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NECESSARY AND SUFFICIENT CONDITIONS FOR A DIFFERENCE CONSTITUTED BY FOUR DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION TO BE COMPLETELY MONOTONIC

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To Magnus Xi-Zhe Qi, my first grandson

(*Communicated by S. Varošanec*)

Abstract. In the paper, by virtue of convolution theorem for the Laplace transforms, Bernstein's theorem for completely monotonic functions, and other techniques, the author finds necessary and sufficient conditions for a difference constituted by four derivatives of a function involving trigamma function to be completely monotonic.

1. Motivations

In the literature [1, Section 6.4], the function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t, \quad \Re(z)>0$$

and its logarithmic derivative $\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$ are respectively called Euler's gamma function and digamma function. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for $k\in\{0\}\cup\mathbb{N}$ are known as polygamma functions, where $\mathbb{N}$ denotes the set of all positive integers.

Recall from Chapter XIII in [7], Chapter 1 in [22], and Chapter IV in [24] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x)\geqslant 0$ for $x\in I$ and $n\in\{0\}\cup\mathbb{N}$, then we call $f(x)$ a completely monotonic function on I.

There are a number of papers and mathematicians dedicated to investigation of complete monotonicity of some functions involving the gamma and polygamma functions. For more information and details, please refer to the papers [2, 4, 5, 18, 20, 27] and closely related references therein.

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Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}^n$. A n-tuple α is said to strictly majorize β (in symbols $\alpha \succ \beta$) if $(\alpha_{[1]}, \alpha_{[2]}, \dots, \alpha_{[n]}) \ne (\beta_{[1]}, \beta_{[2]}, \dots, \beta_{[n]})$, $\sum_{i=1}^{k} \alpha_{[i]} \geqslant \sum_{i=1}^{k} \beta_{[i]}$ for $1 \leqslant k \leqslant n-1$, and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$, where $\alpha_{[1]} \geqslant \alpha_{[2]} \geqslant \cdots \geqslant \alpha_{[n]}$ and $\beta_{[1]} \geqslant \beta_{[2]} \geqslant \cdots \geqslant \beta_{[n]}$ are rearrangements of α and β in a descending order. A real-valued function ϕ defined on a set $\mathscr{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathscr{A}$ if $\boldsymbol{x} \prec \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{A}$ means $\phi(\boldsymbol{x}) < \phi(\boldsymbol{y})$. See [6, p. 8, Definition A.1] and [6, p. 80, Definition A.1]. There have been a lot of literature such as the papers [3, 21, 23, 25, 28, 29] dedicated to investigation of Schur-convexity.

Let

$$G(x) = x\big[x\psi'(x) - 1\big] - \frac{1}{2} = x^2\left[\psi'(x) - \frac{1}{x} - \frac{1}{2x^2}\right], \quad x \in (0, \infty).$$

In [26, Theorem 1], the function $x^{\alpha} G(x)$ was proved to be completely monotonic on $(0, \infty)$ if and only if $\alpha \leqslant 0$. In other words, the completely monotonic degree of the function $\psi'(x) - \frac{1}{x} - \frac{1}{2x^2}$ with respect to x on $(0, \infty)$ is 2. For the notion of completely monotonic degrees, please refer to [10, 26] and closely related references therein.

For $k \in \{0\} \cup \mathbb{N}$ and $\theta_k, \tau_k \in \mathbb{R}$, let

$$\mathscr{G}_{k,\theta_k}(x) = G^{(2k+1)}(x) + \theta_k \big[G^{(k)}(x)\big]^2$$

and

$$\mathfrak{G}_{k,\tau_k}(x) = \frac{G^{(2k+1)}(x)}{\big[(-1)^k G^{(k)}(x)\big]^{\tau_k}}$$

on $(0, \infty)$. In [16, Theorem 3.1 and Theorem 4.1], the author discovered that,

1. if and only if $\theta_k \geqslant \frac{3(2k+2)!}{k!(k+1)!}$, the function $\mathscr{G}_{k,\theta_k}(x)$ is completely monotonic on $(0, \infty)$;

