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SIMPLE ACCURATE BALANCED ASYMPTOTIC APPROXIMATION OF WALLIS' RATIO USING EULER–BOOLE ALTERNATING SUMMATION

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Abstract. For integers $m \geqslant 1$ and $q \geqslant 2$, the Wallis ratio $w_m := \prod_{k=1}^{m} \frac{2k-1}{2k}$ is estimated as

$$\left| w_m - \frac{1}{\sqrt{m\pi}} \exp\left(- \sum_{i=1}^{\lfloor q/2 \rfloor} \frac{\left(1-4^{-i}\right) B_{2i}}{i(2i-1)\cdot m^{2i-1}} \right) \right| < \frac{1}{2} \exp\left(\rho_q^*(m)\right) \cdot \rho_q^*(m),$$

where B_k are the Bernoulli coefficients and

$$\left|\rho_q^*(m)\right| < \frac{\pi(q-2)!}{3(2m\pi)^{q-1}} < \frac{\pi}{3}\sqrt{\frac{2\pi}{q-1}} \cdot \left(\frac{q-1}{2me\pi}\right)^{q-1} \exp\left(\frac{1}{12(q-1)}\right).$$

Some accurate asymptotic estimates of π in terms of w_m are also given.

1. Introduction

In connection with the Wallis sequence $n \mapsto W_n := \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$ is the sequence of the Wallis[1] ratios w_n defined in the literature as

$$w_n := \prod_{k=1}^{n} \frac{2k-1}{2k} \equiv \frac{(2n-1)!!}{(2n)!!}. \tag{1}$$

Both sequences, $\left(W_n\right)_{n\geqslant 1}$ being strictly increasing and $\left(w_n\right)_{n\geqslant 1}$ strictly decreasing, were studied for a long period of time. The Wallis ratio, i.e. the sequence $n \mapsto w_n$, was investigated by many authors, see for example the papers [2, 3, 8, 9, 10, 12, 16, 17, 18, 19, 20, 21, 23, 24]. Perhaps the Wallis ratio additionally attracts mathematicians also because of its direct connections with Catalan numbers $c_n := \frac{1}{n+1}\binom{2n}{n}$, for pure and applied mathematics important objects [11]. Naturally, during the long period a great amount of papers concerning the Wallis ratio have been published. In [24] was presented aesthetically pleasing double inequality

$$Z_1(n) < w_n \leqslant Z_2(n) \tag{2}$$

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[1] John Wallis, 1616 – 1703

true for all $n \in \mathbb{N}$ with

$$Z_1(n) := \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} \quad \text{and} \qquad Z_2(n) := \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}.$$

In [8] was demonstrated the double inequality

$$GXQ_1(n) < w_n \leqslant GXQ_2(n), \tag{3}$$

true for $n \geqslant 2$ with

$$GXQ_1(n) := \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}$$
$$GXQ_2(n) := \frac{4}{3}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}.$$

Recently, in [9] was derived the estimates

$$GFB_1(n) := \left(\frac{2}{3}\right)^{3/2}\left(1-\frac{1}{2n}\right)^{n+1/2}\left(n-\frac{3}{2}\right)^{-1/2} \leqslant w_n \tag{4}$$

and

$$w_n < GFB_2(n) := \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^{n+1/2}\left(n-\frac{3}{2}\right)^{-1/2}, \tag{5}$$

both valid for $n \geqslant 2$. At the same time, in [21, Theorems 4.2 and 5.2] were presented the estimates

$$w_n > QM_1^*(n) := \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2n}\right)^n \exp\left(\frac{1}{24n^2}+\frac{1}{48n^3}+\frac{1}{160n^4}+\frac{1}{960n^5}\right), \tag{6}$$

and

$$QM_1(n) := \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2(n+1/3)}\right)^{n+1/3} < w_n \tag{7}$$

and

$$w_m < QM_2(m) := \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2(n+1/3)}\right)^{n+1/3} \exp\left(\frac{1}{144n^3}\right), \tag{8}$$

all true for $n \geqslant 1$.

Recently, several additional new results were also presented in [4, 5, 19]. For these articles the important reference is thirty years old paper [22], where the author accurately estimates the function $x \mapsto \Gamma(x+1)/\Gamma(x+\frac{1}{2}) \equiv 1/(w_n\sqrt{\pi})$ using its integral representation. Recently was found in [14] an approach, via Stirling's factorial formula, giving more accurate, asymptotic approximation of Wallis' ratio. In our contribution we shall show that similar, accurate and balanced – aesthetically appealing results can be easily obtained using the Euler-Boole alternating summation formula [13].

