \in M(G,M_n)$. The converse by Lemma 2 holds. $\square$

For any $A,B \in M_n$, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Thus, one can write [1, Lemma 5] as follows with a similar proof.

LEMMA 3. *Let G be a locally compact group. The map $\mu \in M(G,M_n^*) \mapsto \mu(\cdot) \in C_0(G,M_n)^*$ defined by*

$$\mu(f) = \mathrm{Tr}\left(\int_G \mathrm{d}\mu f\right),$$

for all $f \in C_0(G,M_n)$ is a linear isometric order-isomorphism.

By the above Lemma we write [2, Lemma 3.1.6] as follows, where the proof is exactly similar.

LEMMA 4. *Let G be a locally compact group and $\Psi : C_0(G,M_n) \longrightarrow M_n$ be a continuous M_n-linear map. Then there is a unique $\mu \in M(G,M_n^*)$ such that*

$$\Psi(f) = \int_G \mathrm{d}\mu f \qquad (f \in C_0(G,M_n)).$$

THEOREM 2. *Let G be a locally compact group with the left Haar measure m_G and $S : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ be a bounded M_n-linear map. Then for some $\mu \in M(G,M_n)$, $S = S_\mu$ if and only if $F_xS = SF_x$, for all $x \in G$ and S maps $C_C(G,M_n)$ into $C_b(G,M_n)$ continuously in the spectrum norm.*

Proof. Similar to the proof of Theorem 1, define an M_n-linear map $\Psi : C_C(G,M_n) \longrightarrow M_n$ by $\Psi(f) = Sf(e)$, for all $f \in C_C(G,M_n)$. By Lemma 4, there is a unique $\mu \in M(G,M_n^*)$ such that

$$\Psi(f) = \int_G \mathrm{d}\mu f \qquad (f \in C_C(G,M_n)).$$

Then by a similar argument in the proof of Theorem 1, the proof is complete. □

3. Matrix valued p-convolution operators

In this section we consider matrix valued left and right p-convolution operators on $L^p(G,M_n)$ in relation to conjugate convolution. We recall the following definition from [6].

DEFINITION 2. Let G be a locally compact group, $1 < p < \infty$. A bounded operator $T : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ is called a matrix valued left p-convolution operator of G if $T({}_af) = {}_aT(f)$, for all $a \in G$ and $f \in L^p(G,M_n)$, where $f_a(\cdot) = f(\cdot a)$ denotes the right translation of f. We denote the set of all matrix valued left p-convolution operators of G by $LCV_p(G,M_n)$. Similarly, we define the right p-convolution operator with entries in M_n, if $T(f_a) = T(f)_a$, for all $a \in G$, $f \in L^p(G,M_n)$ and we denote the set of all such operators by $RCV_p(G,M_n)$. We denote the space of matrix valued p-convolution operators by $CV_p(G,M_n)$ that is $LCV_p(G,M_n) \cap RCV_p(G,M_n)$.

Let $f \in L^1(G,M_n)$, then $f \cdot m_G \in M(G,M_n)$ with the following total variation

$$\|f \cdot m_G\| = |f \cdot m_G|(G) = \int_G \|f(x)\| \,\mathrm{d}m_G(x) = \|f\|_1.$$

We identify $L^1(G,M_n)$ as a closed subspace of $M(G,M_n)$ such that contains all absolutely continuous M_n-valued measures on G and it also is a right ideal of $M(G,M_n)$, because $(f \cdot m_G) * \mu = (f * \mu) \cdot m_G$, for all $f \in L^1(G,M_n)$ and $\mu \in M(G,M_n)$.

In light of (12), we define the following convolution product

$$(g \circledast f)(x) = \int_G g(y^{-1}xy) \otimes \Delta^{\frac{1}{p}}(y) f(y) \,\mathrm{d}m_G(y), \tag{16}$$

for all $g \in L^p(G,M_n)$, $f \in L^1(G,M_n)$ and $x \in G$.