2. if and only if $\theta_k \leqslant 0$, the function $-\mathscr{G}_{k,\theta_k}(x)$ is completely monotonic on $(0, \infty)$;

3. if and only if $\tau_k \geqslant 2$, the function $\mathfrak{G}_{k,\tau_k}(x)$ is decreasing on $(0, \infty)$;

4. if $\tau_k \leqslant 1$, the function $\mathfrak{G}_{k,\tau_k}(x)$ is increasing on $(0, \infty)$;

5. only if

$$\tau_k \leqslant \begin{cases} \psi'(1), & k = 0 \\ -\dfrac{\psi'''(1)}{\psi'(1)\psi''(1)}, & k = 1 \\ \dfrac{k-1}{k} \dfrac{\psi^{(k-1)}(1)\psi^{(2k+1)}(1)}{\psi^{(k)}(1)\psi^{(2k)}(1)}, & k \geqslant 2, \end{cases}$$

   the function $\mathfrak{G}_{k,\tau_k}(x)$ is increasing on $(0, \infty)$;

6. the limits

$$\lim_{x\to 0^+} \mathfrak{G}_{k,\tau_k}(x) = \begin{cases} -2^{\tau_0}, & k=0 \\ 6\psi''(1), & k=1 \\ \dfrac{2(2k+1)}{(k-1)^{\tau_k}k^{\tau_k-1}} \dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k \geqslant 2 \end{cases}$$

and

$$\lim_{x\to\infty} \mathfrak{G}_{k,\tau_k}(x) = \begin{cases} -\infty, & \tau_k > 2 \\ -\dfrac{3(2k+2)!}{k!(k+1)!}, & \tau_k = 2 \\ 0, & \tau_k < 2 \end{cases}$$

are valid;

7. the double inequality

$$-\frac{3(2k+2)!}{k!(k+1)!} < \mathfrak{G}_{k,2}(x) < \begin{cases} -4, & k=0 \\ 6\psi''(1), & k=1 \\ \dfrac{2(2k+1)}{(k-1)^2 k} \dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k \geqslant 2 \end{cases}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.

For $m,n \in \{0\}\cup\mathbb{N}$, let

$$\mathscr{G}_{m,n}(x) = \frac{G^{(m+n+1)}(x)}{G^{(m)}(x)G^{(n)}(x)}$$

and

$$\mathscr{G}_{m,n;\lambda_{m,n}}(x) = G^{(m+n+1)}(x) + \lambda_{m,n}G^{(m)}(x)G^{(n)}(x)$$

on $(0,\infty)$. In [13, Theorems 3.1 and 4.1], the author obtained the following results:

1. the function $\mathscr{G}_{m,n}(x)$ is decreasing in $x\in(0,\infty)$ and maps from $(0,\infty)$,
   (a) if $(m,n)=(0,0)$, onto the interval $(-6,-4)$;
   (b) if $(m,n)\in\{(1,0),(0,1)\}$, onto the interval $\left(-12,-4\psi'(1)\right)$;
   (c) if $(m,n)\in\{(2,0),(0,2)\}$, onto the interval $\left(-18,\frac{6\psi''(1)}{\psi'(1)}\right)$;
   (d) if $(m,n)=(1,1)$, onto the interval $\left(-36,6\psi''(1)\right)$;
   (e) if $(m,n)\in\{(2,1),(1,2)\}$, onto the interval $\left(-72,-\frac{6\psi'''(1)}{\psi'(1)}\right)$;

(f) if $m,n \geqslant 2$, onto the interval

$$\left(-\frac{6(m+n+1)!}{m!n!}, \frac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \frac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}\right).$$

2. the double inequality

$$-\frac{6(m+n+1)!}{m!n!} < \mathscr{G}_{m,n}(x) < \begin{cases} -4, & (m,n) = (0,0) \\ -4\psi'(1), & (m,n) \in \{(1,0),(0,1)\} \\ \dfrac{6\psi''(1)}{\psi'(1)}, & (m,n) \in \{(2,0),(0,2)\} \\ 6\psi''(1), & (m,n) = (1,1) \\ -\dfrac{6\psi'''(1)}{\psi'(1)}, & (m,n) \in \{(2,1),(1,2)\} \\ \dfrac{(m+n+1)(m+n)}{mn(m-1)(n-1)} \dfrac{\psi^{(m+n)}(1)}{\psi^{(m-1)}(1)\psi^{(n-1)}(1)}, & m,n \geqslant 2 \end{cases}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively;

