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QUANTITATIVE WEIGHTED ESTIMATES AND WEIGHTED COMPACTNESS FOR VARIATION OF APPROXIMATE IDENTITIES

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Abstract. In this paper, we give the quantitative weighted BMO estimates and C_q estimates for variation of approximate identities. Meanwhile, we also give a new characterization of $CMO(\mathbb{R}^n)$ via the compactness of the variation operators associated with commutators of approximate identities in weighted Lebesgue spaces.

1. Introduction and main results

The intension of this paper is to establish the quantitative weighted L^∞-BMO estimates, quantitative C_q estimate and the compactness for variation operators associated with commutators of approximate identities. Before we state our main results, let us recall some backgrounds.

The well known extrapolation theorem established by Rubio de Francia [42] showed that if T is an operator bounded on $L^{p_0}(\omega)$ for some $p_0 \in (1,\infty)$ and each $\omega \in A_{p_0}$, then T is bounded on $L^p(\omega)$ for each $\omega \in A_p$ with $p \in (1,\infty)$. The conclusion also holds true if the hypothesis is assumed that T maps $L^1(\omega)$ into $L^{1,\infty}(\omega)$ for any $\omega \in A_1$. While for $p_0 = \infty$, Harboure et al. [28] then established the following extrapolation theorem:

THEOREM 1. *Let T be a sublinear operator defined on $C_c^\infty(\mathbb{R}^n)$, assume that T satisfies*

$$|Q|^{-1}\int_Q |Tf - \langle Tf\rangle_Q| \lesssim \|f/\omega\|_{L^\infty} \operatorname*{ess\,inf}_Q \omega \tag{1}$$

for any cube $Q \subset \mathbb{R}^n$ and $\omega \in A_1$, where $\langle Tf\rangle_Q := |Q|^{-1}\int_Q Tf$ and the implicit constant depends on T and ω. Then T is bounded on $L^p(\omega)$ for any $p \in (1,\infty)$ and $\omega \in A_p$.

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If T is the Hilbert transform and satisfies (1), Muckenhoupt and Whedeen [41] earlier showed (1) holds if and only if $\omega \in A_1$. This result and the extrapolation theorem perhaps are the source of inspiration for Theorem 1. One may wonder how does the implicit constant in (1) depend on ω. Recently, Criado, Pérez and Rivera-Ríos [19] answered this question, they gave a quantitative extension of Theorem 1 and extended the Muckenhoupt-Whedeen's result to Calderón-Zygmund operator.

On the other hand, the other way to obtain the boundedness result of an operator is that one can seek a suitable maximal operator to control it. A classical example of this principle is the following famous Coifman-Feferman inequality [15]:

$$\|T^*f\|_{L^p(\omega)} \leqslant c_{n,p,\omega}\|Mf\|_{L^p(\omega)}, \tag{2}$$

where T^* and M are the maximal Calderón-Zygmund operator and Hardy-Littlewood maximal operator, respecctively, $0<p<\infty$ and $\omega \in A_\infty$. For the necessity condition of (2), Muckenhoupt [40] showed that $\omega \in C_p$ (a larger class than the class of A_∞) is the appropriate condition other than $\omega \in A_\infty$. Later on, Sawyer [43] proved that (2) holds for $1<p<\infty$, $p<q$ and $\omega \in C_q$. However, it is still an open problem that whether $\omega \in C_p$ is the sufficient condition. Recently, Canto et al. [7] provided the quantitative C_q estimates for singular integral operators. Also, we refer readers to see [35] for the recent development of this topic.

Recall that given a locally integral function b and a linear or nonlinear operator T, the commutator $[b,T]$ is defined by

$$[b,T]f(x) := T\big((b(x)-b(\cdot))f\big)(x).$$

And we say that b belongs to $BMO(\mathbb{R}^n)$ spaces if

$$\|b\|_{BMO} := \sup_Q \frac{1}{|Q|}\int_Q |b(x)-\langle b\rangle_Q|dx < \infty.$$

Coifman, Rochberg and Weiss [16] showed that the commutator of Riesz transform is bounded on $L^p(\mathbb{R}^n)$ if and only if the function b is in $BMO(\mathbb{R}^n)$. The compactness of commutators has been started to receive attention with the development of the boundedness of commutators. Uchiyama [44] pointed out that the L^p-boundedness result in [16] could be refined to a compactness one if the space $BMO(\mathbb{R}^n)$ is replaced by $CMO(\mathbb{R}^n)$, which is defined to be the closure of $C_c^\infty(\mathbb{R}^n)$ in the BMO norm. Afterwards, the works on compactness of commutators have been blossomed, for example, [9, 13, 26, 45] et al. However, most of the scholars concerned with the compactness of linear operators, the literature is far from enough regarding the compactness of nonlinear operators, we refer readers to [10, 21] for the commutators of Littlewood-Paley operators and the maximal truncated commutators for singular integrals, etc.

Let $\rho>2$ and $\mathscr{F}(x)=\{F_t(x)\}_{t>0}$ be a family of Lebesgue measurable function, the ρ-variation function $\mathscr{V}_\rho(\mathscr{F})$ of the family $\mathscr{F}$ is defined by

$$\mathscr{V}_\rho(\mathscr{F})(x) = \|\{F_t(x)\}_{t>0}\|_{V_\rho} := \sup_{t_k\downarrow 0}\Big(\sum_{k=1}^{\infty}|F_{t_k}(x)-F_{t_{k+1}}(x)|^\rho\Big)^{1/\rho},$$

where the supremum is taken over all sequences $\{t_k\}$ decreasing to zero. By analogy to the definition of ρ-variation function, assume that $\mathscr{T} = \{T_t\}_{t>0}$ is a family of operators, then the ρ-variation operator is defined by

$$\mathscr{V}_\rho(\mathscr{T} f)(x) = \|\{T_t(f)(x)\}_{t>0}\|_{\mathscr{V}_\rho}.$$

In this paper, we study the variation operators associated with approximate identities. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \phi(x)dx = 1$, where $\mathscr{S}(\mathbb{R}^n)$ is the space of Schwartz functions. We consider the following familiy of approximate identities

$$\Phi \star f(x) := \{\phi_t * f(x)\}_{t>0}, \tag{3}$$

where $\phi_t(x) := t^{-n}\phi(x/t)$. Let $b \in L^1_{loc}(\mathbb{R}^n)$, we will also take into account the corresponding familiy of commutators of approximate identities

$$(\Phi \star f)_b(x) := \{b(x)\phi_t * f(x) - \phi_t * bf(x)\}_{t>0}, \tag{4}$$

where

$$b(x)\phi_t * f(x) - \phi_t * bf(x) = \int_{\mathbb{R}^n} \frac{1}{t^n}\phi\left(\frac{x-y}{t}\right)(b(x)-b(y))f(y)dy.$$

The variation for martingales and several families of operators have been investigated by numerous mathematicians on various fields, such as probability, ergodic theory, and harmonic analysis et al., one may consult [3, 31, 32, 33, 34] for earlier results. Particularly, for $\rho > 2$, the classical work of ρ-variation operators for singular integrals was given in [5], in which the authors gave the L^p-bounds and weak type (1,1) bounds for ρ-variation operators of truncated Hilbert transform and then extended to higher dimensional in [6]. The first quantitative weighted estimates for variation operators were given by Hytönen et al. [29], in which the authors studied variation operators of smooth truncations of singular integrals. Almost at the same time, Ma, Torrea and Xu [39] established the boundedness of ρ-variation operators of Calderón-Zygmund operators, additionally, they proved the variation operators of Calderón-Zygmund operators are bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$, which generalized the result of Crescimbeni et al. [17]. We refer readers to [11, 20, 22] for results of rough kernels and weighted cases. For the variation operators of heat semigroups, Crescimbeni et al. [18] gave the weighed L^p-bounds and weak type (1,1) bounds by using the vector-valued Calderón-Zygmund theory. Liu [38] established the boundedness of variation operators associated with approximate identities on Lebesgue spaces, which covers the results of [18] in the unweighted cases.