From (13), we have the following left convolution product

$$(f \circledast_\ell g)(x) = \int_G \mathrm{d}m_G(y) f(y) g(y^{-1}x) \Delta^{\frac{1}{q}}(y), \tag{17}$$

for all $f \in L^1(G,M_n)$ and $g \in L^p(G,M_n)$. This together Lemma 1 implies that $\|f \circledast_\ell g\|_p \leqslant \|g\|_p \|f\|_1$, for all $f \in L^1(G,M_n)$ and $g \in L^p(G,M_n)$. Thus, $L^p(G,M_n)$ is a left Banach $L^1(G,M_n)$-module.

THEOREM 3. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If $T \in CV_p(G,M_n)$, then $T(f \circledast_\ell g) = f \circledast_\ell T(g)$, for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$.*

Proof. As we discussed the above, $L^p(G,M_n)$ is a left Banach $L^1(G,M_n)$-module with respect to the left conjugate convolution product. Now, suppose that $T \in CV_p(G,M_n)$ with the left conjugate convolution product. Then

$$\begin{aligned}
\langle f \circledast_\ell T(g), h\rangle &= \mathrm{Tr}\left(\int_G (f \circledast_\ell T(g))(x) h(x) \mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G \int_G \mathrm{d}m_G(y) f(y) T(g)(y^{-1}xy) \otimes \Delta^{\frac{1}{q}}(y) h(x) \mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G \int_G \mathrm{d}m_G(y) f(y) T({}_{y^{-1}}g_y)(x) \otimes \Delta^{\frac{1}{q}}(y) h(x) \mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G \mathrm{d}m_G(y) f(y) \otimes \Delta^{\frac{1}{q}}(y) \int_G T({}_{y^{-1}}g_y)(x) h(x) \mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G \mathrm{d}m_G(y) f(y) \otimes \Delta^{\frac{1}{q}}(y) \langle T({}_{y^{-1}}g_y), h\rangle_{M_n}\right) \\
&= \mathrm{Tr}\left(\int_G \mathrm{d}m_G(y) f(y) \otimes \Delta^{\frac{1}{q}}(y) \langle {}_{y^{-1}}g_y, T^*(h)\rangle_{M_n}\right) \\
&= \mathrm{Tr}\left(\int_G \int_G \mathrm{d}m_G(y) f(y) \otimes \Delta^{\frac{1}{q}}(y) g(y^{-1}xy) T^*(h)(x) \mathrm{d}m_G(x)\right) \\
&= \langle f \circledast_\ell g, T^*(h)\rangle = \langle T(f \circledast_\ell g), h\rangle,
\end{aligned}$$

for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$ and $h \in L^q(G,M_n^*)$. Thus, $T(f \circledast_\ell g) = f \circledast_\ell T(g)$, for all $f \in L^1(G,M_n)$ and $g \in L^p(G,M_n)$. $\square$

LEMMA 5. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If for any $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$, $T(f \circledast_\ell g) = f \circledast_\ell T(g)$, then for any $a \in G$,*

(i) ${}_aT(f \circledast_\ell g) = \Delta^{\frac{1}{q}}(a) \otimes ({}_af \circledast_\ell T(g))_a$.

(ii) $T(f \circledast_\ell g)_a = \Delta^{\frac{-1}{q}}(a) \otimes {}_a({}_{a^{-1}}f \circledast_\ell T(g))$.

Proof. (i) For any $a,x \in G$, we have

$$
\begin{aligned}
{}_aT\left(f \circledast_\ell g\right)(x) &= T\left(f \circledast_\ell g\right)(ax) \\
&= (f \circledast_\ell T(g))(ax) \\
&= \int_G \mathrm{d}m_G(y) f(y) T(g)(y^{-1}axy)\Delta^{\frac{1}{q}}(y) \\
&= \int_G \mathrm{d}m_G(y) {}_af(y) T(g)(y^{-1}xay)\Delta^{\frac{1}{q}}(y)\Delta^{\frac{1}{q}}(a) \\
&= \Delta^{\frac{1}{q}}(a) \otimes \left({}_af \circledast_\ell T(g)\right)(xa) \\
&= \Delta^{\frac{1}{q}}(a) \otimes \left({}_af \circledast_\ell T(g)\right)_a(x). \qquad (18)
\end{aligned}
$$

(ii) By a similar argument in (i), the statement (ii) holds. □

A conjugate left bounded approximate identity for $L^1(G)$ is a net such as $(e_\alpha)_\alpha \subseteq L^1(G)$ such that $\|e_\alpha \circledast g - g\|_1 \to 0$, for all $g \in L^1(G)$. This definition is defined by Mohammadzadeh in [11] and he showed that $L^1(G)$ contains a conjugate left bounded approximate identity [11, Corollary 2.3].