2. Approximating w_n accurately and aesthetically pleasing

Walli's sequence and the sequence of Wallis' ratios are closely connected. Namely, for every $m \in \mathbb{N}$ we have

$$\begin{aligned} W_m &= \frac{\prod_{k=1}^{m} 4k^2}{\prod_{k=1}^{m}(2k-1)\prod_{k=1}^{n}(2k+1)} = \frac{1}{2m+1}\left(\prod_{k=1}^{m}\frac{2k}{2k-1}\right)^2 \\ &= \frac{1}{2m+1}\cdot\frac{1}{w_m^2}. \end{aligned}$$

Hence, for all $m \in \mathbb{N}$ we obtain

$$w_m = \frac{1}{\sqrt{(2m+1)W_m}}. \tag{9}$$

Now, we should use several known estimates of the Wallis sequence, for example [6, Theorem 4.1]

$$\begin{aligned} W_m &> \frac{\pi}{2}\left(1 - \frac{\frac{1}{4}}{m+\frac{5}{8}} + \frac{\frac{3}{256}}{(m+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(m+\frac{5}{8})^4} - \frac{\frac{51}{16384}}{(m+\frac{5}{8})^5} - \frac{\frac{75}{65536}}{(m+\frac{5}{8})^6}\right) \\ W_m &< \frac{\pi}{2}\left(1 - \frac{\frac{1}{4}}{m+\frac{5}{8}} + \frac{\frac{3}{256}}{(m+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(m+\frac{5}{8})^4}\right), \end{aligned}$$

true for integer $m \geqslant 1$. But, these estimates are less appropriate when using (9). Fortunately, we are in the position to use [13, Theorem 1, Remark 1], since there is given a nice approximation of Wallis' sequence. Hence, the key to our approach is the next lemma.

LEMMA 1. ([13], Theorem 1 & Remark 1) *For integers $m \geqslant 1$ and $q \geqslant 2$ we have*

$$W_m = \frac{m\pi}{2m+1}\exp\left(2\sigma_q(m)\right)\cdot\exp\left(r_q(m)\right), \tag{10}$$

where

$$\sigma_q(x) \equiv \sum_{i=1}^{\lfloor q/2\rfloor}\frac{\left(1-4^{-i}\right)B_{2i}}{i(2i-1)\cdot x^{2i-1}} \qquad (B_k \textit{ are the Bernoulli coefficients}), \tag{11}$$

and where

$$\left|r_q(m)\right| < r_q^*(m) := \frac{2\pi(q-2)!}{3(2m\pi)^{q-1}} \tag{12}$$

$$< \frac{2\pi}{3}\sqrt{\frac{2\pi}{q-1}}\cdot\exp\left(\frac{1}{12(q-1)}\right)\cdot\left(\frac{q-1}{2me\pi}\right)^{q-1}. \tag{13}$$

Using this lemma and the expression (9), we obtain very accurate approximations of Wallis' ratios given in the next theorem.

THEOREM 1. *For integers $m \geqslant 1$ and $q \geqslant 2$ there holds the equality*

$$w_m = w_q^*(m) \cdot \exp\big(\rho_q(m)\big), \tag{14}$$

where (see (11))

$$w_q^*(m) := \frac{1}{\sqrt{m\pi}} \exp\big(-\sigma_q(m)\big), \tag{15}$$

and

$$|\rho_q(m)| < \rho_q^*(m) := \frac{1}{2} r_q^*(m) = \frac{\pi(q-2)!}{3(2m\pi)^{q-1}} \tag{16}$$

$$< \frac{\pi}{3}\sqrt{\frac{2\pi}{q-1}} \cdot \exp\left(\frac{1}{12(q-1)}\right) \cdot \left(\frac{q-1}{2me\pi}\right)^{q-1}. \tag{17}$$

REMARK 1. The estimate (16) is rather rough as is illustrated on Figure 1 and Figure 2, where are plotted the graphs of the sequences $m \mapsto \rho_q^*(m)/\big|\rho_q(m)\big|$ with $\rho_q(m) \equiv \ln(w_m) + \frac{1}{2}\ln(m\pi) + \sigma_q(m)$ for several values of q.

