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INHOMOGENEOUS LIPSCHITZ SPACES ASSOCIATED WITH FLAG SINGULAR INTEGRALS AND THEIR APPLICATIONS

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Abstract. This note is motivated by Müller, Ricci and Stein's work in [29]. We introduce a new class of inhomogeneous Lipschitz spaces associated with flag singular integrals and characterize these spaces via the Littlewood-Paley theory. We prove that inhomogeneous flag singular integral operators are bounded on these Lipschitz spaces.

1. Introduction

The classical singular integral operators are extension of the Hilbert transform, which have singularity at the origin only. The nature of this singularity leads to the invariance of these singular integral operators under the classical dilations on $\mathbb{R}^n$ given by $\delta x = (\delta x_1,\dots,\delta x_n)$ for $\delta > 0$. On the other hand, the Calderón-Zygmund product theory of singular integral operators on $\mathbb{R}^n$ is concerned with those singular integral operators which are invariant under the n-fold dilations: $\delta x = (\delta_1 x_1, \delta_2 x_2,\dots,\delta_n x_n)$, $\delta_j > 0$ for $1 \leqslant j \leqslant n$. The product theory of $\mathbb{R}^n$ began with the strong maximal function studied by Zygmund, then continued with the Marcinkiewicz multiplier theorem, and more recently has been studied in a variety of directions, for instance, product singular integrals and Hardy and BMO spaces studied by Chang, R. Fefferman, Gundy, Journé, Pipher and Stein et al. (see [1, 2, 3, 7, 8, 9, 11, 15, 16, 27, 34] among others).

To be more precise, R. Fefferman and Stein [11] proved the $L^p(\mathbb{R}^{n+m})$ boundedness of the product convolution operators for $1 < p < \infty$. Chang and R. Fefferman [1, 2, 3] developed a nice theory of multi-parameter Hardy spaces initially introduced by Gundy and Stein [15], including the atomic decompositions and their dual spaces, namely the product Carleson measure spaces. Subsequently, Jourńe [27] introduced product non-convolution operators and showed an $L^2(\mathbb{R}^{n+m})$ boundedness criterion, the $T1$ theorem, for the product non-convolution operators, and many works on the L^p, $1 < p < \infty$, boundedness and H^p boundedness for operators in Journé's class were investigated [9, 19, 34]. By the atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and a geometric covering lemma in [27], R. Fefferman in [9] proved the $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^{n+m})$

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boundedness of Journé product singular integrals. As mentioned by Journé, this method of R. Fefferman in two parameter case breaks down in the setting of three or more parameters. To this end, Pipher [34] proved a Journé type covering lemma in higher dimensions and demonstrated the $H^p(\mathbb{R}^{n_1}\times\cdots\mathbb{R}^{n_k})$ to $L^p(\mathbb{R}^{n_1}\times\cdots\mathbb{R}^{n_k})$ boundedness for singular integral operators in Journé class by considering directly their actions on the atoms supported in arbitrary open sets. In addition, Han et al. [19] obtained the necessary and sufficient conditions for the $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ boundedness of Journé's product singular integrals. We would like to mention that Ricci and Stein also considered the product theory associated with the Zygmund dilations in [35] and see also [10, 20].

A new extension of the multi-parameter analysis came to light with the proof by Müller, Ricci and Stein [29, 30] for the L^p boundedness, $1<p<\infty$, of Marcinkiewicz multipliers on the Heisenberg group $\mathbb{H}^n$. This is surprising since these multipliers are invariant under a two parameter group of dilations on $\mathbb{C}^n\times\mathbb{R}$, which do not reflect any two-parameter group of automorphic dilations on $\mathbb{H}^n$. Moreover, they proved that the Marcinkiewicz multipliers can be characterized by the convolution operator of the form $f*K$, where K is a flag convolution kernel. See Nagel, Ricci and Stein [31] for flag singular integrals on the Euclidean space and applications on certain quadratic CR submanifolds of $\mathbb{C}^n$. Nagel, Ricci, Stein and Wainger [32, 33] generalized the theory of singular integrals with flag kernels to a more general setting, namely, homogeneous group. They proved that on a homogeneous group singular integral operators with flag kernels are bounded on L^p, $1<p<\infty$, and form an algebra.