3. if and only if $\lambda_{m,n} \leqslant 0$, the function $(-1)^{m+n+1}\mathscr{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0,\infty)$;

4. if and only if $\lambda_{m,n} \geqslant \frac{6(m+n+1)!}{m!n!}$, the function $(-1)^{m+n}\mathscr{G}_{m,n;\lambda_{m,n}}(x)$ is completely monotonic on $(0,\infty)$.

In this paper, we would like to consider monotonicity of the function

$$\boldsymbol{G}_{i,j;p,q}(x) = \frac{G^{(i)}(x)G^{(j)}(x)}{G^{(p)}(x)G^{(q)}(x)}$$

and complete monotonicity of the function

$$\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x) = (-1)^{i+j}G^{(i)}(x)G^{(j)}(x) - (-1)^{\ell+m}\Lambda_{i,j;p,q}G^{(p)}(x)G^{(q)}(x) \qquad (1.1)$$

on $(0,\infty)$, where $i,j,p,q \in \{0\} \cup \mathbb{N}$ such that $(i,j) \succ (p,q)$. Figure 1 plotted by the software MATHEMATICA hints that the function $\boldsymbol{G}_{17,11;15,13}(x)$ is not monotonic in $x \in (0,\infty)$.

Therefore, in this paper, we will only consider the functions $\pm\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ and find necessary and sufficient conditions on $\Lambda_{i,j;p,q}$ for $\pm\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ to be completely monotonic on $(0,\infty)$.

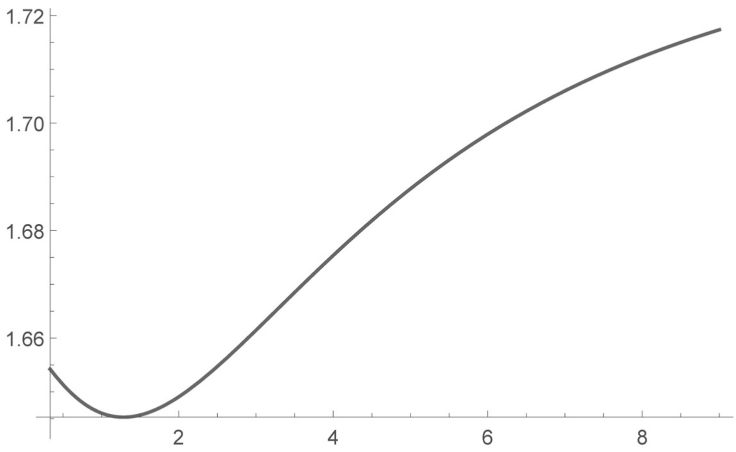

Figure 1: *The graph of the function* $\boldsymbol{G}_{17,11;15,13}(x)$ *on* $\left(\frac{1}{3},9\right)$

2. Lemmas

The following lemmas are necessary in this paper.

LEMMA 2.1. ([13, Lemma 2.3] and [16, Lemma 2.1]) *Let*

$$w(t)=\begin{cases}\dfrac{e^t[(t-2)e^t+t+2]}{(e^t-1)^3}, & t\neq 0;\\ \dfrac{1}{6}, & t=0.\end{cases}$$

Then the following conclusions are valid:

1. *the function* $w(t)$ *is infinitely differentiable, positive, and even on* $(-\infty,\infty)$, *is increasing on* $(-\infty,0)$, *and is decreasing on* $(0,\infty)$;
2. *the function* $w(t)$ *is logarithmically concave on* $(-\infty,\infty)$.