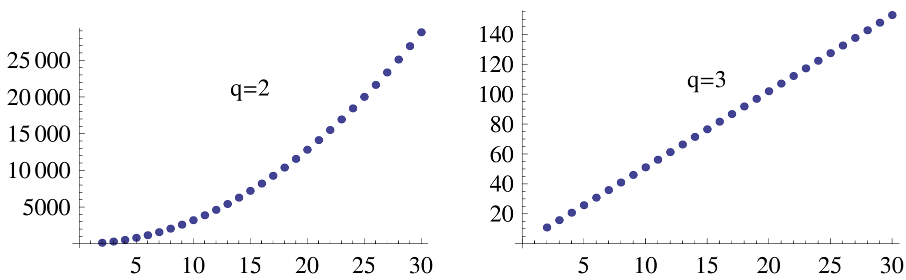

Figure 1: *The graphs of the sequences* $m \mapsto \rho_q^*(m)/\big|\rho_q(m)\big|$, *for* $q \in \{2,3\}$.

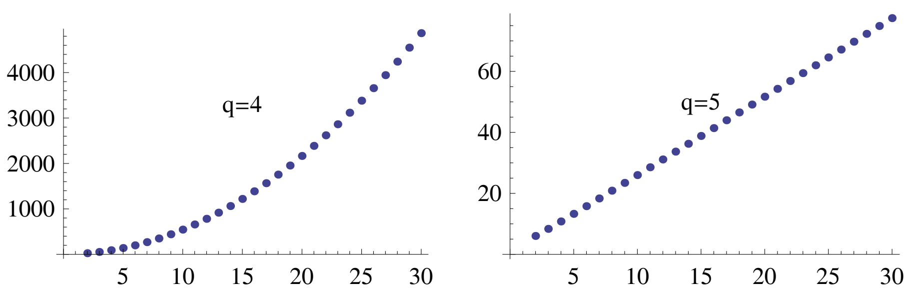

Figure 2: *The graphs of the sequences* $m \mapsto \rho_q^*(m)/\big|\rho_q(m)\big|$, *for* $q \in \{4,5\}$.

COROLLARY 1. (asymptotic expansion) *For* $m \in \mathbb{N}$,

$$\ln\left(w_m\right) \sim \ln\left(\frac{1}{\sqrt{\pi m}}\right) - \sum_{i=1}^{\infty} \frac{(1-4^{-i})B_{2i}}{i(2i-1)m^{2i-1}} \quad \text{as } m \to \infty.$$

REMARK 2. Rather extensive study of the asymptotic expansions of the Wallis ratio and related topics can be found in [6] and [7].

Directly from Theorem 1 we get, using the finite increment theorem, the next corollary.

COROLLARY 2. *The approximation* $w_m \approx w_q^*(m)$ *has the relative error* $\varepsilon_q(m) := \left(w_m - w_q^*(m)\right)/w_m$ *fulfilled, for any* $m \in \mathbb{N}$ *and* $q \geqslant 2$*, the following inequalities (see* (16)–(17)*):*

$$\left|\varepsilon_q(m)\right| = \left|1 - \exp\left(-\rho_q(m)\right)\right| < \exp\left(\rho_q^*(m)\right) \cdot \rho_q^*(m).$$

Setting $q = 2,3,4$ in Theorem 1, we get the next corollary.

COROLLARY 3. *For every* $m \in \mathbb{N}$ *we have the following double asymptotic inequalities:*

$$a(m) := \frac{1}{\sqrt{m\pi}} \exp\left(-\frac{7}{24m}\right) < w_m < \frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{24m}\right) =: b(m)$$

$$\frac{1}{\sqrt{m\pi}} \exp\left(-\frac{1}{8m} - \frac{1}{24m^2\pi}\right) < w_m < \frac{1}{\sqrt{m\pi}} \exp\left(-\frac{1}{8m} + \frac{1}{24m^2\pi}\right)$$

$$\frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{192m^3} - \frac{1}{8m} - \frac{1}{12m^3\pi^2}\right) < w_m < \frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{192m^3} - \frac{1}{8m} + \frac{1}{12m^3\pi^2}\right)$$

In Figure 3 are depicted the graphs of the lower/upper bounds (continuous lines) of the estimates in Corollary 3, together with the graph of the sequence of Wallis' ratios: The left picture is relating to the first line of the inequalities (obtained using $q = 2$) and the right image to the second one ($q = 3$).

