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BESOV–MORREY SPACES AND VOLTERRA INTEGRAL OPERATOR

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Abstract. In this paper, we introduce a class of Besov-Morrey spaces $B_p^\lambda(s)$. For any positive Borel measure μ, we characterize the boundedness and compactness of the identity operator from $B_p^\lambda(s)$ spaces into tent spaces $T_t^q(\mu)$. As an application, the boundedness, compactness and essential norm of the Volterra integral operator T_g from $B_p^\lambda(s)$ spaces to some general function spaces are also investigated.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. For any arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ denote the normalized length of I and $S(I)$ be the Carleson box defined by

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leqslant |z| < 1,\ z/|z| \in I\}.$$

Let $0 < p < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. We say that μ is a p-Carleson measure if

$$\|\mu\|_{CM_p} := \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

When $p = 1$, it gives the classical Carleson measure. μ is said to be a vanishing p-Carleson measure if $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^p} = 0$. When $p = 1$, it gives the vanishing Carleson measure.

Let $0 \leqslant t < \infty$, $0 < q < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Let $T_t^q(\mu)$ be the space of all μ-measurable functions f such that (see, e.g., [16])

$$\|f\|_{T_t^q(\mu)}^q = \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^t}\int_{S(I)} |f(z)|^q d\mu(z) < \infty.$$

For $0 \leqslant s < 1 < p < \infty$, the Besov-type space, denoted by $B_p(s)$, is the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{B_p(s)}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty.$$

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Here dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D})=1$. In particular, $B_p(0)$ is the Besov space, and we always denote it by B_p.

Let $0<p<\infty$, $-2<q<\infty$ and $0\leqslant s<\infty$. The space $F(p,q,s)$, introduced by Zhao in [34], is the space consisting of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^p=|f(0)|^p+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\sigma_a(z)|^2)^s dA(z)<\infty,$$

where $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$. When $q+s>-1$, the space $F(p,q,s)$ is nontrivial. It is easy to see that $F(p,p-2+s,0)$ is the Besov-type space $B_p(s)$, $F(2,0,s)=Q_s$, the Q_s space, and $F(2,0,1)=BMOA$, the space of analytic functions of bounded mean oscillation. $F(p,p,0)$ is just the classical Bergman space A^p. When $s>1$, $F(p,p-2,s)$ is equivalent to the Bloch space ([34]), denoted by $\mathscr{B}$, consisting of all $f\in H(\mathbb{D})$ such that $\|f\|_{\mathscr{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty$. We will denote $F(2,q,0)$ by $\mathscr{D}_q^2$ in this paper.

Let $0\leqslant\lambda\leqslant 1$. The analytic Morrey space $\mathscr{L}^{2,\lambda}$, which was introduced in [33], is the space of all $f\in H^2$ such that

$$\|f\|_{\mathscr{L}^{2,\lambda}}=|f(0)|+\sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{(1-\lambda)}{2}}\|f\circ\sigma_a-f(a)\|_{H^2}<\infty.$$

Clearly, $\mathscr{L}^{2,1}$ coincides with the $BMOA$ space. $\mathscr{L}^{2,0}$ is just the Hardy space H^2 (see [9, 14]). Moreover, $BMOA\subset\mathscr{L}^{2,\lambda}\subset H^2$ for $0<\lambda<1$. The space $\mathscr{L}^{2,\lambda}$ was investigated in [9, 13, 14, 33].

Let $0\leqslant p,\lambda\leqslant 1$. In [4] was introduced the Dirichlet-Morrey space $\mathscr{D}_p^{2,\lambda}$, which consists of all $f\in\mathscr{D}_p^2$ such that

$$\|f\|_{\mathscr{D}_p^{2,\lambda}}=|f(0)|+\sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{p(1-\lambda)}{2}}\|f\circ\sigma_a-f(a)\|_{\mathscr{D}_p^2}<\infty.$$

It is easy to check that $\mathscr{D}_1^{2,\lambda}=\mathscr{L}^{2,\lambda},\mathscr{D}_p^{2,1}=Q_p,\mathscr{D}_p^{2,0}=\mathscr{D}_p^2$ and

$$Q_p\subset\mathscr{D}_p^{2,\lambda}\subset\mathscr{D}_p^2,\qquad 0<\lambda<1.$$

They studied the boundedness and compactness of the Volterra operator T_g on the space $\mathscr{D}_p^{2,\lambda}$. For example, if T_g is bounded on $\mathscr{D}_p^{2,\lambda}$, then $g\in Q_p$, while if $g\in W_p$, then T_g is bounded on $\mathscr{D}_p^{2,\lambda}$. Here the space W_p is the space consisting of all functions $g\in H(\mathbb{D})$ such that (see [4])

$$\int_{\mathbb{D}}|f(z)|^2|g'(z)|^2(1-|z|^2)^p dA(z)\leqslant C\|f\|_{\mathscr{D}_p^2}^2,\ f\in\mathscr{D}_p^2.$$