On the other hand, the variational inequalities for the commutators of singular integrals also have been intensively studied. In 2013, Betancor et al. [1] studied the mapping property of variation operators for the commutators of Riesz transforms in Euclidean and Schrödinger setting. Few years later, Liu and Wu [37] obtained the weighted L^p-boundedness for variation operators of commutators of truncated singular integrals with the Calderón-Zygmund kernels. Recently, variation operators of commutators with rough kernels were also obtained in [12]. While, for the compactness of variation operators of commutators, the result is rare, in 2019, Guo et al. [24] first gave a characterization of $CMO(\mathbb{R}^n)$ via the compactness of variation operators of commutators of singular integrals. Recently, Guo et al. [25] gave a new characterization of $BMO(\mathbb{R}^n)$ via the boundedness of variation operators associated with commutators of approximate identities.

From the previous known facts about variation inequalities, none of the quantitative weighted estimates of endpoint case $p=\infty$ for variation operators have been considered before. Inspired by the work of [25], one may wonder whether a new characterization of $CMO(\mathbb{R}^n)$ can be established. In this paper, we settle these problems as follows. Firstly, we give the quantitative weighted L^∞-BMO bounds for variation operators associated with convolutions with approximation of identities. As applications, we give a simper proof of the L^p-boundedness of variation operators associated with convolutions with approximation of identities than in [38] and extend it to weighted cases. Secondly, we obtain the quantitative C_q estimates for variation operators, which has never been considered before for variation operators. Thirdly, we give a new characterization of $CMO(\mathbb{R}^n)$. We state our results as follows.

THEOREM 2. *Let $\Phi\star f$ be given by (3), ω be a weight (see its definition in Section 2.1) and $\rho>2$. Assume that $f\in L^p(\mathbb{R}^n)$ for some $p\geqslant 1$ and $|f|\lesssim\omega$ almost everywhere. Then for all cubes $Q\subset\mathbb{R}^n$ and all $1<r<\infty$,*

$$\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)-\langle\mathscr{V}_\rho(\Phi\star f)\rangle_Q|\lesssim r'\|f/\omega\|_{L^\infty}\operatorname*{ess\,inf}_Q M_r\omega, \tag{5}$$

where $\omega\in L^r_{loc}$ such that the right-hand side is finite, $1/r+1/r'=1$ and $M_r(f):=(M(|f|^r))^{1/r}$. Specially, if $\omega\in A_\infty$,

$$\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)-\langle\mathscr{V}_\rho(\Phi\star f)\rangle_Q|\lesssim[\omega]_{A_\infty}\|f/\omega\|_{L^\infty}\operatorname*{ess\,inf}_Q M\omega. \tag{6}$$

Moreover, if $\omega\in A_1$,

$$\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)-\langle\mathscr{V}_\rho(\Phi\star f)\rangle_Q|\lesssim[\omega]_{A_1}[\omega]_{A_\infty}\|f/\omega\|_{L^\infty}\operatorname*{ess\,inf}_Q \omega. \tag{7}$$

REMARK 1. We can restate (5)–(7) as the following norm forms:

$$\left\|\frac{M^\sharp(\mathscr{V}_\rho(\Phi\star f))}{M_r\omega}\right\|_{L^\infty}\lesssim r'\|f/\omega\|_{L^\infty};$$

$$\left\|\frac{M^\sharp(\mathscr{V}_\rho(\Phi\star f))}{M\omega}\right\|_{L^\infty}\lesssim[\omega]_{A_\infty}\|f/\omega\|_{L^\infty};$$

$$\left\|\frac{M^\sharp(\mathscr{V}_\rho(\Phi\star f))}{\omega}\right\|_{L^\infty}\lesssim[\omega]_{A_1}[\omega]_{A_\infty}\|f/\omega\|_{L^\infty}.$$

The weighted $BMO(\mathbb{R}^n)$ space $BMO_\omega(\mathbb{R}^n)$, which was first introduced in [41] and developed by Bloom [2], is the set of all locally integrable functions f on $\mathbb{R}^n$ with $\|f\|_{BMO_\omega} < \infty$, where $\|f\|_{BMO_\omega} := \omega(Q)^{-1}\int_Q |f(x) - \langle f\rangle_Q| dx < \infty$ and $\omega \in A_\infty$. By (7), we have the following corollary.

COROLLARY 1. *Under the same assumptions as in Theorem 2, then for $\rho > 2$ and $\omega \in A_1$,*

$$\|\mathscr{V}_\rho(\Phi\star f)\|_{BMO_\omega} \lesssim [\omega]_{A_1}[\omega]_{A_\infty}\|f/\omega\|_{L^\infty(\mathbb{R}^n)}.$$

By Theorem 1 and Theorem 2, we obtain the weighted L^p-boundedness of variation operators associated with approximate identities.

COROLLARY 2. *Let $\rho > 2$, $1 < p < \infty$ and $\omega \in A_p$,*

$$\|\mathscr{V}_\rho(\Phi\star f)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}.$$

REMARK 2. Corollary 2 extends the result in [38] to the weighted, providing a simpler proof.

THEOREM 3. *Let $\Phi\star f$ be given by (3) and ω be a weight. Assume that $f \in L^p(\mathbb{R}^n)$ for some $p \geqslant 1$ and $|f| \lesssim \omega$ almost everywhere. Then for all cubes $Q \subset \mathbb{R}^n$ and any $\varepsilon \in (0,1)$,*

$$\inf_{c\in\mathbb{C}}\left(\frac{1}{|Q|}\int_Q |\mathscr{V}_\rho(\Phi\star f)(x) - c|^\varepsilon dy\right)^{1/\varepsilon} \leqslant C\|f/\omega\|_{L^\infty} \operatorname*{ess\,inf}_Q M\omega, \tag{8}$$

where C depends on ε and $\|\mathscr{V}_\rho\|_{L^1\to L^{1,\infty}}$. Moreover,

$$\left\|\frac{M_\varepsilon^\sharp(\mathscr{V}_\rho(\Phi\star f))}{M\omega}\right\|_{L^\infty} \leqslant C\|f/\omega\|_{L^\infty}. \tag{9}$$

Specially, if $\omega \in A_1$,

$$\left\|\frac{M_\varepsilon^\sharp(\mathscr{V}_\rho(\Phi\star f))}{\omega}\right\|_{L^\infty} \leqslant C[\omega]_{A_1}\|f/\omega\|_{L^\infty}. \tag{10}$$

REMARK 3. Theorem 3 is an improved version of Theorem 2 in the sense that a better dependence on the A_1–A_∞ constant is given.

THEOREM 4. *Under the same assumptions as in Theorem 3, then for $\rho > 2$,*

$$\left\|\frac{M_\varepsilon^\sharp(\mathscr{V}_\rho(\Phi\star f))}{\nu}\right\|_{L^\infty} \leqslant C[(\mu,\nu)]_{A_1}\|f/\mu\|_{L^\infty}. \tag{11}$$

Moreover, if $\mu \in A_\infty$, then

$$\left\|\frac{M^\sharp(\mathscr{V}_\rho(\Phi\star f))}{\nu}\right\|_{L^\infty} \leqslant C[\mu]_{A_\infty}[(\mu,\nu)]_{A_1}\|f/\mu\|_{L^\infty}, \tag{12}$$

where the definition of $[(\mu,\nu)]_{A_1}$ is given in Section 2.