LEMMA 6. *Let G be a locally compact group and m_G be the left Haar measure on G. Then $L^1(G,M_n)$ has a conjugate left bounded approximate identity, respect to $\circledast_\ell$ and has a conjugate right bounded approximate identity, respect to $\circledast$.*

Proof. Let $(E_\alpha)_\alpha \subseteq L^1(G)$ be the left conjugate bounded approximate identity for $L^1(G)$, then it is easy to see that

$$
E_\alpha = \begin{pmatrix} e_\alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_\alpha \end{pmatrix}
$$

is a conjugate left (right) bounded approximate identity for $L^1(G,M_n)$ respect to $\circledast_\ell$ ($\circledast$). Indeed, for any α, the support of e_α is compact and one can suppose that e_α on its support is at most 1. Then, without loss of generality, we can suppose that $\Delta(y) = 1$, for all $y \in \mathrm{Supp}(e_\alpha)$. Then by the construction of $(e_\alpha)_\alpha$ in [11, Corollary 2.3], the rest of proof is clear. □

Note that E_α is diagonal and diagonal matrices are in the center of the algebra of $n \times n$ matrices. Moreover, we can assume that $\|E_\alpha\|_1 \leqslant 1$ and according to the construction of E_α's, the support of each E_α is compact. It is natural to ask if the converse of Theorem 3 holds. We investigate the question in the special cases.

THEOREM 4. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If $T \in RCV_p(G,M_n)$ and $T\left(f \circledast_\ell g\right) = f \circledast_\ell T(g)$, for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$, then $T \in CV_p(G,M_n)$.*

Proof. Let $(E_\alpha)_\alpha$ be a conjugate left bounded approximate identity for $L^1(G,M_n)$ with $\|E_\alpha\|_1 \leqslant 1$. Set $f = E_\alpha$. Clearly f is in $C_C(G,M_n)$ and $\|f\|_{L^1(G,M_n)} \leqslant 1$. Thus, for any $\varepsilon > 0$ and $g \in C_C(G,M_n)$, we have $\|f \circledast_\ell g - g\|_p < \varepsilon_1$, where ε_1 depends on ε. Since, $C_C(G,M_n)$ is dense in $L^p(G,M_n)$, we get that for any $\varepsilon > 0$ and $g \in L^p(G,M_n)$, $\|f \circledast_\ell g - g\|_p < 2\varepsilon_1$. Hence, for any $a \in G$, we have $\|_a(f \circledast_\ell g) - {}_a g\| < 2\varepsilon_1$. So, for each $a \in G$, we get

$$\|_a(T(f \circledast_\ell g)) - {}_aT(g)\|_p = \|_a(f \circledast_\ell T(g)) - {}_aT(g)\|_p < 2\varepsilon_1. \tag{19}$$

From boundedness of T, for each $a \in G$, we also get

$$\|T({}_a(f \circledast_\ell g)) - T({}_a g)\|_p < 2\|T\|\varepsilon_1. \tag{20}$$

Moreover, for any $g \in L^p(G,M_n)$, $h \in L^q(G,M_n^*)$ and $a \in G$,

$$\begin{aligned}
\langle T({}_a(f \circledast_\ell g)), h\rangle &= \langle {}_a(f \circledast_\ell g), T^*(h)\rangle \\
&= \mathrm{Tr}\left(\int_G {}_a(f \circledast_\ell g)(x) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y) f(y) g(y^{-1}axy) \otimes \Delta^{\frac{1}{q}}(y) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y)\, {}_a f(y) g(y^{-1}xay) \otimes \Delta^{\frac{1}{q}}(ay) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G ({}_a f \circledast_\ell g)_a(x) \otimes \Delta^{\frac{1}{q}}(a) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \Delta^{\frac{1}{q}}(a)\langle ({}_a f \circledast_\ell g)_a, T^*(h)\rangle \\
&= \langle \Delta^{\frac{1}{q}}(a) T(({}_a f \circledast_\ell g)_a), h\rangle.
\end{aligned} \tag{21}$$