In this paper, we introduce a class of Morrey spaces, which we will call Besov-Morrey spaces, and denote them by $B_p^\lambda(s)$. Let $0<s,\lambda<1<p<\infty$. We say that an $f\in B_p(s)$ belongs to the Besov-Morrey space $B_p^\lambda(s)$, if

$$\|f\|_{B_p^\lambda(s)}=|f(0)|+\sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{s(1-\lambda)}{p}}\|f\circ\sigma_a(z)-f(a)\|_{B_p(s)}<\infty.$$

Under the above norm, $B_p^\lambda(s)$ is a Banach space. It is easy to see that $B_p^1(s) = F(p, p-2, s)$ and $B_p^0(s) = B_p(s)$. Moreover,

$$F(p,p-2,s) \subset B_p^\lambda(s) \subset B_p(s), \qquad 0 < \lambda < 1.$$

Let $f, g \in H(\mathbb{D})$. The Volterra integral operator T_g is defined by

$$T_g f(z) = \int_0^z f(\xi) g'(\xi) d\xi, \ z \in \mathbb{D}.$$

In [17], Pommerenke showed that T_g is bounded on H^2 if and only if $g \in BMOA$. In [1], Aleman and Siskakis proved that T_g is bounded on H^p $(p \geqslant 1)$ if and only if $g \in BMOA$. In [2], the authors showed that T_g is bounded on the Bergman space A^p if and only if $g \in \mathscr{B}$. See [7, 8, 10, 12, 13, 18, 19, 21, 22, 23, 24, 25, 26, 29, 35] and the references therein for more study of the operator T_g.

The rest of this paper is organized as follows. In Section 2, some basic properties of Besov-Morrey spaces were studied. In Section 3, we study the boundedness and compactness of the identity operator Id from $B_p^\lambda(s)$ to tent spaces $T_t^q(\mu)$. As an application, the boundedness, compactness and the essential norm of the Volterra integral operator $T_g : B_p^\lambda(s) \to F(q, q-2+\frac{qs(1-\lambda)}{p}, t)$ are given in Section 4.

In this paper, we say that $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leqslant Cg$. Denote by $f \asymp g$ whenever $f \lesssim g \lesssim f$.

2. Some basic properties

PROPOSITION 1. *Let* $0 < s < 1$, $0 < \lambda \leqslant 1 < p < \infty$ *and* $f \in H(\mathbb{D})$. *Then* $f \in B_p^\lambda(s)$ *if and only if*

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^{s\lambda}} \int_{S(I)} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty. \tag{1}$$

Proof. Assume that $f \in B_p^\lambda(s)$. Given any arc $I \subset \partial\mathbb{D}$, let $a = (1-|I|)\xi$, where ξ is the center of I. We have

$$|1-\bar{a}z| \approx 1-|a|^2 \approx |I|, \ z \in S(I).$$

Note that

$$\begin{aligned}
\|f\|^p_{B_p^\lambda(s)} &\geqslant (1-|a|^2)^{s(1-\lambda)} \|f\circ\sigma_a - f(a)\|^p_{B_p(s)} \\
&= (1-|a|^2)^{s(1-\lambda)} \int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^p (1-|z|^2)^{p-2+s} dA(z) \\
&= (1-|a|^2)^{s(1-\lambda)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^s dA(z) \\
&= (1-|a|^2)^{s(1-\lambda)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}} dA(z) \\
&\geqslant \frac{1}{|I|^{s\lambda}} \int_{S(I)} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z),
\end{aligned}$$

which implies the desired result by the arbitrariness of I.

Conversely, suppose that (1) holds. Let $d\mu_f(z) = |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z)$. Then

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu_f(S(I))}{|I|^{s\lambda}} = \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{s\lambda}} \int_{S(I)} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z) < \infty.$$

So μ_f is an $s\lambda$-Carleson measure. Then for $a\in\mathbb{D}$,

$$\begin{aligned}\|f\circ\sigma_a - f(a)\|_{B_p(s)}^p &= \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^s dA(z)\\ &= \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}\frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}dA(z)\\ &= \int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}d\mu_f(z).\end{aligned}$$

Thus

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{s(1-\lambda)}\|f\circ\sigma_a - f(a)\|_{B_p(s)}^p &= \sup_{a\in\mathbb{D}}(1-|a|^2)^{s(1-\lambda)}\int_{\mathbb{D}} \frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}d\mu_f(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^{2s-s\lambda}}{|1-\bar{a}z|^{2s}}d\mu_f(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|a|^2)^{q}}{|1-\bar{a}z|^{p+q}}d\mu_f(z)\\ &< \infty,\end{aligned}$$

where $q=(2-\lambda)s>0$, $p=s\lambda>0$. The last inequality used the Lemma 2.2 in [16]. The proof is complete. □

