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AN APPLICATION OF THE AFFINE SHORTENING FLOW

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Abstract. In this paper, using the affine curve shortening flow, we prove the following inequality: if C is a smooth closed and convex curve with affine perimeter $\mathscr{L}$ and enclosed area $\mathscr{A}$, then

$$\mu_{\max} \geqslant \frac{\mathscr{L}}{2\mathscr{A}},$$

where $\mu_{\max}$ is the maximum affine curvature of C.

1. Introduction

Let γ be a smooth Jordan curve in $\mathbb{R}^2$. Pestov and Ionin [11] showed the following interesting inequality,

$$\kappa_{\max} \geqslant \sqrt{\frac{\pi}{A}}, \tag{1}$$

where $\kappa_{\max}$ and A are the maximum curvature and the enclosed area of γ. Equality holds in (1) if and only if γ is a circle.

Various proofs and generalizations of inequality (1) aroused much interest. Howard and Treibergs [7] gave a proof of (1) by analytical methods. Pankrashkin [9] obtained another proof of (1) through the curve shortening flow (see Gage-Hamilton [2] and Grayson [4]). For a smooth simple closed curve on surfaces, Yang and Fang [15] got an analog of (1) by the curve shortening flow on surfaces (see Gage [3] and Grayson [5]). Other aspects of this topic can be found in Ferone-Nitsch-Trombetti [1], Pankrashkin-Popoff [10] and Ritoré-Sinestrari [12].

Following the work on the curve shortening flow in the plane (see Gage-Hamilton [2], Grayson [4]), Sapiro and Tannenbaum [13] studied the affine curve shortening problem. Inspired by the idea of Pankrashkin [9], Yang [14] showed an inequality for the minimum affine curvature through the affine curve shortening flow. In this short paper, we get the affine analog of (1) through the affine curve shortening flow.

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THEOREM 1. *If C is a smooth closed and convex curve in $\mathbb{R}^2$, then*

$$\mu_{\max} \geqslant \frac{\mathscr{L}}{2\mathscr{A}}, \tag{2}$$

and equality holds if and only if C is an ellipse, where $\mu_{\max}$ and $\mathscr{L}$ are the maximum affine curvature and the affine perimeter of C, and $\mathscr{A}$ is its enclosed area.

2. A maximum curvature inequality

Let $C : S^1 \to \mathbb{R}^2$ be a smooth embedded curve with parameter p (where S^1 denotes the unit circumference). A reparametrization of $C(p)$ to a new parameter s can be performed such that,

$$[C_s, C_{ss}] = 1, \tag{3}$$

where $[X,Y]$ stands for the determinant of the 2×2 matrix whose columns are given by the vectors $X, Y \in \mathbb{R}^2$, where C_s and C_{ss} are the first and the second derivative of C with respect to s. Furthermore, C_s is called the *affine tangent vector* and C_{ss} the *affine normal vector* of the curve C. The relation is invariant under proper affine transformations, and the parameter s is called the *affine arc-length*. Denoting by

$$g(p) = [C_p, C_{pp}]^{\frac{1}{3}},$$

the parameter s is explicitly given by

$$s(p) = \int_0^p g(\xi) d\xi.$$

The *affine perimeter* of C is given by

$$\mathscr{L} = \int_C ds.$$

By differentiating (3), one has

$$[C_s, C_{sss}] = 0,$$

which implies that C_s and C_{sss} are linearly dependent and thus, there exists $\mu \in \mathbb{R}$ such that

$$C_{sss} + \mu C_s = 0.$$

Combining with (3) yields

$$\mu = [C_{ss}, C_{sss}],$$

and μ is called the *affine curvature* of C. A more comprehensive exposition of various aspects of the Affine Differential Geometry can be found in [13].

Let $C(p,t) : S^1 \times [0, \omega) \to \mathbb{R}^2$ be a family of smooth closed and convex curves where t parameterizes the family and p parameterizes each curve, and ω is the maximal time that the family of curves exists. Inspired by the curve shortening flow, Sapiro and Tannenbaum [13] composed the affine curve shortening flow:

$$\begin{cases} \frac{\partial C(p,t)}{\partial t} = C_{ss}(p,t), \\ C(\cdot, 0) = C_0(\cdot), \end{cases} \tag{4}$$

where $s := s(p) = \int [C_p, C_{pp}]^{\frac{1}{3}} dp$ and $C_0(\cdot)$ denotes a smooth, closed and convex curve. Under the affine curve shortening flow (4), they showed that the affine curvature $\mu(\cdot,t)$ is always positive and the limit shape (i.e., the curve obtained when $t \to \omega$) is an "elliptical point", with convergence in C^∞ norm.

Next, we will apply the affine curve shortening flow (4) to deal with Theorem 1. First, we give two lemmas which will be used in the rest of this paper.

LEMMA 1. ([13]) *The evolution equations of the affine perimeter $\mathscr{L}(t)$, the enclosed area $\mathscr{A}(t)$ and the affine curvature μ with respect to the evolving curves $C(\cdot,t)$ are*

$$\mathscr{L}_t = -\frac{2}{3}\oint \mu(s,t)ds, \tag{5}$$

$$\mathscr{A}_t = -\mathscr{L}(t), \tag{6}$$

$$\frac{\partial \mu(s,t)}{\partial t} = \frac{1}{3}\mu_{ss}(s,t) + \frac{4}{3}\mu^2(s,t). \tag{7}$$

LEMMA 2. ([13]) *If C is a smooth closed and convex curve, then the affine isoperimetric inequality*

$$2\oint \mu(s,t)ds \leqslant \frac{\mathscr{L}^2(t)}{\mathscr{A}(t)} \tag{8}$$

holds, with equality if and only if C is an ellipse.