THEOREM 5. *Let $1 < p < q < \infty$ and $\rho > 2$, then for some $\delta \in (p/q, 1)$ and any $\omega \in C_q$, we have*

$$\|\mathscr{V}_\rho(\Phi \star f)\|_{L^p(\omega)} \lesssim \Big(\frac{pq}{\delta q - p}\max\{1, [\omega]_{C_q}\log^+[\omega]_{C_q}\}\Big)^{1/\delta}\|Mf\|_{L^p(\omega)}.$$

THEOREM 6. *Let $(\Phi \star f)_b$ be given by (4) and $1 < p < \infty$. Then for $\rho > 2$ and $\omega \in A_p$, $\mathscr{V}_\rho((\Phi \star f)_b)$ is compact on $L^p(\omega)$ if and only if $b \in CMO(\mathbb{R}^n)$.*

REMARK 4. We point out that all the results above also hold for variation of heat semigroups and Poisson semigroups. Thus, we extend main results in [18].

We organize the rest of the paper as follows. We give preliminaries in Section 2. Section 3 is devoted to proving Theorems 2, 3, 4 and 5. In Section 4, we give the proof of Theorem 6.

We make some conventions at the end of this section. In this paper, we omit the constant which is independent of the main parameters. We denote $f \lesssim g$, $f \sim g$ if $f \leqslant Cg$ and $f \lesssim g \lesssim f$, respectively. For any ball $Q \subset \mathbb{R}^n$, $\langle f \rangle_Q$ means the mean value of f over Q, χ_Q represents the characteristic function of Q and c_Q denotes the center of the cube Q.

2. Preliminaries

2.1. Weights

A weight ω is a nonnegative and locally integrable function on $\mathbb{R}^n$. Given a weight ω, we say that $w \in A_p$ $(1 < p < \infty)$, if for all cubes $Q \subset \mathbb{R}^n$,

$$[\omega]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(y)dy\Big)\Big(\frac{1}{|Q|}\int_Q w(y)^{1-p'}dy\Big)^{p-1} < \infty.$$

When $p = 1$, we say that $\omega \in A_1$ if

$$[\omega]_{A_1} := \|M\omega/\omega\|_{L^\infty} < \infty.$$

When $p = \infty$, we define $A_\infty := \cup_{1 \leqslant p < \infty} A_p$, and the constant of A_∞ is defined by

$$[\omega]_{A_\infty} := \sup_Q \frac{1}{\omega(Q)}\int_Q M(\omega\chi_Q)(x)dx < \infty.$$

The doubling property of weight will be used in this paper: for $\lambda > 1$, and all cubes Q, if $\omega \in A_p$, we have $\omega(\lambda Q) \leqslant \lambda^{np}[\omega]_{A_p}\omega(Q)$. For two weights μ, ν, we say $(\mu, \nu) \in A_1$ if

$$[(\mu, \nu)]_{A_1} := \|M\mu/\nu\|_{L^\infty} < \infty.$$

The following lemma is the well known reverse Hölder inequality.

LEMMA 1. (cf. [30]) *Let* $\omega \in A_\infty$*, there is a positive constant* τ_n *such that for each* $\delta \in [0, 1/(\tau_n[\omega]_{A_\infty})]$ *and each cube* $Q \subset \mathbb{R}^n$*,*

$$\Big(\frac{1}{|Q|}\int_Q \omega(x)^{1+\delta}dx\Big)^{1/(1+\delta)} \leqslant \frac{2}{|Q|}\int_Q \omega(x)dx.$$

Next we introduce the C_p class of weights. Recall that a weight $\omega \in C_p$, if there are $C, \varepsilon > 0$ such that for each cube Q and each measurable $E \subset Q$,

$$\omega(E) \leqslant C\Big(\frac{|E|}{|Q|}\Big)^{\varepsilon}\int_{\mathbb{R}^n} M(\chi_Q)(x)^p\omega(x)dx.$$

And the C_p constant $[\omega]_{C_p}$ is defined in [7] by

$$[\omega]_{C_p} := \sup_Q \frac{\int_Q M(\omega\chi_Q)(x)dx}{\int_{\mathbb{R}^n} M(\chi_Q)(x)^p\omega(x)dx}.$$

From the definitions above, one can see that the C_p class of weights is larger than the A_∞ class of weights. In [8], the authors gave the following lemma that concerned with the quantitative C_q estimates for Hardy-Littlewood maximal function.

LEMMA 2. (cf. [8]) *Let* $1 < p < q < \infty$ *and* $\omega \in C_q$*. Then for any* $f \in L_c^\infty(\mathbb{R}^n)$*, we have*

$$\|Mf\|_{L^p(\omega)} \leqslant c_n\frac{pq}{q-p}\max\{1, [\omega]_{C_q}\log^+[\omega]_{C_q}\}\|M^\sharp f\|_{L^p(\omega)}.$$

2.2. Sharp maximal functions

Let $Q \subset \mathbb{R}^n$ be a cube with sides parallel to the axes. The sharp maximal function is defined by

$$M^\sharp(f)(x) := \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y) - \langle f\rangle_Q|dy \sim \sup_{Q\ni x}\inf_{c\in\mathbb{C}}\frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

For $\varepsilon \in (0,1)$, we also define the modified sharp maximal function by

$$M_\varepsilon^\sharp(f)(x) := (M^\sharp(|f|^\varepsilon)(x))^{1/\varepsilon}.$$

3. Quantitative weighted estimates for variation operators

In this section, we give the proof of Theorems 2, 3, 4 and 5. Before we prove Theorem 2, we need to establish the following lemma.

LEMMA 3. *Let* $\rho > 2$*, then for any* $1 < p < \infty$*,*

$$\|\mathscr{V}_\rho(\Phi\star f)\|_{L^p(\mathbb{R}^n)} \lesssim pp'\|f\|_{L^p(\mathbb{R}^n)},$$

where the implicit constant is independent of p, p'*.*

Proof. This conclusion was proved in [38], here, we track the constant that depends on p and p' for our convenience. By using the result in [25, Lemma 2.3] and the standard steps in [36, Theorem 1.1], we can prove that for any $1<p<\infty$,

$$\|\mathscr{V}_\rho(\Phi\star f)\|_{L^p(\mathbb{R}^n)} \lesssim p\|Mf\|_{L^p(\mathbb{R}^n)},$$

where the implicit constant is independent of p, p'. Then the result follows by

$$\|Mf\|_{L^p(\mathbb{R}^n)} \lesssim p'\|f\|_{L^p(\mathbb{R}^n)}. \qquad \square$$

Proof of Theorem 2. Let f be given as Theorem 2, given a cube Q, write $f = g+h$, where $g := f\chi_{2Q}$. We first prove (5).