REMARK 1. From the proof of Proposition 1, we see that

$$\|f\|_{B_p^\lambda(s)}^p \approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{s\lambda}} \int_{S(I)} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z).$$

PROPOSITION 2. *Let* $0<s,\lambda<1<p<\infty$ *and* $f\in H(\mathbb{D})$. *For any* $f\in B_p^\lambda(s)$,

$$|f(z)| \lesssim \frac{\|f\|_{B_p^\lambda(s)}}{(1-|z|^2)^{\frac{s(1-\lambda)}{p}}},\ z\in\mathbb{D}.$$

Proof. Suppose that $f\in B_p^\lambda(s)$. By Lemma 4.12 in [36], for any analytic function g on $\mathbb{D}$,

$$|g(0)|^p \leqslant (p-1+s)\int_{\mathbb{D}} |g(z)|^p(1-|z|^2)^{p-2+s}dA(z).$$

Apply the function $g = (f\circ\sigma_a - f(a))'$ to the above inequality, we obtain

$$\begin{aligned}|f'(a)|^p(1-|a|^2)^p &\leqslant (p-1+s)\int_{\mathbb{D}}|(f\circ\sigma_a)'(z)|^p(1-|z|^2)^{p-2+s}dA(z)\\ &= \frac{(p-1+s)}{(1-|a|^2)^{s(1-\lambda)}}(1-|a|^2)^{s(1-\lambda)}\|f\circ\sigma_a - f(a)\|^p_{B_p(s)}\\ &\lesssim \frac{\|f\|^p_{B^\lambda_p(s)}}{(1-|a|^2)^{s(1-\lambda)}},\end{aligned}$$

for each $a\in\mathbb{D}$. Thus

$$|f'(a)| \lesssim \frac{\|f\|_{B^\lambda_p(s)}}{(1-|a|^2)^{\frac{s(1-\lambda)}{p}+1}},\ a\in\mathbb{D}.$$

Since $f(z)-f(0)=\int_0^z f'(\xi)d\xi$, by integrating both sides of the last inequality, we obtain the desired result. □

3. Embedding $B^\lambda_p(s)$ into tent spaces

In this section, we study the boundedness and compactness of the identity operator $Id : B^\lambda_p(s) \to T^q_t(\mu)$. We say that Id is compact if

$$\lim_{n\to\infty}\frac{1}{|I|^t}\int_{S(I)}|f_n(z)|^q d\mu(z) = 0,$$

where $I\subset\partial\mathbb{D}$, $\{f_n\}$ is a bounded sequence in $B^\lambda_p(s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$.

LEMMA 1. [16, Corollary 2.5] *Let* $a,b\in\mathbb{D}$ *and* $r>-1$, $s,t>0$ *such that* $0<s+t-r-2<s$. *Then*

$$\int_{\mathbb{D}}\frac{(1-|z|^2)^r}{|1-\bar{a}z|^s|1-\bar{b}z|^t}dA(z) \lesssim \frac{1}{(1-|a|^2)^{s+t-r-2}}.$$

LEMMA 2. *Let* $0<s<1$, $0<\lambda<1<p<\infty$ *and* $b\in\mathbb{D}$. *Then the function*

$$f_b(z) = \frac{1}{(1-\bar{b}z)^{\frac{s(1-\lambda)}{p}}},$$

belongs to $B^\lambda_p(s)$.

Proof. By Lemma 1, we obtain

$$
\begin{aligned}
\|f_b\|^p_{B^\lambda_p(s)} &\approx \sup_{a\in\mathbb{D}}(1-|a|^2)^{s(1-\lambda)}\|f_b\circ\sigma_a-f(a)\|^p_{B_p(s)}\\
&=\sup_{a\in\mathbb{D}}(1-|a|^2)^{s(1-\lambda)}\int_{\mathbb{D}}|(f_b\circ\sigma_a)'(z)|^p(1-|z|^2)^{p-2+s}dA(z)\\
&=\sup_{a\in\mathbb{D}}(1-|a|^2)^{s(1-\lambda)}\int_{\mathbb{D}}|f_b'(z)|^p(1-|z|^2)^{p-2+s}\frac{(1-|a|^2)^s}{|1-\bar{a}z|^{2s}}dA(z)\\
&\lesssim\sup_{a\in\mathbb{D}}(1-|a|^2)^{s(1-\lambda)}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-2+s}(1-|a|^2)^s}{|1-\bar{b}z|^{p+s(1-\lambda)}|1-\bar{a}z|^{2s}}dA(z)\\
&=\sup_{a\in\mathbb{D}}(1-|a|^2)^{s(2-\lambda)}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-2+s}}{|1-\bar{b}z|^{p+s(1-\lambda)}|1-\bar{a}z|^{2s}}dA(z)\\
&<\infty,
\end{aligned}
$$

as desired. □

LEMMA 3. [3] *Let $1<p<\infty$, $s>-1$, $t\geqslant 0$ such that $t<2+s$. If $f\in H(\mathbb{D})$, then*

$$\int_{\mathbb{D}}|f(z)-f(0)|^p\frac{(1-|z|^2)^s}{|1-\bar{w}z|^t}dA(z)\lesssim\int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|^2)^{s+p}}{|1-\bar{w}z|^t}dA(z).$$

Now we are in a position to state and prove the main results in this section.