Proof of Theorem 1. Set $Q(s,t) = \mu^2(s,t)$ and $Q_{\max}(t) = \max\{Q(s,t) \mid s \in [0,\mathscr{L}]\}$. Then

$$\frac{\partial Q(s,t)}{\partial t} = 2\mu(s,t)\frac{\partial \mu(s,t)}{\partial t}, \quad \frac{\partial Q(s,t)}{\partial s} = 2\mu(s,t)\frac{\partial \mu(s,t)}{\partial s},$$

$$\frac{\partial^2 Q(s,t)}{\partial s^2} = 2\left(\frac{\partial \mu(s,t)}{\partial s}\right)^2 + 2\mu(s,t)\frac{\partial^2 \mu(s,t)}{\partial s^2},$$

which together with (7) yields

$$\begin{aligned}\frac{\partial Q(s,t)}{\partial t} &= \frac{2}{3}\mu(s,t)\frac{\partial^2 \mu(s,t)}{\partial s^2} + \frac{8}{3}\mu^3(s,t)\\ &= \frac{1}{3}\frac{\partial^2 Q(s,t)}{\partial s^2} - \frac{1}{6Q(s,t)}\left(\frac{\partial Q(s,t)}{\partial s}\right)^2 + \frac{8}{3}Q^{\frac{3}{2}}(s,t).\end{aligned}$$

Since $Q_{\max}(t)$ is Lipschitz continuous, it is differentiable almost everywhere. Let s^* be the point such that $Q(s^*,t) = Q_{\max}(t)$. From Hamilton's technique of the maximum principle [6, p. 159], it follows that

$$\begin{aligned}(Q_{\max}(t))_t &\leqslant \frac{\partial Q}{\partial t}(s^*,t)\\ &= \frac{1}{3}\frac{\partial^2 Q}{\partial s^2}(s^*,t) - \frac{1}{6Q(s^*,t)}\left(\frac{\partial Q}{\partial s}(s^*,t)\right)^2 + \frac{8}{3}Q^{\frac{3}{2}}(s^*,t)\end{aligned}$$

and at the point (s^*,t),

$$\frac{\partial^2 Q}{\partial s^2}(s^*,t) \leqslant 0, \quad \frac{\partial Q}{\partial s}(s^*,t) = 0.$$

Hence,

$$(Q_{\max}(t))_t \leqslant \frac{8}{3} Q_{\max}^{\frac{3}{2}}(t), \tag{9}$$

which implies that

$$\left(Q_{\max}(t)^{-\frac{1}{2}}\right)_t \geqslant -\frac{4}{3}.$$

Let ω be the maximal existence time. Integrating the above expression on $[t,\omega)$ (where ω is finite [8, p. 1190]) and that $Q_{\max}(t) \to \infty$ when $t \to \omega$ (see [13, Theorem 15.1]), one gets

$$\frac{1}{\sqrt{Q_{\max}(t)}} \leqslant \frac{4}{3}(\omega - t). \tag{10}$$

It follows from (5), (6) and (8) that

$$\begin{aligned} \left(\frac{\mathscr{A}(t)}{\mathscr{L}(t)}\right)_t = \frac{\mathscr{L}(t)\mathscr{A}_t - \mathscr{A}(t)\mathscr{L}_t}{\mathscr{L}^2(t)} &= -1 + \frac{\mathscr{A}(t)}{\mathscr{L}^2(t)}\left(\frac{2}{3}\oint \mu(s,t)ds\right) \\ &\leqslant -1 + \frac{\mathscr{A}(t)}{\mathscr{L}^2(t)}\frac{\mathscr{L}^2(t)}{3\mathscr{A}(t)} = -\frac{2}{3}, \end{aligned} \tag{11}$$

which implies that

$$\frac{\mathscr{A}(\omega)}{\mathscr{L}(\omega)} - \frac{\mathscr{A}(t)}{\mathscr{L}(t)} \leqslant -\frac{2}{3}(\omega - t).$$

Since $\mathscr{A}(t)$ is an infinitesimal of order higher than $\mathscr{L}(t)$ as $t \to \omega$, $\frac{\mathscr{A}(t)}{\mathscr{L}(t)} \to 0$ as $t \to \omega$, hence

$$\omega - t \leqslant \frac{3}{2}\frac{\mathscr{A}(t)}{\mathscr{L}(t)}. \tag{12}$$

From (10), (12) and $\mu_{\max}(t) = Q_{\max}^{\frac{1}{2}}(t)$, (2) follows and equality holds in (2) only when the equalities hold in (9) and (11). By the affine isoperimetric inequality (8), the equality holds in (11) only when the evolving curve is an ellipse. □

REMARK 1. Since the affine inequality (8) plays a significant role in the proof of Theorem 1, the condition on the curve C cannot be reduced to being a smooth Jordan curve by using this method.

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