For $y\in(2Q)^c$, $x\in Q$, by Minkowski's inequality, we have

$$\begin{aligned} &|\mathscr{V}_\rho(\Phi\star h)(x)-\mathscr{V}_\rho(\Phi\star h)(c_Q)| \\ &\quad\leqslant \|\{\phi_t*h(x)-\phi_t*h(c_Q)\}_{t>0}\|_{\mathscr{V}_\rho} \\ &\quad= \sup_{t_k\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n\setminus 2Q}\{[\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y)] \\ &\qquad\qquad -[\phi_{t_k}(c_Q-y)-\phi_{t_{k+1}}(c_Q-y)]\}f(y)dy\Big|^\rho\Big)^{1/\rho} \\ &\quad\leqslant \int_{\mathbb{R}^n\setminus 2Q}|f(y)|\|\{\phi_t(x-y)-\phi_t(c_Q-y)\}_{t>0}\|_{\mathscr{V}_\rho}dy. \end{aligned} \tag{13}$$

For $y\in(2Q)^c$, $x\in Q$, using the mean value theorem, we deduce that

$$\begin{aligned} &\|\{\phi_t(x-y)-\phi_t(c_Q-y)\}_{t>0}\|_{\mathscr{V}_\rho} \\ &\quad\leqslant \sup_{t_k\downarrow 0}\Big(\sum_k\Big|[\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y)]-[\phi_{t_k}(c_Q-y)-\phi_{t_{k+1}}(c_Q-y)]\Big|\Big) \\ &\quad= \sup_{t_k\downarrow 0}\Big(\sum_k\Big|\int_{t_{k+1}}^{t_k}\frac{\partial}{\partial t}(\phi_t(x-y)-\phi_t(c_Q-y))dt\Big|\Big) \\ &\quad\leqslant \int_0^\infty\Big|\frac{\partial}{\partial t}(\phi_t(x-y)-\phi_t(c_Q-y))\Big|dt \\ &\quad\lesssim |x-c_Q|\int_0^\infty\frac{1}{t^{n+2}}\Big(1+\frac{|y-c_Q|}{t}\Big)^{-(n+2)}dt \\ &\quad= \frac{|x-c_Q|}{|y-c_Q|^{n+1}}\int_0^\infty\frac{t^n}{(t+1)^{n+2}}dt \sim \frac{|x-c_Q|}{|y-c_Q|^{n+1}}. \end{aligned} \tag{14}$$

From the assumption on f and the L^p-boundedness of $\mathscr{V}_\rho(\Phi\star h)$ (see [38]), we know that $\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q<\infty$. Hence, from (13) and (14), we obtain

$$\begin{aligned} &\int_Q|\mathscr{V}_\rho(\Phi\star h)(x)-\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q|dx \\ &\quad= \int_Q|\mathscr{V}_\rho(\Phi\star h)(x)-\mathscr{V}_\rho(\Phi\star h)(c_Q)+\mathscr{V}_\rho(\Phi\star h)(c_Q)-\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q|dx \end{aligned} \tag{15}$$

$$
\begin{aligned}
&\leqslant 2\int_Q |\mathscr{V}_\rho(\Phi\star h)(x)-\mathscr{V}_\rho(\Phi\star h)(c_Q)|dx\\
&\lesssim \int_Q\int_{\mathbb{R}^n\setminus 2Q}\frac{|x-c_Q|}{|y-c_Q|^{n+1}}|f(y)|dydx\\
&\leqslant \|f/\omega\|_{L^\infty}\int_Q\sum_{j=1}^\infty\int_{2^{j+1}Q\setminus 2^jQ}\frac{\sqrt{n}l_Q}{(2^{j-1}l_Q)^{n+1}}\omega(y)dydx\\
&\lesssim \|f/\omega\|_{L^\infty}\int_Q\sum_{j=1}^\infty 2^{-j}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\omega(y)dydx\\
&\lesssim \|f/\omega\|_{L^\infty}|Q|\operatorname*{ess\,inf}_{x\in Q}M_r\omega(x).
\end{aligned}
$$

On the other hand, by Lemma 3, we have

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q\mathscr{V}_\rho(\Phi\star g)(x)dx &\leqslant \Big(\frac{1}{|Q|}\int_Q\mathscr{V}_\rho(\Phi\star g)(x)^r dx\Big)^{1/r}\\
&\lesssim rr'\Big(\frac{1}{|2Q|}\int_{2Q}|f(y)|^r dy\Big)^{1/r}\\
&\lesssim rr'\|f/\omega\|_{L^\infty}\Big(\frac{1}{|2Q|}\int_{2Q}|\omega(y)|^r dy\Big)^{1/r}\\
&\lesssim rr'\|f/\omega\|_{L^\infty}\operatorname*{ess\,inf}_{x\in Q}M_r\omega(x).
\end{aligned} \tag{16}
$$

Now observe that

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)(x)-\langle\mathscr{V}_\rho(\Phi\star f)\rangle_Q|dx\\
&\quad=\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)(x)-\mathscr{V}_\rho(\Phi\star h)(x)+\mathscr{V}_\rho(\Phi\star h)(x)-\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q\\
&\qquad\qquad+\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q-\langle\mathscr{V}_\rho(\Phi\star f)\rangle_Q|dx\\
&\quad\leqslant\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star h)(x)-\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q|dx+\frac{2}{|Q|}\int_Q\mathscr{V}_\rho(\Phi\star g)(x)dx
\end{aligned}
$$

Hence, (5) follows by (15) and (16).

Next, we turn to prove (6). Choose $r=1+1/(\tau_n[\omega]_{A_\infty})$, then $r'\sim[\omega]_{A_\infty}$ and $r\sim c$. Since $\omega\in A_\infty$, applying Lemma 1, we get (6).

Finally, (7) is a consequence of (6) and the definition of A_1. Theorem 2 is proved. □

Proof of Theorem 3. We first prove (8). Fix a cube Q, write $f=g+h$ with $g:=f\chi_{2Q}$. Then for $\varepsilon\in(0,1)$, one can see that

$$
\begin{aligned}
&\inf_{c\in\mathbb{C}}\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)(x)-c|^\varepsilon dx\\
&\quad\leqslant\frac{1}{|Q|}\int_Q|\mathscr{V}_\rho(\Phi\star f)(x)-\langle\mathscr{V}_\rho(\Phi\star h)\rangle_Q|^\varepsilon dx
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star f)(x)-\mathcal{V}_\rho(\Phi\star h)(x)|^\varepsilon dx\\
&\quad+\frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star h)(x)-\langle\mathcal{V}_\rho(\Phi\star h)\rangle_Q|^\varepsilon dx\\
&\leqslant \frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star g)(x)|^\varepsilon dx+\frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star h)(x)-\langle\mathcal{V}_\rho(\Phi\star h)\rangle_Q|^\varepsilon dx\\
&=: I+II.
\end{aligned}$$

For II, by (15) and Jesen's inequality, we have

$$\begin{aligned}
II^{1/\varepsilon} &\leqslant \frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star h)(x)-\langle\mathcal{V}_\rho(\Phi\star h)\rangle_Q| dx\\
&\lesssim \|f/\omega\|_{L^\infty}\operatorname*{ess\,inf}_{x\in Q} M\omega(x).
\end{aligned}$$

To estimate I, we use the Kolmogorov inequality and the weak type $(1,1)$ of $\mathcal{V}_\rho(\Phi\star g)$ (see [38]), then

$$\begin{aligned}
I^{1/\varepsilon} &= \Big(\frac{1}{|Q|}\int_Q \mathcal{V}_\rho(\Phi\star g)(x)^\varepsilon dx\Big)^{1/\varepsilon}\\
&\leqslant \Big(\frac{1}{1-\varepsilon}\Big)^{1/\varepsilon}\|f/\omega\|_{L^\infty}\|\mathcal{V}_\rho(\Phi)\|_{L^1\to L^{1,\infty}}\frac{1}{|Q|}\int_{2Q}\omega(y)dy\\
&\leqslant \Big(\frac{1}{1-\varepsilon}\Big)^{1/\varepsilon}\|f/\omega\|_{L^\infty}\|\mathcal{V}_\rho(\Phi)\|_{L^1\to L^{1,\infty}}\operatorname*{ess\,inf}_{x\in Q} M\omega(x).
\end{aligned}$$

Hence, by the estimate of $I^{1/\varepsilon}$ and $II^{1/\varepsilon}$, we get the desired results.