THEOREM 1. *Let μ be a positive Borel measure on $\mathbb{D}$, $0<s<1$, $0<\lambda<1<p<q<\infty$, $0<t<\infty$ such that $\frac{pt}{q}+s\lambda>s$ and $\frac{pt}{q}-s\lambda\geqslant 0$. Then the identity operator $Id: B^\lambda_p(s)\to T^q_t(\mu)$ is bounded if and only if μ is a $(t+\frac{qs(1-\lambda)}{p})$-Carleson measure.*

Proof. First we suppose that $Id: B^\lambda_p(s)\to T^q_t(\mu)$ is bounded. For any $I\subset\partial\mathbb{D}$, let ξ be the midpoint of I and $a=(1-|I|)\xi$. Set

$$f_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^{\frac{s(1-\lambda)+p}{p}}},\ z\in\mathbb{D}.$$

By Lemma 2, we see that $f_a\in B^\lambda_p(s)$ with $\sup_{a\in\mathbb{D}}\|f_a\|_{B^\lambda_p(s)}\lesssim 1$. Since $|1-\bar{a}z|\approx 1-|a|^2\approx|I|$, $z\in S(I)$, we get

$$\frac{\mu(S(I))}{|I|^{t+\frac{qs(1-\lambda)}{p}}}\approx\frac{1}{|I|^t}\int_{S(I)}|f_a(z)|^q d\mu(z)\lesssim\|f_a\|^q_{B^\lambda_p(s)}<\infty,$$

which implies that μ is a $(t+\frac{qs(1-\lambda)}{p})$-Carleson measure.

Conversely, let μ be a $(t+\frac{qs(1-\lambda)}{p})$-Carleson measure. Let $f\in B_p^\lambda(s)$. For any $I\subset\partial\mathbb{D}$, let ξ be the midpoint of I and $a=(1-|I|)\xi$. Note that

$$\begin{aligned}\frac{1}{|I|^t}\int_{S(I)}|f(z)|^q d\mu(z)&\lesssim\frac{1}{|I|^t}\int_{S(I)}|f(a)|^q d\mu(z)+\frac{1}{|I|^t}\int_{S(I)}|f(z)-f(a)|^q d\mu(z)\\&:=E_1+E_2.\end{aligned}$$

By Proposition 2, $|f(z)|\lesssim\frac{\|f\|_{B_p^\lambda(s)}}{(1-|z|^2)^{\frac{s(1-\lambda)}{p}}}$. Hence

$$\begin{aligned}E_1=\frac{1}{|I|^t}\int_{S(I)}|f(a)|^q d\mu(z)&\lesssim\frac{1}{|I|^t}\int_{S(I)}\frac{\|f\|^q_{B_p^\lambda(s)}}{(1-|a|^2)^{\frac{qs(1-\lambda)}{p}}}d\mu(z)\\&\lesssim\frac{\mu(S(I))}{|I|^{t+\frac{qs(1-\lambda)}{p}}}\|f\|^q_{B_p^\lambda(s)}\\&\lesssim\|f\|^q_{B_p^\lambda(s)}.\end{aligned}$$

Using Theorem 1 of [6] and the assumption that $\frac{pt}{q}+s(1-\lambda)>0$, we see that μ is a $(t+\frac{qs(1-\lambda)}{p})$-Carleson measure if and only if $\mathscr{D}^p_{p-2+s+\frac{pt}{q}-s\lambda}\subset L^q(\mu)$. Note that $f\in B_p^\lambda(s)\subset\mathscr{D}^p_{p-2+s+\frac{pt}{q}-s\lambda}$. We obtain

$$\begin{aligned}E_2&=\frac{1}{|I|^t}\int_{S(I)}|f(z)-f(a)|^q d\mu(z)\\&\lesssim(1-|a|^2)^t\int_{S(I)}\left|\frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{2t}{q}}}\right|^q d\mu(z)\\&\lesssim\left((1-|a|^2)^{\frac{pt}{q}}\int_{\mathbb{D}}\left|\frac{d}{dz}\frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{2t}{q}}}\right|^p(1-|z|^2)^{p-2+s+\frac{pt}{q}-s\lambda}dA(z)\right)^{\frac{q}{p}}.\end{aligned}$$