At the extreme values of p, $p=1,\infty$, it is natural to hope that certain Hardy space and BMO bounds are available. However, the flag singular integrals are not invariant under the n-fold dilations mentioned above, but satisfy instead an implicit multiparameter structure. In [22] Han, Lu and Sawyer developed a theory of the flag Hardy spaces H^p_{flag} $(0<p\leqslant 1)$ on the Heisenberg group $\mathbb{H}^n$ via the discrete Littlewood-Paley square function, and proved that singular integrals with flag kernels, which include the aforementioned Marcinkiewicz multipliers, are bounded on $H^p_{flag}(\mathbb{H}^n)$, as well as from $H^p_{flag}(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$. More recently, Han, Lee and Li et al. [18] developed various characterizations of the Hardy spaces in the multi-parameter flag setting. At the endpoint case of $p=\infty$, Han, Han, Li and Tan [17] constructed flag Lipschitz spaces on Heisenberg groups and prove that Marcinkiewicz multipliers are bounded on them. For other results associated with flag kernels, we refer the reader to [4, 6, 21, 24, 26, 38], among others.

On the other hand, it is well-known that for the one-parameter setting, the classical Hardy spaces $H^p(\mathbb{R}^n)$ $(0<p\leqslant 1)$ are well suited for the applications to PDEs with constant coefficients. However, the Hardy spaces $H^p(\mathbb{R}^n)$ $(0<p\leqslant 1)$ are not stable under multiplication by test functions and thus it is not well played when it comes to PDEs with variable coefficients. To overcome those drawbacks, Goldberg [14] introduced the class of inhomogeneous Hardy spaces $h^p(\mathbb{R}^n)$ for $0<p\leqslant 1$, namely, local Hardy spaces. Moreover, he showed that a class of inhomogeneous Calderón-Zygmund operators with a mild additional size condition are bounded on them.

Motivated by those works, in this article, we first introduced a class of inhomogeneous flag singular integral kernel $K(x,y)$ on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Similar to [29], these flag kernels can be obtained trough a projection of a inhomogeneous product kernel $K^{\sharp}(x,y,z)$ on $\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}$. To do this, we begin with recalling the definitions of a class of inhomogeneous distributions on Euclidean space $\mathbb{R}^N$. Following closely from [31], a k-normalized bump function on a space $\mathbb{R}^N$ is a C^k function supported on the unit ball with C^k norm bounded by 1. However, the definitions given below are independent of the choice of $k\geqslant 1$, and thus we usually speak of normalized bump functions rather than k-normalized bump functions.

For the sake of simplicity of presentations, we will restrict our considerations to the case $\mathbb{R}^N:=\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}$.

DEFINITION 1. A inhomogeneous product kernel on $\mathbb{R}^N$ is a distribution $\mathscr{K}$ on $\mathbb{R}^N$ which coincides with a C^∞ function away from the coordinate subspace $x_j=0$ for $j=1,2,3$ and which satisfies:

(1) (Differential Inequalities) For each multi-indices $\alpha=(\alpha_1,\cdots,\alpha_{n_1})$, $\beta=(\beta_1,\cdots,\beta_{n_2})$, $\gamma=(\gamma_1,\cdots,\gamma_{n_2})$, there exists a $\delta>0$ such that