Since $T \in RCV_p(G,M_n)$, (21) implies that

$$\begin{aligned}
T({}_a(f \circledast_\ell g)) &= \Delta^{\frac{1}{q}}(a) T(({}_a f \circledast_\ell g)_a) \\
&= \Delta^{\frac{1}{q}}(a) T({}_a f \circledast_\ell g)_a \\
&= \Delta^{\frac{1}{q}}(a) ({}_a f \circledast_\ell T(g))_a,
\end{aligned} \tag{22}$$

for all $a \in G$. On the other hand, by (18), ${}_aT(f \circledast_\ell g) = \Delta^{\frac{1}{q}}(a) \otimes ({}_a f \circledast_\ell T(g))_a$, for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$ and $a \in G$. Thus, (18) and (22) imply that

$${}_aT(f \circledast_\ell g) = T({}_a(f \circledast_\ell g)), \tag{23}$$

for all $a \in G$. We set $\varepsilon_1 = \varepsilon/2(\|T\|+1)$. Then (19), (20) and (23) imply that

$$\begin{aligned}
\|T({}_a g) - {}_aT(g)\|_p &\leqslant \|T({}_a g) - T({}_a(f \circledast_\ell g))\|_p + \|T({}_a(f \circledast_\ell g)) - {}_aT(f \circledast_\ell g)\|_p \\
&\quad + \|{}_aT(f \circledast_\ell g) - {}_aT(g)\|_p \\
&< \varepsilon.
\end{aligned}$$

This shows that $T \in LCV_p(G,M_n)$. $\square$

THEOREM 5. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If $T \in LCV_p(G,M_n)$ and $T(f \circledast_\ell g) = f \circledast_\ell T(g)$, for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$, then $T \in CV_p(G,M_n)$.*

Proof. By the same reasons in the proof of Theorem 4, for every $\varepsilon > 0$ there exists $f \in L^1(G,M_n)$ with $\|f\|_{L^1(G,M_n)} \leqslant 1$ such that for every $g \in L^p(G,M_n)$, $\|f \circledast_\ell g - g\|_p < 2\varepsilon_1$, where ε_1 depends on ε. Hence, for any $a \in G$, we have $\|{}_a(f \circledast_\ell g) - {}_ag\| < 2\varepsilon_1$. So, for each $a \in G$, we get

$$\|(T(f \circledast_\ell g))_a - T(g)_a\|_p = \|(f \circledast_\ell T(g))_a - T(g)_a\|_p < 2\varepsilon_1, \tag{24}$$

and

$$\|T({}_a(f \circledast_\ell g)) - T({}_ag)\|_p < 2\|T\|\varepsilon_1. \tag{25}$$

Moreover, for any $g \in L^p(G,M_n)$, $h \in L^q(G,M_n^*)$ and $a \in G$,

$$\begin{aligned}
\langle T((f \circledast_\ell g)_a), h\rangle &= \langle (f \circledast_\ell g)_a, T^*(h)\rangle \\
&= \mathrm{Tr}\left(\int_G (f \circledast_\ell g)_a(x) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y) f(y) g(y^{-1}xay) \otimes \Delta^{\frac{1}{q}}(y) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G\int_G \mathrm{d}m_G(y)\, {}_{a^{-1}}f(y) g(y^{-1}axy) \otimes \Delta^{\frac{1}{q}}(a^{-1}y) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \mathrm{Tr}\left(\int_G {}_a({}_{a^{-1}}f \circledast_\ell g)(x) \otimes \Delta^{\frac{-1}{q}}(a) T^*(h)(x)\,\mathrm{d}m_G(x)\right) \\
&= \Delta^{\frac{-1}{q}}(a) \langle {}_a({}_{a^{-1}}f \circledast_\ell g), T^*(h)\rangle \\
&= \langle \Delta^{\frac{-1}{q}}(a) T({}_a({}_{a^{-1}}f \circledast_\ell g)), h\rangle.
\end{aligned} \tag{26}$$