Since

$$\frac{d}{dz}\frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{2t}{q}}}=\frac{f'(z)(1-\bar{a}z)^{\frac{2t}{q}}+\bar{a}(\frac{2t}{q})(f(z)-f(a))(1-\bar{a}z)^{\frac{2t}{q}-1}}{(1-\bar{a}z)^{\frac{4t}{q}}},$$

we deduce that $E_2\lesssim(I+J)^{\frac{q}{p}}$, where

$$I=(1-|a|^2)^{\frac{pt}{q}}\int_{\mathbb{D}}\frac{|f'(z)|^p}{|1-\bar{a}z|^{\frac{2pt}{q}}}(1-|z|^2)^{p-2+s+\frac{pt}{q}-s\lambda}dA(z)$$

and

$$J=(1-|a|^2)^{\frac{pt}{q}}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^{\frac{2pt}{q}+p}}(1-|z|^2)^{p-2+s+\frac{pt}{q}-s\lambda}dA(z).$$

Since $\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|\sigma_a(z)|^2)$, by the assumption that $\frac{pt}{q} - s\lambda \geqslant 0$, we have

$$\begin{aligned}
I &= (1-|a|^2)^{\frac{pt}{q}} \int_{\mathbb{D}} \frac{|f'(z)|^p}{|1-\bar{a}z|^{\frac{2pt}{q}}} (1-|z|^2)^{p-2+s+\frac{pt}{q}-s\lambda} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s-s\lambda} (1-|\sigma_a(z)|^2)^{\frac{pt}{q}} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s-s\lambda} (1-|\sigma_a(z)|^2)^{s\lambda} dA(z) \\
&\lesssim \sup_{I\subseteq \partial\mathbb{D}} \frac{1}{|I|^{s\lambda}} \int_{S(I)} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) \\
&\approx \|f\|^p_{B^\lambda_p(s)}.
\end{aligned}$$

Making the change of variable $w = \sigma_a(z)$, by Lemma 3 and the assumption that $\frac{pt}{q} + s\lambda > s$ and $\frac{pt}{q} - s\lambda \geqslant 0$ we obtain

$$\begin{aligned}
J &= (1-|a|^2)^{\frac{pt}{q}} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^{\frac{2pt}{q}+p}} (1-|z|^2)^{p-2+s+\frac{pt}{q}-s\lambda} dA(z) \\
&= \int_{\mathbb{D}} |f\circ\sigma_a(w) - f\circ\sigma_a(0)|^p \frac{(1-|w|^2)^{p-2+s+\frac{pt}{q}-s\lambda}(1-|a|^2)^{s-s\lambda}}{|1-\bar{a}w|^{p+2s-2s\lambda}} dA(w) \\
&\lesssim \int_{\mathbb{D}} |(f\circ\sigma_a)'(w)|^p \frac{(1-|w|^2)^{2p-2+s+\frac{pt}{q}-s\lambda}(1-|a|^2)^{s-s\lambda}}{|1-\bar{a}w|^{p+2s-2s\lambda}} dA(w) \text{ (Lemma 3)} \\
&= \int_{\mathbb{D}} |f'(\sigma_a(w))|^p (1-|\sigma_a(w)|^2)^p \frac{(1-|w|^2)^{p-2+s+\frac{pt}{q}-s\lambda}(1-|a|^2)^{s-s\lambda}}{|1-\bar{a}w|^{p+2s-2s\lambda}} dA(w) \\
&= \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \frac{(1-|\sigma_a(z)|^2)^{p-2+s+\frac{pt}{q}-s\lambda}(1-|a|^2)^{s-s\lambda}}{|1-\bar{a}\sigma_a(z)|^{p+2s-2s\lambda}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \\
&= \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|a|^2)^{\frac{pt}{q}}(1-|z|^2)^{2p-2+s+\frac{pt}{q}-s\lambda}}{|1-\bar{a}z|^{p+\frac{2pt}{q}}} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s-s\lambda} (1-|\sigma_a(z)|^2)^{\frac{pt}{q}} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s-s\lambda} (1-|\sigma_a(z)|^2)^{s\lambda} dA(z) \\
&\lesssim \|f\|^p_{B^\lambda_p(s)}.
\end{aligned}$$

Hence, $E_2 \lesssim \|f\|^q_{B^\lambda_p(s)}$. Therefore,

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^t} \int_{S(I)} |f(z)|^q d\mu(z) \lesssim \|f\|^q_{B^\lambda_p(s)},$$

which implies the desired result. The proof is complete. □

THEOREM 2. *Let* $0 < s < 1$, $0 < \lambda < 1 < p < q < \infty$, $0 < t < \infty$ *such that* $\frac{pt}{q} + s\lambda > s$ *and* $\frac{pt}{q} - s\lambda \geqslant 0$. *Let* μ *be a positive Borel measure on* $\mathbb{D}$ *such that point evaluation is a bounded functional in* $T_t^q(\mu)$. *Then the identity operator* $Id : B_p^\lambda(s) \to T_t^q(\mu)$ *is compact if and only if* μ *is a vanishing* $(t + \frac{qs(1-\lambda)}{p})$*-Carleson measure.*