To prove (9), we use $\big||a|^\varepsilon-|b|^\varepsilon\big|\leqslant|a-b|^\varepsilon$ for $\varepsilon\in(0,1)$, then

$$\begin{aligned}
&\inf_{c\in\mathbb{C}}\frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star f)(x)^\varepsilon-c^\varepsilon| dx\\
&\quad\leqslant \frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star f)(x)^\varepsilon-\langle\mathcal{V}_\rho(\Phi\star h)\rangle_Q^\varepsilon| dx\\
&\quad\leqslant \frac{1}{|Q|}\int_Q |\mathcal{V}_\rho(\Phi\star f)(x)-\langle\mathcal{V}_\rho(\Phi\star h)\rangle_Q|^\varepsilon dx,
\end{aligned}$$

from the definition of $M_\varepsilon^\sharp(\mathcal{V}_\rho(\Phi\star f))$, we find that (9) can be proved by following the steps of the proof (8).

Finally, (10) follows by (9) and the definition of A_1. Theorem 3 is proved. □

Proof of Theorem 4. Take $\omega=\mu$ in (9) and use $M\mu\leqslant[(\mu,\nu)]_{A_1}\nu$, we get (11). Take $\omega=\mu$ in the second norm inequality in Remark 1 and again use $M\mu\leqslant[(\mu,\nu)]_{A_1}\nu$, we get (12). This completes the proof of Theorem 4. □

Proof of Theorem 5. By a minor modification of the proof of Theorem 3, we can prove that for any $\varepsilon\in(0,1)$,

$$M_\varepsilon^\sharp(\mathcal{V}_\rho(\Phi\star f))(x)\leqslant c_{n,\varepsilon,\mathcal{V}_\rho(\Phi)}Mf(x).$$

Since $p/\delta < q$, from Lemma 2, we obtain

$$\begin{aligned}\|\mathscr{V}_\rho(\Phi\star f)\|_{L^p(\omega)} &\leqslant \|M_\delta(\mathscr{V}_\rho(\Phi\star f))\|_{L^p(\omega)} = \|M((\mathscr{V}_\rho(\Phi\star f))^\delta)\|^{1/\delta}_{L^{p/\delta}(\omega)}\\ &\lesssim \Big(\frac{pq}{\delta q-p}\max\{1,[\omega]_{C_q}\log^+[\omega]_{C_q}\}\Big)^{1/\delta}\|M^\sharp((\mathscr{V}_\rho(\Phi\star f))^\delta)\|^{1/\delta}_{L^{p/\delta}(\omega)}\\ &= \Big(\frac{pq}{\delta q-p}\max\{1,[\omega]_{C_q}\log^+[\omega]_{C_q}\}\Big)^{1/\delta}\|M^\sharp_\delta((\mathscr{V}_\rho(\Phi\star f)))\|_{L^p(\omega)}\\ &\lesssim \Big(\frac{pq}{\delta q-p}\max\{1,[\omega]_{C_q}\log^+[\omega]_{C_q}\}\Big)^{1/\delta}\|Mf\|_{L^p(\omega)},\end{aligned}$$

where in the third inequality, we used the following result proved by [46]:

$$\|Mf\|_{L^p(\omega)} \lesssim \|M^\sharp f\|_{L^p(\omega)}, \quad 1<p<q<\infty,\ \omega\in C_q.$$

This completes the proof of Theorem 5. □

4. The characterization of $CMO(\mathbb{R}^n)$

This section is devoted to proving Theorem 6. We first recall the following definitions.

DEFINITION 1. For a complex-valued measurable function f, the local mean oscillation of f over a cube Q is defined by

$$a_\lambda(f;Q) := \inf_{c\in\mathbb{C}}((f-c)\chi_Q)^*(\lambda|Q|) \qquad (0<\lambda<1),$$

where f^* denotes the non-increasing rearrangement of f.

DEFINITION 2. By a median value of a real-valued measurable function f over a measure set E of positive finite measure, we mean a possibly non-unique, real number $m_f(E)$ such that

$$\max(|\{x\in E: f(x)>m_f(E)\}|,\ |\{x\in E: f(x)<m_f(E)\}|) \leqslant |E|/2.$$

To prove our theorem, we need the following lemmas. To be more precise, we use Lemma 4 and Lemma 6 to prove the sufficiency of Theorem 6, and Lemmas 5–8 are applied to prove the necessity of Theorem 6.

LEMMA 4. (cf. [27]) *Let $p\in(0,\infty)$ and ω be a weight, a subset E of $L^p(\mathbb{R}^n)$ is precompact (or totally bounded) if the following statements hold:*

(a) *E is uniformly bounded, i.e., $\sup_{f\in E}\|f\|_{L^p(\omega)}\lesssim 1$;*

(b) *E uniformly vanishes at infinity, that is,*

$$\lim_{N\to\infty}\int_{|x|>N}|f(x)|^p\omega(x)dx = 0,$$

uniformly for all $f \in E$*;*

(c) E *is uniformly equicontinuous, that is,*

$$\lim_{\rho\to 0}\sup_{y\in B(0,\rho)}\int_{\mathbb{R}^n}|f(x+y)-f(x)|^p\omega(x)dx=0,$$

uniformly for all $f \in E$.

LEMMA 5. (cf. [26]) *Let* $b \in BMO(\mathbb{R}^n)$. *Then* $b \in CMO(\mathbb{R}^n)$ *if and only if the following three conditions hold:*

(1) $\lim_{d\to 0}\sup\limits_{|Q|=d} a_\lambda(b;Q)=0$,

(2) $\lim_{d\to +\infty}\sup\limits_{|Q|=d} a_\lambda(b;Q)=0$,

(3) $\lim_{d\to +\infty}\sup\limits_{|Q|\cap[-d,d]^n=\emptyset} a_\lambda(b;Q)=0$.

LEMMA 6. (cf. [25]) *Let* $\phi \in \mathscr{S}(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n}\phi(x)dx=1$, $1<p<\infty$, $\omega\in A_p$. *Then for* $\rho>2$, $\mathscr{V}_\rho((\Phi\star f)_b)$ *is bounded on* $L^p(\omega)$ *if and only if* $b\in BMO(\mathbb{R}^n)$.

LEMMA 7. *Let* $\omega\in A_p$ *and* b *be a real-valued measurable function. Given a cube* Q*, there exist sets* E *and* F *associated with* Q *such that for* $f=\left(\int_F\omega(x)dx\right)^{-1/p}\chi_F$ *and any measurable set* B *with* $|B|\leqslant\lambda/8|Q|$,

$$\|\mathscr{V}_\rho((\Phi\star f)_b)\|_{L^p(E\backslash B,\omega)}\gtrsim a_\lambda(b;Q),$$

where the implicit constant is independent of Q.