Since $T \in LCV_p(G,M_n)$, (26) implies that

$$T((f \circledast_\ell g)_a) = \Delta^{\frac{-1}{q}}(a)\, {}_a({}_{a^{-1}}f \circledast_\ell T(g)), \tag{27}$$

for all $a \in G$. Then by Lemma 5(ii) and (27), we have

$$T(f \circledast_\ell g)_a = T((f \circledast_\ell g)_a), \tag{28}$$

for all $a \in G$. We set $\varepsilon_1 = \varepsilon/2(\|T\|+1)$. Then (24), (25) and (28) imply that

$$\|T(g_a) - T(g)_a\|_p < \varepsilon.$$

This shows that $T \in RCV_p(G,M_n)$. $\square$

THEOREM 6. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If $T \in CV_p(G,M_n)$, then $T(g \circledast f) = T(g) \circledast f$, for all $f \in L^1(G,M_n)$ and $g \in L^p(G,M_n)$.*

Proof. By Lemma 1 we get $\|g \circledast f\|_p \leqslant \|g\|_p \|f\|_1$, for all $g \in L^p(G,M_n)$ and $f \in L^1(G,M_n)$. This shows that $L^p(G,M_n)$ is a right Banach $L^1(G,M_n)$-module respect to the right conjugate convolution product. Assume that $T \in CV_p(G,M_n)$. By a similar argument in the proof of Theorem 3, we have

$$\langle T(g) \circledast f, h\rangle = \langle T(g \circledast f), h\rangle,$$

for all $g \in L^p(G,M_n)$, $h \in L^q(G,M_n^*)$ and $f \in L^1(G,M_n)$. □

The proof of the following result is similar to the proof of Lemma 5 and we omit it.

LEMMA 7. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If, for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$, $T(g \circledast f) = T(g) \circledast f$, then, for any $a \in G$,*

(i) ${}_aT(g \circledast f) = \Delta^{\frac{1}{p}}(a) \otimes (T(g) \circledast {}_af)_a$.

(ii) $T(g \circledast f)_a = \Delta^{\frac{-1}{p}}(a) \otimes_a (T(g) \circledast {}_{a^{-1}}f)$.

Similar to Theorem 4, we now consider the converse of Theorem 6.

THEOREM 7. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G,M_n))$. If $T \in RCV_p(G,M_n)$ and $T(g \circledast f) = T(g) \circledast f$, for all $f \in L^1(G,M_n)$, $g \in L^p(G,M_n)$, then $T \in CV_p(G,M_n)$.*

Proof. Similar to the proof of Theorem 4, let $(E_\alpha)_\alpha$ be the obtained conjugate right bounded approximate identity for $(L^1(G,M_n),\circledast)$ with $\|E_\alpha\|_1 \leqslant 1$ in Lemma 6. Set $f = E_\alpha$. Then, for any $\varepsilon > 0$ and $g \in C_C(G,M_n)$, we have $\|g \circledast f - g\|_p < \varepsilon_1$, where ε_1 depends on ε. Hence, for any $\varepsilon > 0$ and $g \in L^p(G,M_n)$, $\|g \circledast f - g\|_p < 2\varepsilon_1$. Thus, for any $a \in G$, we have $\|{}_a(g \circledast f) - {}_ag\| < 2\varepsilon_1$. So, for each $a \in G$, we get

$$\|{}_a(T(g \circledast f)) - {}_aT(g)\|_p = \|{}_a(T(g) \circledast f) - {}_aT(g)\|_p < 2\varepsilon_1. \tag{29}$$

From boundedness of T, for each $a \in G$, we also get

$$\|T({}_a(g \circledast f)) - T({}_ag)\|_p < 2\|T\|\varepsilon_1. \tag{30}$$

Moreover, for any $g \in L^p(G,M_n)$, $h \in L^q(G,M_n^*)$ and $a \in G$, similar to (21), we have

$$\langle T({}_a(g \circledast f)), h\rangle = \langle \Delta^{\frac{1}{p}}(a) T((g \circledast {}_af)_a), h\rangle. \tag{31}$$