Proof. First, we suppose that $Id : B_p^\lambda(s) \to T_t^q(\mu)$ is compact. Let $\{I_n\}$ be a sequence arcs with $\lim_{n\to\infty} |I_n| = 0$. Set $b_n = (1-|I_n|)\xi_n$, where ξ_n is the midpoint of I_n. Take

$$f_n(z) = \frac{1-|b_n|^2}{(1-\bar{b}_n z)^{\frac{s(1-\lambda)+p}{p}}},\ z \in \mathbb{D}.$$

We know that $f_n \in B_p^\lambda(s)$ and $\{f_n\}$ converges to 0 uniformly on every compact subset of $\mathbb{D}$ when $n \to \infty$. Then we have

$$\frac{\mu(S(I_n))}{|I_n|^{t+\frac{qs(1-\lambda)}{p}}} \approx \frac{1}{|I_n|^t} \int_{S(I)} |f_n(z)|^q d\mu(z) \to 0 \quad (n \to \infty),$$

which implies that μ is a vanishing $(t + \frac{qs(1-\lambda)}{p})$-Carleson measure.

Conversely, suppose that μ is a vanishing $(t + \frac{qs(1-\lambda)}{p})$-Carleson measure. From [16] we get

$$\|\mu - \mu_r\|_{CM_{t+\frac{qs(1-\lambda)}{p}}} \to 0,\ r \to 1.$$

Here $\mu_r(z) = \mu(z)$ for $|z| < r$ and $\mu_r(z) = 0$ for $r \leqslant |z| < 1$. Let $f_n \in B_p^\lambda(s)$ such that $\|f_n\|_{B_p^\lambda(s)} \lesssim 1$ and $\{f_n\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\begin{aligned}
&\frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |f_n(z)|^q d\mu(z) \\
\lesssim &\frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |f_n(z)|^q d\mu_r(z) + \frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |f_n(z)|^q d(\mu-\mu_r)(z) \\
\lesssim &\frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |f_n(z)|^q d\mu_r(z) + \|\mu-\mu_r\|_{CM_{t+\frac{qs(1-\lambda)}{p}}} \|f_n\|^q_{B_p^\lambda(s)} \\
\lesssim &\frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |f_n(z)|^q d\mu_r(z) + \|\mu-\mu_r\|_{CM_{t+\frac{qs(1-\lambda)}{p}}}.
\end{aligned}$$

Letting $n \to \infty$ and then $r \to 1$, we have $\lim_{n\to\infty} \|f_n\|_{T_t^q(\mu)} = 0$. Therefore $Id : B_p^\lambda(s) \to T_t^q(\mu)$ is compact. The proof is complete. □

4. An application

In this section, by using Theorem 1, we completely characterize the boundedness, compactness and essential norm of the operator $T_g : B_p^\lambda(s) \to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$.

THEOREM 3. *Let $g \in H(\mathbb{D})$, $0<s<1$, $0<\lambda<1<p<q<\infty$, $0<t<1$ such that $\frac{pt}{q}+s\lambda > s$ and $\frac{pt}{q}-s\lambda \geqslant 0$. Then $T_g : B_p^\lambda(s) \to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$ is bounded if and only if $g \in F(q,q-2,t+\frac{qs(1-\lambda)}{p})$.*

Proof. Assume that $T_g : B_p^\lambda(s) \to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$ is bounded. For any $I \subset \partial\mathbb{D}$, let ξ be the midpoint of I and $a=(1-|I|)\xi$. Set $f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{s(1-\lambda)+p}{p}}}$, $z\in\mathbb{D}$. Then $f_a \in B_p^\lambda(s)$ and $\|f_a\|_{B_p^\lambda} \lesssim 1$. Thus,

$$\|T_g f_a\|_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)} \lesssim \|T_g\|\|f_a\|_{B_p^\lambda(s)} \lesssim \|T_g\|.$$

We have

$$\begin{aligned}
\infty &> \|T_g f_a\|^q_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)} \\
&= \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |(T_g f_a)'(z)|^q (1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2}(1-|\sigma_b(z)|^2)^t dA(z) \\
&\gtrsim \int_{\mathbb{D}} |f_a(z)|^q |g'(z)|^q (1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2}(1-|\sigma_a(z)|^2)^t dA(z) \\
&\gtrsim \frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |g'(z)|^q (1-|z|^2)|^{q-2+t+\frac{qs(1-\lambda)}{p}} dA(z),
\end{aligned}$$

which implies that $g \in F(q,q-2,t+\frac{qs(1-\lambda)}{p})$ by Proposition 1(take $\lambda=1$).