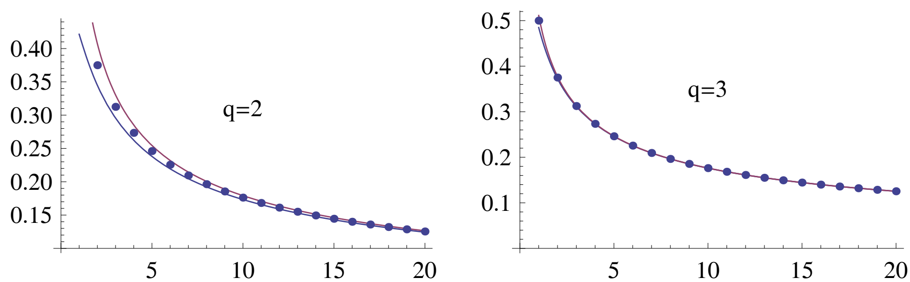

Figure 3: *The graphs of the lower/upper bounds (continuous lines) of the estimates in Corollary 3, together with the graph of the sequence of Wallis' ratios: The left picture is relating to the first line of the estimates and the right image to the second line.*

Putting $q = 7$ in Theorem 1, we obtain the following corollary.

COROLLARY 4. *For any $m \in \mathbb{N}$ we have the following inequalities:*

$$A(m) := \frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{192m^3} - \frac{1}{8m} - \frac{1}{640m^5} - \frac{5}{8m^6\pi^5}\right) < w_m \tag{18}$$

and

$$w_m < B(m) := \frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{192m^3} - \frac{1}{8m} - \frac{1}{640m^5} + \frac{5}{8m^6\pi^5}\right). \tag{19}$$

Setting $q = 11$ in Theorem 1, we get the following corollary.

COROLLARY 5. *For any $m \in \mathbb{N}$ we estimate w_m in the following way:*

$$\frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{192m^3} - \frac{1}{8m} - \frac{1}{640m^5} + \frac{17}{14336m^7} - \frac{31}{18432m^9} - \frac{945}{8m^{10}\pi^9}\right) < w_m$$

and

$$w_m < \frac{1}{\sqrt{m\pi}} \exp\left(\frac{1}{192m^3} - \frac{1}{8m} - \frac{1}{640m^5} + \frac{17}{14336m^7} - \frac{31}{18432m^9} + \frac{945}{8m^{10}\pi^9}\right).$$

COROLLARY 6. *For integers $m \geqslant 1$ and $q \geqslant 2$ such that $\rho_q^*(m) < 1$ (see (16)), we have*

$$\left|w_m - w_q^*(m)\right| < \frac{\pi}{6}\sqrt{\frac{2\pi}{q-1}} \cdot \exp\left(\frac{1}{12(q-1)}\right) \cdot \left(\frac{q-1}{2me\pi}\right)^{q-1}.$$

Proof. Referring to the finite increment theorem we have $\left|e^x - 1\right| = e^{\vartheta x}|x| \leqslant e^{|x|}|x|$, for any $x \in \mathbb{R}$ and some $\vartheta \in (0,1)$. Therefore, invoking Theorem 1 and supposing that $\rho_q^*(m) < 1$, we obtain,

$$\begin{aligned}\left|w_m - w_q^*(m)\right| &= \left|w_m - w_m \exp\left(-\rho_q(m)\right)\right| = w_m\left|1 - \exp\left(-\rho_q(m)\right)\right| \\ &\leqslant w_1 \cdot \exp\left(\left|\rho_q(m)\right|\right) \cdot \left|\rho_q(m)\right| < \frac{1}{2} \cdot \exp\left(\rho_q^*(m)\right) \cdot \rho_q^*(m) \\ &< \frac{1}{2} \cdot e \cdot \rho_q^*(m),\end{aligned}$$

where we consider (13). □

From Corollary 6 follows the next corollary.