Proof. Without loss of generality, we may assume that b and ϕ are real valued, $\phi(z)\geqslant 1$, where $z\in B(z_0,\delta)$ with $|z_0|=1$ and $\delta>0$ is a small constant. For any cube Q, denote by

$$P:=Q-10\sqrt{n}\delta^{-1}l_Qz_0$$

the cube associated with Q. By the definition of $a_\lambda(f;Q)$, there exists a subset $\tilde{Q}$ of Q, such that $|\tilde{Q}|=\lambda|Q|$, and according to the definition of $m_b(P)$, there exist subsets $E\subset\tilde{Q}$ and $F\subset P$ such that

$$|E|=|\tilde{Q}|/2=\lambda|Q|/2,\ |F|=|P|/2=|Q|/2.$$

By the Hölder inequality, we have

$$\int_{E\backslash B}\mathscr{V}_\rho((\Phi\star f)_b)(x)dx\leqslant\Big(\int_{E\backslash B}\mathscr{V}_\rho((\Phi\star f)_b)(x)^p\omega(x)dx\Big)^{1/p}\Big(\int_Q\omega(x)^{-p'/p}\Big)^{1/p'}. \tag{17}$$

On the other hand, for $x \in E$, it was proved in [25] that

$$\mathscr{V}_\rho((\Phi \star f)_b)(x) \gtrsim a_\lambda(b;Q)\Big(\int_F \omega(x)dx\Big)^{-1/p},$$

then

$$\begin{aligned}\int_{E\setminus B} \mathscr{V}_\rho((\Phi \star f)_b)(x)dx &\gtrsim |E\setminus B|\Big(\int_P \omega(x)dx\Big)^{-1/p} a_\lambda(b;Q) \\ &\geqslant 3\lambda/8|Q|\Big(\int_P \omega(x)dx\Big)^{-1/p} a_\lambda(b;Q).\end{aligned} \tag{18}$$

Hence, by (17) and (18), we deduce that

$$\begin{aligned}&\Big(\int_{E\setminus B} \mathscr{V}_\rho((\Phi \star f)_b)(x)^p \omega(x)dx\Big)^{1/p} \\ &\geqslant \Big(\int_Q \omega(x)^{-p'/p}dx\Big)^{-1/p'}\Big(\int_{E\setminus B} \mathscr{V}_\rho((\Phi \star f)_b)(x)dx\Big) \\ &\gtrsim \Big(\int_Q \omega(x)^{-p'/p}dx\Big)^{-1/p'}\Big(\int_P \omega(x)dx\Big)^{-1/p} a_\lambda(b;Q)|Q| \gtrsim a_\lambda(b;Q),\end{aligned}$$

where we use the definition of A_p and $P \subset KQ$ for some $K > 0$ in the last inequality. This is the desired result. □

LEMMA 8. *Let $\omega \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Given a cube Q, let P, E, F be the sets associated with Q mentioned in Lemma 7. Set $f = \Big(\int_F \omega(x)dx\Big)^{-1/p}\chi_F$. Then there is a $\delta > 0$ such that*

$$\|\mathscr{V}_\rho((\Phi \star f)_b)\|_{L^p(2^{d+1}Q\setminus 2^dQ,\omega)} \lesssim 2^{-\delta dn/p}d,$$

for d large enough, where the implicit constant is independent of d and Q.

Proof. Note that

$$\begin{aligned}\|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho} &\leqslant \|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_1} \\ &= \sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{t_{k+1}}^{t_k}\frac{\partial}{\partial t}(\phi_t(x-y))dt\Big|\Big) \\ &\lesssim \int_0^\infty \frac{1}{t^{n+1}(1+\frac{|x-y|}{t})^{n+1}}dt + \int_0^\infty \frac{|x-y|}{t^{n+2}(1+\frac{|x-y|}{t})^{n+2}}dt \\ &= \frac{1}{|x-y|^n}\Big(\int_0^\infty \frac{t^{n-1}}{(1+t)^{n+1}}dt + \int_0^\infty \frac{t^n}{(1+t)^{n+2}}dt\Big) \\ &\sim |x-y|^{-n},\end{aligned} \tag{19}$$

and observe that the set F can be chosen so that

$$f(x) \lesssim \Big(\int_P \omega(x)dx \Big)^{-1/p} \chi_P(x).$$

Then by Minkowski's inequality,

$$\begin{aligned} \mathscr{V}_\rho((\Phi \star f)_b)(x) &\leqslant \int_P |b(x)-b(y)| \|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho} dy \Big(\int_P \omega(x)dx \Big)^{-1/p} \\ &\lesssim \int_P |b(x)-b(y)||x-y|^{-n} dy \Big(\int_P \omega(x)dx \Big)^{-1/p} \\ &\leqslant \int_P |\langle b\rangle_P - b(y)||x-y|^{-n} dy \Big(\int_P \omega(x)dx \Big)^{-1/p} \\ &\quad + |b(x)-\langle b\rangle_P| \int_P |x-y|^{-n} dy \Big(\int_P \omega(x)dx \Big)^{-1/p}. \end{aligned} \tag{20}$$

For $x \in 2^{d+1}Q \setminus 2^d Q$ and $y \in P$, since $|x-y| \sim 2^d l_Q$ and $|Q| = |P|$, we have

$$\begin{aligned} &\int_P |b(y)-\langle b\rangle_P||x-y|^{-n} dy \\ &\quad \sim \frac{1}{2^{dn}|P|} \int_P |b(y)-\langle b\rangle_P| dy \leqslant 2^{-dn} \|b\|_{BMO(\mathbb{R}^n)}. \end{aligned} \tag{21}$$

To estimate $\|b(\cdot)-\langle b\rangle_P\|_{L^p(2^{d+1}Q\setminus 2^dQ,\omega)}$, let v be a positive constant independent of Q satisfy $2Q \subset 2^v P$, by the Hölder inequality and reverse Hölder inequality, one can compute that

$$\begin{aligned} &\Big(\int_{2^{d+1}Q\setminus 2^dQ} |b(x)-\langle b\rangle_P|^p \omega(x)dx \Big)^{1/p} \\ &\leqslant \Big(\int_{2^{d+v}P} |b(x)-\langle b\rangle_P|^p \omega(x)dx \Big)^{1/p} \\ &\leqslant |2^{d+v}P|^{1/p} \Big(\frac{1}{|2^{d+v}P|} \int_{2^{d+v}P} |b(x)-\langle b\rangle_P|^{p(1+\varepsilon)'} \Big)^{\frac{1}{p(1+\varepsilon)'}} \\ &\quad \times \Big(\frac{1}{|2^{d+v}P|} \int_{2^{d+v}P} \omega(x)^{1+\varepsilon} dx \Big)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim |2^{d+v}P|^{1/p} (d + \|b\|_{BMO(\mathbb{R}^n)}) \Big(\frac{1}{|2^{d+v}P|} \int_{2^{d+v}P} \omega(x)dx \Big)^{1/p} \\ &\lesssim |2^{d+v}P|^{1/p} d \Big(\frac{1}{|2^{d+v}P|} \int_{2^{d+v}P} \omega(x)dx \Big)^{1/p}. \end{aligned} \tag{22}$$

Therefore, by (20)–(22), we have

$$\begin{aligned} &\|\mathscr{V}_\rho((\Phi \star f)_b)\|_{L^p(2^{d+1}Q\setminus 2^dQ,\omega)} \\ &\lesssim 2^{dn(1/p-1)} d \Big(\frac{1}{|2^{d+v}P|} \int_{2^{d+v}P} \omega(x)dx \Big)^{1/p} \Big(\frac{1}{|P|} \int_P \omega(x)dx \Big)^{-1/p} \end{aligned}$$

Since $\omega \in A_p$, there is a $\delta > 0$ such that $\omega \in A_{p-\delta}$. Then the doubling property of $A_{p-\delta}$ yields that

$$\Big(\frac{1}{|P|}\int_P \omega(x)dx\Big)^{-1/p} \lesssim 2^{-dn/p}2^{dn(1-\delta/p)}\Big(\frac{1}{|2^{d+v}P|}\int_{2^{d+v}P}\omega(x)dx\Big)^{-1/p}.$$

We immediately have

$$\|\mathscr{V}_\rho((\Phi\star f)_b)\|_{L^p(2^{d+1}Q\backslash 2^dQ,\omega)} \lesssim 2^{-\delta dn/p}d. \quad \square$$

Now, we are in the position to prove Theorem 6.