Conversely, suppose that $g \in F(q,q-2,t+\frac{qs(1-\lambda)}{p})$. From [36] and Proposition 1 we obtain

$$\begin{aligned}
\|g\|^q_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})} &\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{t+\frac{qs(s-\lambda)}{p}} dA(z) \\
&\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{t+\frac{qs(1-\lambda)}{p}}} \int_{S(I)} |g'(z)|^q (1-|z|^2)^{t+\frac{qs(1-\lambda)}{p}+q-2} dA(z) \\
&\approx \sup_{I\subset\partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^{t+\frac{qs(1-\lambda)}{p}}},
\end{aligned}$$

which means that μ_g is a $(t+\frac{qs(1-\lambda)}{p})$-Carleson measure, where $\mu_g = |g'(z)|^q(1-|z|^2)^{t+\frac{qs(1-\lambda)}{p}+q-2}dA(z)$. By Theorem 1, the identity operator $Id : B_p^\lambda(s) \to T_t^q(\mu)$ is bounded. Let $f \in B_p^\lambda(s)$. We deduce that

$$\begin{aligned}
\|T_g f\|^q_{F(q,\frac{qs(1-\lambda)}{p}+q-2,t)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2} (1-|\sigma_a(z)|^2)^t dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^q |g'(z)|^q (1-|z|^2)^{t+\frac{qs(1-\lambda)}{p}+q-2} \frac{(1-|a|^2)^t}{|1-\bar{a}z|^{2t}} dA(z) \\
&\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^t} \int_{\mathbb{D}} |f(z)|^q d\mu_g(z) \\
&= \|f\|^q_{T_t^q(\mu)} \lesssim \|f\|^q_{B_p^\lambda(s)} \|g\|^q_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})} < \infty.
\end{aligned}$$

Therefore $T_g : B_p^\lambda(s) \to F(q, q-2+\frac{qs(1-\lambda)}{p}, t)$ is bounded. The proof is complete. $\square$

Next, we give an estimation for the essential norm of T_g. First, we recall some relevant definitions. The essential norm of $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf_K \{\|T-K\|_{X\to Y} : K \text{ is a compact operator from } X \text{ to } Y\},$$

where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces and $T : X \to Y$ is a bounded linear operator. It is clear that $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. For some results about the essential norm of operator T_g and some related ones see, for example, [5, 8, 11, 15, 24, 27, 28, 29, 30, 31, 35].

For a closed subspaces A of X, given $f \in X$, the distance from f to A denoted by $\mathrm{dist}_X(f,A)$, is defined by $\mathrm{dist}_X(f,A) = \inf_{g\in A} \|f-g\|_X$.

Let $F_0(p,q,s)$ denote the space of all $f \in F(p,q,s)$ such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) = 0.$$

We need the following lemma, which can be found in [20].

LEMMA 4. *Let* $1 < q < \infty$, $0 < \alpha < \infty$. *If* $g \in F(q, q-2, \alpha)$, *then*

$$\begin{aligned}
&\mathrm{dist}_{F(q,q-2,\alpha)}(g, F_0(q,q-2,\alpha)) \approx \limsup_{r\to 1^-} \|g-g_r\|_{F(q,q-2,\alpha)} \\
&\approx \limsup_{|a|\to 1} \left(\int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^\alpha dA(z) \right)^{\frac{1}{q}}.
\end{aligned}$$

Here $g_r(z) = g(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

LEMMA 5. *Let* $0 < s < 1$, $0 < \lambda < 1 < p < q < \infty$ *and* $0 < t < \infty$. *If* $0 < r < 1$ *and* $g \in F(q, q-2, t+\frac{qs(1-\lambda)}{p})$, *then* $T_{g_r} : B_p^\lambda(s) \to F(q, q-2+\frac{qs(1-\lambda)}{p}, t)$ *is compact.*

Proof. Let $\{f_n\} \subset B_p^\lambda(s)$ such that $\{f_n\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$ and $\sup_n \|f_n\|_{B_p^\lambda(s)} \lesssim 1$. By Proposition 2 and the fact that

$F(q,q-2,t+\frac{qs(1-\lambda)}{p})\subset\mathscr{B}$, we have

$$\begin{aligned}
&\|T_{g_r}f_n\|^q_{F(q,\frac{qs(1-\lambda)}{p}+q-2,t)}\\
=&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_n(z)|^q|g_r'(z)|^q(1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2}(1-|\sigma_a(z)|^2)^t dA(z)\\
\lesssim&\frac{\|g\|^q_{\mathscr{B}}}{(1-r^2)^q}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_n(z)|^q(1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2}(1-|\sigma_a(z)|^2)^t dA(z)\\
\lesssim&\frac{\|g\|^q_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}}{(1-r^2)^q}\int_{\mathbb{D}}|f_n(z)|^q(1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2}dA(z)\\
\lesssim&\frac{\|g\|^q_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}}{(1-r^2)^q}\|f_n\|^q_{B_p^\lambda(s)}\int_{\mathbb{D}}(1-|z|^2)^{q-2}dA(z).
\end{aligned}$$