Since $T \in RCV_p(G,M_n)$, (31) implies that

$$\begin{aligned} T({}_a(g \circledast f)) &= \Delta^{\frac{1}{p}}(a) T((g \circledast {}_af)_a) \\ &= \Delta^{\frac{1}{p}}(a) T(g \circledast {}_af)_a \\ &= \Delta^{\frac{1}{p}}(a) (T(g) \circledast {}_af)_a, \end{aligned} \tag{32}$$

for all $a \in G$. Then by Lemma 7(i) and (32), we get that

$$ {}_aT(g \circledast f) = T({}_a(g \circledast f)), \tag{33} $$

for all $a \in G$. We set $\varepsilon_1 = \varepsilon/2(\|T\|+1)$. Then (29), (30) and (33) imply that

$$ \begin{aligned} \|T({}_ag) - {}_aT(g)\|_p &\leqslant \|T({}_ag) - T({}_a(g \circledast f))\|_p + \|T({}_a(g \circledast f)) - {}_aT(g \circledast f)\|_p \\ &\quad + \|{}_aT(g \circledast f) - {}_aT(g)\|_p \\ &< \varepsilon. \end{aligned} $$

This shows that $T \in LCV_p(G, M_n)$. $\square$

THEOREM 8. *Let G be a locally compact group, m_G be the left Haar measure on G and $T \in B(L^p(G, M_n))$. If $T \in LCV_p(G, M_n)$ and $T(g \circledast f) = T(g) \circledast f$, for all $f \in L^1(G, M_n)$, $g \in L^p(G, M_n)$, then $T \in CV_p(G, M_n)$.*

Proof. Similar to the proof of Theorem 7, for any $\varepsilon > 0$ there exists $f \in L^1(G, M_n)$ with norm less than 1 such that for any $g \in L^p(G, M_n)$, $\|g \circledast f - g\|_p < \varepsilon_1$, where ε_1 depends on ε. Thus, for any $a \in G$, we have $\|{}_a(g \circledast f) - {}_ag\| < 2\varepsilon_1$. So, for each $a \in G$, we get

$$ \|(T(g \circledast f))_a - T(g)_a\|_p < 2\varepsilon_1, \tag{34} $$

and

$$ \|T((g \circledast f)_a) - T(g_a)\|_p < 2\|T\|\varepsilon_1. \tag{35} $$

Moreover, for any $g \in L^p(G, M_n)$, $h \in L^q(G, M_n^*)$ and $a \in G$, we have

$$ \langle T((g \circledast f)_a), h\rangle = \langle \Delta^{\frac{-1}{p}}(a) \otimes T({}_a(g \circledast {}_{a^{-1}}f)), h\rangle. \tag{36} $$

On the other hand $T \in LCV_p(G, M_n)$, so (36) implies that

$$ T((g \circledast f)_a) = \Delta^{\frac{-1}{p}}(a) \otimes {}_a(T(g) \circledast {}_{a^{-1}}f), \tag{37} $$

for all $a \in G$. Then by Lemma 7(ii) and (37), we get that

$$ T(g \circledast f)_a = T((g \circledast f)_a), \tag{38} $$

for all $a \in G$. We set $\varepsilon_1 = \varepsilon/2(\|T\|+1)$. Then (34), (35) and (38) imply that

$$ \|T(g_a) - T(g)_a\|_p < \varepsilon. $$

Thus $T \in RCV_p(G, M_n)$. $\square$

4. Problems

In this section, we ask some questions that they have important role in the notion of the left (right) conjugate convolution operators on $L^p(G,M_n)$, where G is a locally compact group and $1 \leqslant p < \infty$.

1. Under which conditions a left (right) conjugate convolution operator on $L^p(G,M_n)$ is (weakly) compact?

2. The spectrum and eigenvalue sets of convolution operators on $L^p(G,M_n)$ are characterized in [2]. How we can characterize these sets for the left (right) conjugate convolution operators on $L^p(G,M_n)$?

3. Let $\{\sigma_t\}_{t>0}$ be a (one-parameter) convolution semigroup M_n-valued measures on G (for definition, see [2, Chapter 4]). Define $T_{t>0} : L^p(G,M_n) \longrightarrow L^p(G,M_n)$ by $T_t(f) = f \circledast \sigma_t$ and

$$\bigcap_{t>0} H_c\left(T_t, L^p(G,M_n)\right) = \{f \in L^p(G,M_n) : f = f \circledast \sigma_t \text{ for all } t > 0\}.$$

What is the dual space of $\bigcap_{t>0} H_c\left(T_t, L^p(G,M_n)\right)$?

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