LEMMA 2.2. (Convolution theorem for the Laplace transforms [24, pp. 91–92]) *Let* $f_k(t)$ *for* $k=1,2$ *be piecewise continuous in arbitrary finite intervals included in* $(0,\infty)$. *If there exist some constants* $M_k>0$ *and* $c_k\geqslant 0$ *such that* $|f_k(t)|\leqslant M_k e^{c_k t}$ *for* $k=1,2$, *then*

$$\int_0^\infty\left[\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}u\right]e^{-st}\,\mathrm{d}t=\int_0^\infty f_1(u)e^{-su}\,\mathrm{d}u\int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}v.$$

LEMMA 2.3. ([11, Lemma 2.6]) *For* $m,n,p,q\in\mathbb{N}$ *such that* $(p,q)\succ(m,n)$, *the function*

$$\frac{s^{m-1}(1-s)^{n-1}+(1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1}+(1-s)^{p-1}s^{q-1}}$$

is increasing in $s \in \left(0, \frac{1}{2}\right)$.

LEMMA 2.4. (Bernstein's theorem [24, p. 161, Theorem 12b]) *A function* $f(x)$ *is completely monotonic on* $(0,\infty)$ *if and only if*

$$f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\sigma(t), \quad x \in (0,\infty), \tag{2.1}$$

where $\sigma(s)$ *is non-decreasing and the integral in* (2.1) *converges for* $x \in (0,\infty)$.

LEMMA 2.5. ([9, Lemma 2.4]) *For* $i, j, \ell, m \in \{0\} \cup \mathbb{N}$ *with* $(i, j) \succ (\ell, m)$, *the inequality* $i!j! > \ell!m!$ *is valid.*

LEMMA 2.6. ([10, Theorem 6.1]) *If* $f(x)$ *is differentiable and logarithmically concave on* $(-\infty,\infty)$, *then the product* $f(x)f(x_0 - x)$ *for any fixed number* $x_0 \in \mathbb{R}$ *is increasing in* $x \in \left(-\infty, \frac{x_0}{2}\right)$ *and decreasing in* $x \in \left(\frac{x_0}{2},\infty\right)$.

3. Necessary and sufficient conditions of complete monotonicity

In this section, we find necessary and sufficient conditions on $\Lambda_{i,j;p,q}$ for the functions $\pm\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ defined by (1.1) to be completely monotonic on $(0,\infty)$.

THEOREM 3.1. *For* $i, j, p, q \in \{0\} \cup \mathbb{N}$ *such that* $(i, j) \succ (p, q)$,

1. *if* $\Lambda_{i,j;p,q} \leqslant 1$, *the function* $\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ *defined by* (1.1) *is completely monotonic on* $(0,\infty)$;
2. *if and only if* $\Lambda_{i,j;p,q} \geqslant \frac{i!j!}{p!q!}$, *the function* $-\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ *is completely monotonic on* $(0,\infty)$;
3. *the double inequality*

$$1 < \frac{G^{(i)}(x)G^{(j)}(x)}{G^{(p)}(x)G^{(q)}(x)} < \frac{i!j!}{p!q!} \tag{3.1}$$

is valid on $(0,\infty)$ *and the right hand side inequality is sharp in the sense that the number* $\frac{i!j!}{p!q!}$ *can not be replaced by any smaller one.*

Proof. In the proof of [17, Theorem 4], the author derived an integral representation