Proof of Theorem 6. Assume that $b \in CMO(\mathbb{R}^n)$, we first show that $\mathscr{V}_\rho((\Phi\star f)_b)$ is compact on $L^p(\omega)$. According the definition of compact operator, we need to check that

$$A(\mathscr{V}_\rho((\Phi\star f)_b)) := \{\mathscr{V}_\rho((\Phi\star f)_b) : \|f\|_{L^p(\omega)} \leqslant 1\}$$

is precompact. By Lemma 6, it suffices to verify $A(\mathscr{V}_\rho((\Phi\star f)_b))$ is precompact for $b \in C_c^\infty(\mathbb{R}^n)$. Without loss of generality, we assume that b is supported in a cube Q centered at the origin. Now, let us proceed a further reduction. Choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ supported on $B(0,1)$ such that $\varphi = 1$ on $B(0,1/2)$ and $0 \leqslant \varphi \leqslant 1$, denote $\varphi_\delta(x) = \varphi(x/\delta)$ with $\delta > 0$. Define

$$\mathscr{V}_\rho^\delta((\Phi\star f)_b)(x) = \sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}(\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y))(1-\varphi_\delta(x-y)) \times (b(x)-b(y))f(y)dy\Big|^\rho\Big)^{1/\rho}.$$

We claim that it suffices to check that

$$A(\mathscr{V}_\rho^\delta((\Phi\star f)_b)) := \{\mathscr{V}_\rho^\delta((\Phi\star f)_b) : \|f\|_{L^p(\omega)} \leqslant 1\}$$

is precompact.

Indeed, the sublinearity of variation operator, the Minkowski inequality, $b \in C_c^\infty(\mathbb{R}^n)$ and (19) yield that

$$\begin{aligned}
&|\mathscr{V}_\rho((\Phi\star f)_b)(x) - \mathscr{V}_\rho^\delta((\Phi\star f)_b)(x)| \\
&\leqslant \sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}(\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y))\varphi_\delta(x-y)(b(x)-b(y))f(y)dy\Big|^\rho\Big)^{1/\rho} \\
&\leqslant \int_{\mathbb{R}^n}\|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho}|\varphi_\delta(x-y)||b(x)-b(y)||f(y)|dy \\
&\lesssim \int_{|x-y|\leqslant\delta}\frac{|f(y)|}{|x-y|^{n-1}}dy \leqslant \sum_{j=0}^{\infty}\int_{2^{-j-1}\delta<|x-y|\leqslant 2^{-j}\delta}\frac{|f(y)|}{|x-y|^{n-1}}dy \lesssim \delta Mf(x).
\end{aligned}$$

Hence, by the $L^p(\omega)$-boundedness of M,

$$\|\mathscr{V}_\rho((\Phi\star f)_b) - \mathscr{V}_\rho^\delta((\Phi\star f)_b)\|_{L^p(\omega)\to L^p(\omega)} \lesssim \delta,$$

which implies the claim by letting $\delta \to 0$.

Now, in the following, we prove that $A(\mathscr{V}_\rho^\delta((\Phi\star f)_b))$ is a precompact set. Invoking Lemma 4, we need to check that conditions $(a)-(c)$ of Lemma 4 for $A(\mathscr{V}_\rho((\Phi\star f)_b))$. It is easy to see that (a) of Lemma 4 holds by Lemma 6 and the $L^p(\omega)$-boundedness of M.

For $x\in(2Q)^c$, recall that $\operatorname{supp} b\subset Q$, make use of Minkowski's inequality, (19) and $b\in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\mathscr{V}_\rho((\Phi\star f)_b)(x)\\
&\quad=\sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}(\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y))b(y)(1-\varphi_\delta(x-y))f(y)dy\Big|^\rho\Big)^{1/\rho}\\
&\quad\lesssim\|b\|_{L^\infty(\mathbb{R}^n)}\int_Q\frac{|f(y)|}{|x-y|^n}dy\lesssim|x|^{-n}\int_Q|f(y)|dy\\
&\quad\lesssim|x|^{-n}\|f\|_{L^p(\omega)}\Big(\int_Q\omega(x)^{-p'/p}dx\Big)^{1/p'}.
\end{aligned}$$

Choose $N>2$, since $\omega\in A_{p-\varepsilon}$ for some $\varepsilon>0$, using the doubling property of $A_{p-\varepsilon}$ and the definition of A_p, we have

$$\begin{aligned}
&\Big(\int_{(2^NQ)^c}\mathscr{V}_\rho((\Phi\star f)_b)(x)^p\omega(x)dx\Big)^{1/p}\\
&\quad\lesssim\Big(\int_{(2^NQ)^c}\omega(x)|x|^{-np}dx\Big)^{1/p}\Big(\int_Q\omega(x)^{-p'/p}dx\Big)^{1/p'}\\
&\quad\leqslant\Big(\sum_{d=N}^\infty\int_{2^{d+1}Q\backslash 2^dQ}\omega(x)|x|^{-np}dx\Big)^{1/p}\Big(\int_Q\omega(x)^{-p'/p}dx\Big)^{1/p'}\\
&\quad\lesssim\Big(\sum_{d=N}^\infty\omega(2^{d+1}Q)2^{-dnp}|Q|^{-p}\Big)^{1/p}\Big(\int_Q\omega(x)^{-p'/p}dx\Big)^{1/p'}\\
&\quad\lesssim\Big(\sum_{d=N}^\infty 2^{(d+1)(p-\varepsilon)n}2^{-dnp}\Big)^{1/p}\Big(\frac{1}{|Q|}\int_Q\omega(x)dx\Big)^{1/p}\Big(\frac{1}{|Q|}\int_Q\omega(x)^{-p'/p}dx\Big)^{1/p'}\\
&\quad\lesssim(2^{-Nn\varepsilon})^{1/p},
\end{aligned}$$

which tends to 0 as $N\to\infty$. Hence, condition (b) of Lemma 4 holds.