COROLLARY 7. *For all integers $m \geqslant 1$ the following estimates hold:*

$$\left|w_m - w_2^*(m)\right| < \frac{1}{5\,m}$$
$$\left|w_m - w_8^*(m)\right| < \frac{1}{500\,m^7}$$
$$\left|w_m - w_{11}^*(m)\right| < \frac{1}{250\,m^{10}}$$
$$\left|w_m - w_{18}^*(m)\right| < \frac{1}{m^{17}}\,.$$

3. Approximating π using Wallis' sequence

From Theorem 1 we read the next theorem.

THEOREM 2. *For integers $m \geqslant 1$ and $q \geqslant 2$ we have*

$$\pi = \pi_q^*(m) \cdot \exp\big(-2\rho_q(m)\big) \tag{20}$$

with

$$\pi_q^*(m) := \frac{1}{m \cdot w_m^2} \cdot \exp\big(-2\sigma_q(m)\big) \tag{21}$$

and

$$\left|2\rho_q(m)\right| < \frac{2\pi(q-2)!}{3(2m\pi)^{q-1}}\,. \tag{22}$$

From Corollary 5 we read the next corollary.

COROLLARY 8. *For every $m \in \mathbb{N}$ there hold the following inequalities:*

$$\pi > \frac{1}{m \cdot w_m^2} \exp\left(\frac{1}{96m^3} - \frac{1}{4m} - \frac{1}{320m^5} + \frac{17}{7168m^7} - \frac{31}{9216m^9} - \frac{945}{4m^{10}\pi^9}\right)$$
$$=: \lambda_1(m)$$

and

$$\pi < \frac{1}{m \cdot w_m^2} \exp\left(\frac{1}{96m^3} - \frac{1}{4m} - \frac{1}{320m^5} + \frac{17}{7168m^7} - \frac{31}{9216m^9} + \frac{945}{4m^{10}\pi^9}\right)$$
$$=: \lambda_2(m)$$

Putting $m = 100$ in the inequalities of Corollary 8 and using Mathematica [25], we obtain the estimate

$$3.141\,592\,653\,589\,793\,238\,462\,391\ldots < \pi < 3.141\,592\,653\,589\,793\,238\,462\,889\ldots.$$

This way we find $\pi = 3.141\,592\,653\,589\,793\,238\,462\ldots.$

REMARK 3. Mortici [15, Th. 2, p. 2617] provided the double inequality

$$M_1(m) < \pi < M_2(m),$$

where

$$M_1(m) := \left(\frac{m+1/4}{m^2+m/2+3/32} + \frac{9}{2048m^5} - \frac{45}{8192m^6}\right)\left(\frac{(2m)!!}{(2m-1)!!}\right)^2$$

and

$$M_2(m) := \left(\frac{m+1/4}{m^2+m/2+3/32} + \frac{9}{2048m^5}\right)\left(\frac{(2m)!!}{(2m-1)!!}\right)^2.$$

We have $\lambda_1(1) < M_1(1)$, but $\lambda_1(m) > M_1(m)$ for $m \in \{2,3,4,\ldots,100\}$ because for the quotients $q_1(m) := \lambda_1(m)/M_1(m)$, using Mathematica [25], we get $q_1(1) < 1$ and $q_1(m) > 1$, for $m \in \{2,3,4,\ldots,100\}$. Similarly we find $\lambda_2(m) < M_2(m)$ for $m \in \{2,3,4,\ldots,100\}$.

4. Conclusion: A comparison with some previous results

Approximations provided by Theorem 1, surpass the estimates (2)–(8) stated in the Introduction, in terms of both accuracy and aesthetics. For an illustration we give several examples.

In Figure 4 is depicted, on both sides, the graph of the sequence of Wallis ratios together with the graphs of the bounds $a(m)$ and $b(m)$ (continuous lines) from the Corollary 3. On the left are plotted also the graphs of the bounds $GXQ_1(m)$ and $GXQ_2(m)$ and on the right are added the graphs of the bounds (see (3)–(5)) $GFB_1(m)$ and $GFB_2(m)$ (dashed lines). The estimates (3)–(5) are evidently not balanced.