By the dominated convergence theorem, we get the desired result. The proof is complete. □

THEOREM 4. *Let $g\in H(\mathbb{D})$, $0<s<1$, $0<\lambda<1<p<q<\infty$, $0<t<1$ such that $\frac{pt}{q}+s\lambda>s$ and $\frac{pt}{q}-s\lambda\geqslant 0$. If $T_g: B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$ is bounded, then*

$$\|T_g\|_{e,B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)}\approx \operatorname{dist}_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}\Big(g,F_0\Big(q,q-2,t+\frac{qs(1-\lambda)}{p}\Big)\Big).$$

Proof. Let $\{I_n\}\subset\partial\mathbb{D}$ and $\lim_{n\to\infty}|I_n|=0$. Suppose $e^{i\theta_n}$ is the center of I_n and $c_n=(1-|I_n|)e^{i\theta_n}$. For each n, let $f_n(z)=\frac{1-|c_n|^2}{(1-\bar{c_n}z)^{\frac{s(1-\lambda)+p}{p}}}$. Then $\{f_n\}$ is bounded in $B_p^\lambda(s)$ and $\{f_n\}$ converges to zero uniformly on every compact subsets of $\mathbb{D}$. Given a compact operator $K: B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$. Using Lemma 2.10 in [32], we have $\lim_{n\to\infty}\|Kf_n\|_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)}=0$. So

$$\begin{aligned}
&\|T_g-K\|\gtrsim\limsup_{n\to\infty}\|(T_g-K)f_n\|_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)}\\
\gtrsim&\limsup_{n\to\infty}\Big(\|T_gf_n\|_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)}-\|Kf_n\|_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)}\Big)\\
=&\limsup_{n\to\infty}\|T_gf_n\|_{F(q,q-2+\frac{qs(1-\lambda)}{p},t)}\\
\geqslant&\limsup_{n\to\infty}\Big(\int_{\mathbb{D}}|f_n(z)|^q|g'(z)|^q(1-|z|^2)^{\frac{qs(1-\lambda)}{p}+q-2}(1-|\sigma_{c_n}(z)|^2)^t dA(z)\Big)^{\frac{1}{q}}\\
\gtrsim&\limsup_{n\to\infty}\left(\frac{1}{|I_n|^{t+\frac{qs(1-\lambda)}{p}}}\int_{S(I_n)}|g'(z)|^q(1-|z|^2)^{q-2+t+\frac{qs(1-\lambda)}{p}}dA(z)\right)^{\frac{1}{q}},
\end{aligned}$$

which implies that

$$\begin{aligned}&\|T_g\|_{e,B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)}\\&\gtrsim \limsup_{n\to\infty}\left(\frac{1}{|I_n|^{t+\frac{qs(1-\lambda)}{p}}}\int_{S(I_n)}|g'(z)|^q(1-|z|^2)^{q-2+t+\frac{qs(1-\lambda)}{p}}dA(z)\right)^{\frac{1}{q}}.\end{aligned}$$

By Lemma 4 and the arbitrariness of n, we have

$$\|T_g\|_{e,B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)} \gtrsim \mathrm{dist}_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}\Big(g,F_0\Big(q,q-2,t+\frac{qs(1-\lambda)}{p}\Big)\Big).$$

On the other hand, by Lemma 5, we see that $T_{g_r}: B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$ is compact. Then

$$\begin{aligned}\|T_g\|_{e,B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)} &\leqslant \|T_g-T_{g_r}\| = \|T_{g-g_r}\|\\ &\approx \|g-g_r\|_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}.\end{aligned}$$

Using Lemma 4 again, we obtain

$$\begin{aligned}\|T_g\|_{e,B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)} &\lesssim \limsup_{r\to1^-}\|g-g_r\|_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}\\ &\approx \mathrm{dist}_{F(q,q-2,t+\frac{qs(1-\lambda)}{p})}\Big(g,F_0\Big(q,q-2,t+\frac{qs(1-\lambda)}{p}\Big)\Big).\end{aligned}$$

The proof is complete. □

The following result can be deduced by Theorem 4 directly.

COROLLARY 1. *Let $g\in H(\mathbb{D})$, $0<s<1$, $0<\lambda<1<p<q<\infty$, $0<t<1$ such that $\frac{pt}{q}+s\lambda>s$ and $\frac{pt}{q}-s\lambda\geqslant 0$. Then the operator $T_g: B_p^\lambda(s)\to F(q,q-2+\frac{qs(1-\lambda)}{p},t)$ is compact if and only if $g\in F_0(q,q-2,t+\frac{qs(1-\lambda)}{p})$.*

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