Finally, let us check condition (c) of Lemma 4 holds. For $|z|\leqslant\delta/8$, a careful computation shows that

$$\begin{aligned}
&|\mathscr{V}_\rho((\Phi\star f)_b)(x+z)-\mathscr{V}_\rho((\Phi\star f)_b)(x)| \qquad (23)\\
&\quad\leqslant\sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}\Big[(\phi_{t_k}(x+z-y)-\phi_{t_{k+1}}(x+z-y))(1-\varphi_\delta(x+z-y))\\
&\qquad\qquad-(\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y))(1-\varphi_\delta(x-y))\Big](b(x+z)-b(y))f(y)dy\Big|^\rho\Big)^{1/\rho}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}(\phi_{t_k}(x-y)-\phi_{t_{k+1}}(x-y))(1-\varphi_\delta(x-y))\\
&\qquad\times(b(x+z)-b(x))f(y)dy\Big|^\rho\Big)^{1/\rho}\\
&=: I+II.
\end{aligned}
$$

We first consider I. Note that

$$
\begin{aligned}
I\leqslant&\sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}\Big(\int_{t_{k+1}}^{t_k}\frac{\partial}{\partial t}(\phi_t(x+z-y)-\phi_t(x-y))dt\Big)(1-\varphi_\delta(x+z-y))\\
&\qquad\times(b(x+z)-b(y))f(y)dy\Big|^\rho\Big)^{1/\rho}\\
&+\sup_{\{t_k\}\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}\Big(\int_{t_{k+1}}^{t_k}\frac{\partial}{\partial t}(\phi_t(x-y))dt\Big)(\varphi_\delta(x-y)-\varphi_\delta(x+z-y))\\
&\qquad\times(b(x+z)-b(y))f(y)dy\Big|^\rho\Big)^{1/\rho}\\
=:&\ I_1+I_2.
\end{aligned}
$$

Observe that $1-\varphi_\delta(x+z-y)$ vanishes when $|x-y|\leqslant 3\delta/8$, thus, by $b\in C_c^\infty(\mathbb{R}^n)$, Minkowski's inequality, (14) and the mean value theorem with $\theta\in(0,1)$, we deduce that

$$
\begin{aligned}
I_1&\lesssim\|b\|_{L^\infty(\mathbb{R}^n)}\int_{|x-y|>3\delta/8}\|\{\phi_t(x+z-y)-\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho}|f(y)|dy\\
&\lesssim\int_{|x-y|>3\delta/8}\frac{|z|}{|(x-y)+z\theta|^{n+1}}|f(y)|dy\\
&\lesssim|z|\int_{|x-y|>3\delta/8}\frac{|f(y)|}{|x-y|^{n+1}}|f(y)|dy\\
&\leqslant|z|\sum_{j=0}^\infty\frac{1}{(2^j3\delta/8)^{n+1}}\int_{2^j3\delta/8<|x-y|\leqslant 2^{j+1}3\delta/8}|f(y)|dy\lesssim\frac{|z|}{\delta}Mf(x).
\end{aligned}
$$

For I_2, note that $|\varphi_\delta(x-y)-\varphi_\delta(x+z-y)|$ vanishes when $|x-y|\leqslant 3\delta/8$ or $|x-y|>9\delta/8$. And by mean value theorem, for some $\theta\in(0,1)$, we have

$$
|\varphi_\delta(x-y)-\varphi_\delta(x+z-y)|\leqslant\frac{|z|}{\delta}\Big|\nabla\varphi\Big(\frac{(1-\theta)x+\theta(x+z)-y}{\delta}\Big)\Big|.
$$

Since

$$
|\nabla\varphi(x)|\lesssim\chi_{1/2\leqslant|x|\leqslant 1}(x)
$$

and when $3\delta/8<|x-y|\leqslant 9\delta/8$ and $|z|<\delta/8$,

$$
|(1-\theta)x+\theta(x+z)-y|\sim|x-y|.
$$

It follows that

$$|\varphi_\delta(x-y)-\varphi_\delta(x+z-y)| \leqslant \frac{|z|}{|x-y|}.$$

From this, again by $b \in C_c^\infty(\mathbb{R}^n)$, Minkowski's inequality and (19), we get that

$$\begin{aligned} I_2 &\lesssim |z| \int_{3\delta/8<|x-y|\leqslant 9\delta/8} \|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho} |x-y|^{-1}\| |f(y)| dy \\ &\lesssim \frac{|z|}{\delta} \int_{3\delta/8<|x-y|\leqslant 9\delta/8} \frac{|f(y)|}{|x-y|^n} dy \lesssim \frac{Mf(x)}{\delta}. \end{aligned}$$

Hence, we obtain

$$\|I\|_{L^p(\omega)} \leqslant \|I_1\|_{L^p(\omega)} + \|I_2\|_{L^p(\omega)} \lesssim \frac{|z|}{\delta}. \tag{24}$$

In the following, we deal with II. One can see that

$$\begin{aligned} II &\lesssim |z| \sup_{\delta>0} \Big\| \{ \int_{|x-y|>\delta} \phi_t(x-y) f(y) dy\}_{t>0} \Big\|_{\mathscr{V}_\rho} \\ &\quad + |z| \int_{\delta/2<|x-y|\leqslant\delta} \|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho} |f(y)| dy \\ &=: II_1 + II_2. \end{aligned}$$

Now, we consider the operator:

$$\begin{aligned} &T : L^2(\mathbb{R}^n) \to L^2_{\mathscr{V}_\rho}(\mathbb{R}^n) \\ &f \to Tf(x) := \int_{\mathbb{R}^n} \phi_t(x-y) f(y) dy. \end{aligned}$$

From Corollary 2, we know that T is bounded from $L^2(\mathbb{R}^n)$ into $L^2_{\mathscr{V}_\rho}(\mathbb{R}^n)$. Enjoy the same estimate as (19) and (14), we can prove that

$$\|\{\phi_t(x-y)\}_{t>0}\|_{\mathscr{V}_\rho} \lesssim |x-y|^{-n}, \quad x,y \in \mathbb{R}^n, x \neq y;$$

$$\Big\| \Big\{ \frac{\partial}{\partial x}(\phi_t(x-y)) \Big\}_{t>0} \Big\|_{\mathscr{V}_\rho} + \Big\| \Big\{ \frac{\partial}{\partial y}(\phi_t(x-y)) \Big\}_{t>0} \Big\|_{\mathscr{V}_\rho} \lesssim |x-y|^{-n-1}, \ x,y \in \mathbb{R}^n, \ x \neq y;$$

Hence, applying the vector-valued Calderón-Zygmund theory, we obtain that T is bounded from $L^p(\omega)$ into $L^p_{\mathscr{V}_\rho}(\omega)$ for $\omega \in A_p$. Denote

$$T^* f(x) := \sup_{\eta>0} \Big\| \Big\{ \int_{|x-y|>\eta} \phi_t(x-y) f(y) dy \Big\}_{t>0} \Big\|_{\mathscr{V}_\rho},$$

then for any $r > 1$ and $x \in \mathbb{R}^n$, $T^* f(x) \lesssim M(\|Tf(\cdot)\|^r_{\mathscr{V}_\rho})(x)^{1/r} + Mf(x)$ (see [14]), which yields that T^* is bounded from $L^p(\omega)$ to $L^p(\omega)$ provided that $\omega \in A_p$. Hence,

$$\|II_1\|_{L^p(\omega)} \lesssim |z|.$$

For II_2, by (19), we have

$$II_2 \leqslant |z| \int_{\delta/2<|x-y|\leqslant\delta} |x-y|^{-n}|f(y)|dy \lesssim |z|Mf(x).$$

Combining with the estimate of II_1 and II_2, we conclude that

$$\|II\|_{L^p(\omega)} \leqslant \|II_1\|_{L^p(\omega)} + \|II_2\|_{L^p(\omega)} \lesssim |z|. \tag{25}$$

Hence, by (24) and (25), we obtain

$$\|\mathscr{V}_\rho^\delta((\Phi\star f)_b)(\cdot+z) - \mathscr{V}_\rho^\delta((\Phi\star f)_b)(\cdot)\|_{L^p(\omega)} \to 0,$$

as $|z| \to 0$, uniformly for all f with $\|f\|_{L^p(\omega)} \leqslant 1$. This completes the proof of the sufficiency of Theorem 6. □

While to prove the necessity of Theorem 6, we use Lemmas 5–8, then the conclusion follows by the standard steps in [26, Theorem 1.4], we omit the details. This completes th proof of Theorem 6. □

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