$$G(x) = \int_0^\infty w(t)e^{-xt} \,\mathrm{d}t, \tag{3.2}$$

where $w(t)$ is defined in Lemma 2.1. Combining (3.2) with Lemma 2.2 gives

$$\begin{aligned}\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x) = &\int_0^\infty w(t)t^i e^{-xt} \,\mathrm{d}t \int_0^\infty w(t)t^j e^{-xt} \,\mathrm{d}t \\ &- \Lambda_{i,j;p,q} \int_0^\infty w(t)t^p e^{-xt} \,\mathrm{d}t \int_0^\infty w(t)t^q e^{-xt} \,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \left[\int_0^t u^i (t-u)^j w(u) w(t-u) \mathrm{d}u\right] e^{-xt} \mathrm{d}t \\
&\quad - \Lambda_{i,j;p,q} \int_0^\infty \left[\int_0^t u^p (t-u)^q w(u) w(t-u) \mathrm{d}u\right] e^{-xt} \mathrm{d}t \\
&= \int_0^\infty \left[\frac{\int_0^t u^i (t-u)^j w(u) w(t-u) \mathrm{d}u}{\int_0^t u^p (t-u)^q w(u) w(t-u) \mathrm{d}u} - \Lambda_{i,j;p,q}\right] \\
&\quad \times \left[\int_0^t u^p (t-u)^q w(u) w(t-u) \mathrm{d}u\right] e^{-xt} \mathrm{d}t \\
&= \int_0^\infty \left[\frac{\int_0^1 s^i (1-s)^j w(st) w((1-s)t) \mathrm{d}s}{\int_0^1 s^p (1-s)^q w(st) w((1-s)t) \mathrm{d}s} - \Lambda_{i,j;p,q}\right] \\
&\quad \times \left[\int_0^t u^p (t-u)^q w(u) w(t-u) \mathrm{d}u\right] e^{-xt} \mathrm{d}t.
\end{aligned}
$$

By Lemma 2.3, we obtain that the double inequality

$$
1 < \frac{s^i (1-s)^j + (1-s)^i s^j}{s^p (1-s)^q + (1-s)^p s^q} < \infty \tag{3.3}
$$

is valid and sharp for $s \in \left(0, \frac{1}{2}\right)$ and $(i,j) \succ (p,q)$. Hence, we have

$$
\begin{aligned}
\frac{\int_0^1 s^i (1-s)^j w(st) w((1-s)t) \mathrm{d}s}{\int_0^1 s^p (1-s)^q w(st) w((1-s)t) \mathrm{d}s} &= \frac{\int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] w(st) w((1-s)t) \mathrm{d}s}{\int_0^{1/2} [s^p (1-s)^q + s^q (1-s)^p] w(st) w((1-s)t) \mathrm{d}s} \\
&> 1.
\end{aligned} \tag{3.4}
$$

Consequently, by Lemma 2.4, when $\Lambda_{i,j;p,q} \leqslant 1$, the function $\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ is completely monotonic on $(0,\infty)$.

By virtue of Lemma 2.1, we acquire

$$
\begin{aligned}
\lim_{t \to 0^+} \frac{\int_0^1 s^i (1-s)^j w(st) w((1-s)t) \mathrm{d}s}{\int_0^1 s^p (1-s)^q w(st) w((1-s)t) \mathrm{d}s} &= \frac{\int_0^1 s^i (1-s)^j \mathrm{d}s}{\int_0^1 s^p (1-s)^q \mathrm{d}s} \\
&= \frac{B(i+1, j+1)}{B(p+1, q+1)} = \frac{i!j!}{p!q!}.
\end{aligned}
$$

Let

$$
\begin{aligned}
S_{i,j;p,q}(t) &= \int_0^{1/2} \left[s^i (1-s)^j + s^j (1-s)^i\right] w(st) w((1-s)t) \mathrm{d}s \\
&\quad - \frac{i!j!}{p!q!} \int_0^{1/2} [s^p (1-s)^q + s^q (1-s)^p] w(st) w((1-s)t) \mathrm{d}s \\
&= \int_0^{1/2} \bigg(\left[s^i (1-s)^j + s^j (1-s)^i\right] \\
&\quad - \frac{i!j!}{p!q!} [s^p (1-s)^q + s^q (1-s)^p] \bigg) w(st) w((1-s)t) \mathrm{d}s
\end{aligned}
$$

$$= \int_0^{1/2} \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!} \right] \\ \times [s^p(1-s)^q + s^q(1-s)^p] w(st) w((1-s)t) \mathrm{d}s$$

for $t \in (0,\infty)$ and $(i,j) \succ (p,q)$. By Lemma 2.3, Lemma 2.5, and the sharp inequality (3.3), we find that the function

$$\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!}$$

is decreasing in $s \in \left(0, \frac{1}{2}\right)$ and has a unique zero $s_0 \in \left(0, \frac{1}{2}\right)$ for $(i,j) \succ (p,q)$. As a result, utilizing Lemmas 2.1 and 2.6, we have