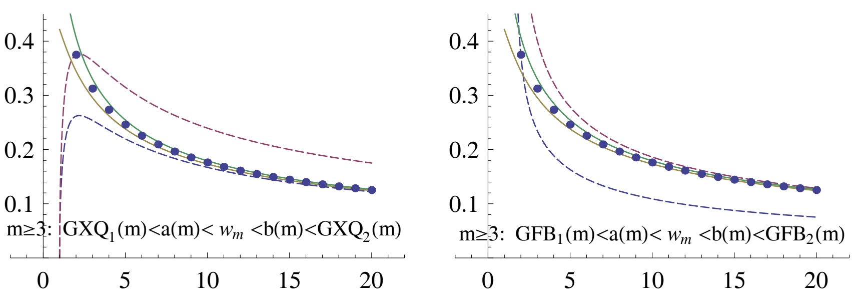

Figure 4: *The comparison of the accuracy of the first double inequality of Corollary 3 (continuous lines) with the estimates (3), on the left, and with the inequalities (4)–(5), on the right (dashed lines).*

Figure 5 shows, on the left (see (7), (8) and Corollary 4), the graphs of the functions $m \mapsto m^4\big(QM_1(m) - A(m)\big)$ (dashed line) and $m \mapsto m^6\big(QM_2(m) - B(m)\big)$ (continuous line) and on the right the graph of the function $m \mapsto m^{10}\big(QM_1^*(m) - w_{11}^*(m)\exp\big(-\rho_{11}^*(m)\big)\big)$, illustrating that the lower bound in Corollary 5 is greater[2] than $QM_1^*(m)$.

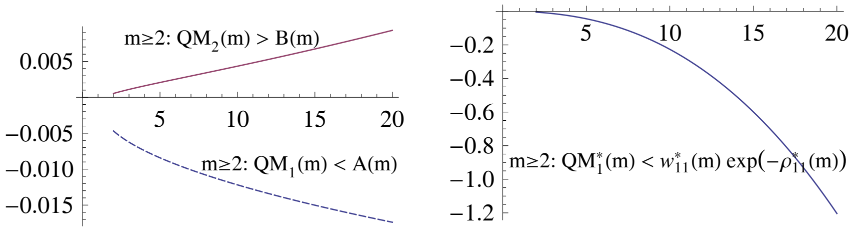

Figure 5: *On the left are the graphs of the functions* $m \mapsto m^4\big(QM_1(m) - A(m)\big)$ *(dashed line) and* $m \mapsto m^6\big(QM_2(m) - B(m)\big)$ *(continuous line) and on the right is the graph of the function* $m \mapsto m^{10}\big(QM_1^*(m) - w_{11}^*(m)\exp\big(-\rho_{11}^*(m)\big)\big)$*, illustrating that the lower bound in Corollary 5 is greater than* $QM_1^*(m)$.

Zhang's estimates (2) are quite accurate and also balanced as is shown on Figure 6, where are plotted, on the left, the graphs of the functions $Z_1(m)$ and $Z_2(m)$, together with the graph of the sequence $m \mapsto w_m$. On the right of this figure are plotted the graphs of the functions $m \mapsto m^5\big(Z_1(m) - A(m)\big)$ and $m \mapsto m^5\big(Z_2(m) - B(m)\big)$ (see (2) and Corollary 4).

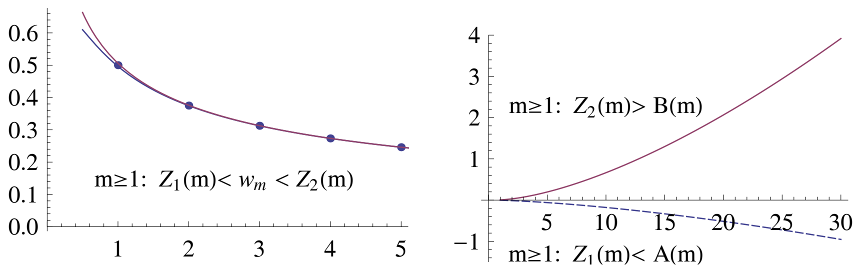

Figure 6: *Left: Accurate Zhang's estimates (2) is illustrated by the graphs of the functions* $Z_1(m)$ *and* $Z_2(m)$ *together with the graph of the sequence* $m \mapsto w_m$*. Right: The graphs of the functions* $m \mapsto m^5\big(Z_1(m) - A(m)\big)$ *and* $m \mapsto m^5\big(Z_2(m) - B(m)\big)$ *(see (2) and Corollary 4).*

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[2] Even for $q = 8$ we get the similar result: $QM_1^*(m) < w_8^*(m)\exp(-\rho_8^*(m))$, for $m \in \{1, \ldots, 20\}$.

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