$$\begin{aligned} S_{i,j;p,q}(t) &= \left(\int_0^{s_0} + \int_{s_0}^{1/2} \right) \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!} \right] \\ &\quad \times [s^p(1-s)^q + s^q(1-s)^p] w(st) w(t-st) \mathrm{d}s \\ &< w(s_0t) w(t-s_0t) \int_0^{1/2} \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^p(1-s)^q + s^q(1-s)^p} - \frac{i!j!}{p!q!} \right] \\ &\quad \times [s^p(1-s)^q + s^q(1-s)^p] \mathrm{d}s \\ &= w(s_0t) w(t-s_0t) \int_0^{1/2} \bigg(s^i(1-s)^j + s^j(1-s)^i \\ &\quad - \frac{i!j!}{p!q!} [s^p(1-s)^q + s^q(1-s)^p] \bigg) \mathrm{d}s \\ &= w(s_0t) w(t-s_0t) \bigg(\int_0^{1/2} \left[s^i(1-s)^j + s^j(1-s)^i \right] \mathrm{d}s \\ &\quad - \frac{i!j!}{p!q!} \int_0^{1/2} [s^p(1-s)^q + s^q(1-s)^p] \mathrm{d}s \bigg) \\ &= w(s_0t) w(t-s_0t) \left[B(i+1,j+1) - \frac{i!j!}{p!q!} B(p+1,q+1) \right] \\ &= w(s_0t) w(t-s_0t) \left[\frac{i!j!}{(i+j+1)!} - \frac{i!j!}{p!q!} \frac{p!q!}{(p+q+1)!} \right] \\ &= 0. \end{aligned}$$

Accordingly, the inequality

$$\frac{\int_0^{1/2} \left[s^i(1-s)^j + s^j(1-s)^i \right] w(st) w((1-s)t) \mathrm{d}s}{\int_0^{1/2} [s^p(1-s)^q + s^q(1-s)^p] w(st) w((1-s)t) \mathrm{d}s} < \frac{i!j!}{p!q!}$$

is valid and sharp for $t \in (0,\infty)$ and $(i,j) \succ (p,q)$.

Therefore, by the equality in (3.4) and Lemma 2.4, for $(i,j) \succ (p,q)$, if and only if $\Lambda_{i,j;p,q} \geqslant \frac{i!j!}{p!q!}$, the function $-\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ is completely monotonic on $(0,\infty)$.

The double inequality (3.1) follows from complete monotonicity of the functions $\pm \boldsymbol{G}_{i,j;p,q;\Omega_{i,j;p,q}}(x)$ on $(0,\infty)$. The sharpness of the right hand side inequality in (3.1) follows from the limit

$$\lim_{x\to\infty}\left[(-1)^k x^{k+1} G^{(k)}(x)\right] = \frac{k!}{6}.$$

in [16, Lemma 2.2], where $k \geqslant 0$. The proof of Theorem 3.1 is complete. □

4. Remarks

Finally, we list several remarks.

REMARK 4.1. What is the necessary and sufficient condition on $\Lambda_{i,j;p,q}$ such that the function $\boldsymbol{G}_{i,j;p,q;\Lambda_{i,j;p,q}}(x)$ defined in (1.1) is completely monotonic on $(0,\infty)$? What is the sharp lower bound of the left hand side inequality in (3.1)?

REMARK 4.2. For $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $n > 2$, and two nonnegative integer tuples $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}_0^n$ with $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$, let

$$\boldsymbol{G}_{\boldsymbol{\alpha},\boldsymbol{\beta};C_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(x) = \prod_{r=1}^{n}\left[(-1)^{\alpha_r} G^{(\alpha_r)}(x)\right] - C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \prod_{r=1}^{n}\left[(-1)^{\beta_r} G^{(\beta_r)}(x)\right]$$

on $(0,\infty)$. One can discuss necessary and sufficient conditions on $C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{R}$ such that the functions $\pm \boldsymbol{G}_{\boldsymbol{\alpha},\boldsymbol{\beta};C_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(x)$ are respectively completely monotonic on $(0,\infty)$.

REMARK 4.3. This paper is a revised version of the electronic preprint [8] and the tenth one in a series of articles including [15, 9, 11, 12, 13, 14, 16, 17, 19].

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