$$
\begin{aligned}
&|\mathscr{K}(x_1,x_2,x_3)|\leqslant C\min\left\{(|x_1|+|x_2|)^{-n_1-n_2},(|x_1|+|x_2|)^{-n_1-n_2-\delta}\right\}\\
&\qquad\qquad\qquad\qquad\qquad \min\left\{|x_3|^{-n_2},|x_3|^{-n_2-\delta}\right\};\\
&|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\mathscr{K}(x_1,x_2,x_3)|\leqslant C(|x_1|+|x_2|)^{-n_1-n_2-|\alpha|-|\beta|}\min\left\{|x_3|^{-n_2},|x_3|^{-n_2-\delta}\right\};\\
&|\partial_{x_3}^{\gamma}\mathscr{K}(x_1,x_2,x_3)|\leqslant C\min\left\{(|x_1|+|x_2|)^{-n_1-n_2},(|x_1|+|x_2|)^{-n_1-n_2-\delta}\right\}|x_3|^{-n_2-|\gamma|};\\
&|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\partial_{x_3}^{\gamma}\mathscr{K}(x_1,x_2,x_3)|\leqslant C(|x_1|+|x_2|)^{-n_1-n_2-|\alpha|-|\beta|}|x_3|^{-n_2-|\gamma|}.
\end{aligned}\tag{1}
$$

(2) (Cancellation Condition)

(i) For each multi-indices α, β and any given normalized bump function φ on $\mathbb{R}^{n_2}$ and any $r>0$, there exists a $\delta>0$ so that

$$
\Big|\int_{\mathbb{R}_2^n}\mathscr{K}(x_1,x_2,x_3)\varphi(rx_3)dx_3\Big|\leqslant C\min\left\{(|x_1|+|x_2|)^{-n_1-n_2},(|x_1|+|x_2|)^{-n_1-n_2-\delta}\right\}\tag{2}
$$

and

$$
\Big|\int_{\mathbb{R}_2^n}\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\mathscr{K}(x_1,x_2,x_3)\varphi(rx_3)dx_3\Big|\leqslant C(|x_1|+|x_2|)^{-n_1-n_2-|\alpha|-|\beta|}.\tag{3}
$$

(ii) For each multi-index γ and any given normalized bump function φ on $\mathbb{R}^{n_1+n_2}$ and any $r>0$, there exists a $\delta>0$ so that

$$
\Big|\int_{\mathbb{R}^{n_1+n_2}}\mathscr{K}(x_1,x_2,x_3)\varphi(rx_1,rx_2)dx_1dx_2\Big|\leqslant C\min\left\{|x_3|^{-n_2},|x_3|^{-n_2-\delta}\right\}\tag{4}
$$

and

$$
\Big|\int_{\mathbb{R}^{n_1+n_2}}\partial_{x_3}^{\gamma}\mathscr{K}(x_1,x_2,x_3)\varphi(rx_1,rx_2)dx_1dx_2\Big|\leqslant C|x_3|^{-m-|\gamma|}.\tag{5}
$$

(iii) For any given normalized bump function φ on $\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}$ and any $r_1,r_2>0$, we have

$$
\Big|\int_{\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}}\mathscr{K}(x_1,x_2,x_3)\varphi(r_1x_1,r_1x_2,r_2x_3)dx_1dx_2dx_3\Big|\leqslant C.\tag{6}
$$

Moreover, the corresponding constants that appear in these differential inequalities are independent of the scaling parameters and depend only on α,β,γ and δ.

We will rephrase Definition 2.3.2 in [31] of a flag kernel in the inhomogeneous case as follows.

DEFINITION 2. A inhomogeneous flag kernel on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ is a distribution $\mathscr{K}$ on $\mathbb{R}^{n_1+n_2}$ which coincides with a C^∞ function away from the coordinate subspace $x_1=0$ and which satisfies:

(1) (Differential Inequalities) For each multi-indices $\alpha=(\alpha_1,\cdots,\alpha_{n_1})$, $\beta=(\beta_1,\cdots,\beta_{n_2})$, there exists a regularity exponent $\delta>0$ such that

$$
\begin{aligned}
&|\mathscr{K}(x_1,x_2)|\leqslant C\min\left\{|x_1|^{-n_1},|x_1|^{-n_1-\delta}\right\}\min\left\{(|x_1|+|x_2|)^{-n_2},(|x_1|+|x_2|)^{-n_2-\delta}\right\};\\
&|\partial_{x_1}^{\alpha}\mathscr{K}(x_1,x_2)|\leqslant C|x_1|^{-n_1-|\alpha|}\min\left\{(|x_1|+|x_2|)^{-n_2},(|x_1|+|x_2|)^{-n_2-\delta}\right\};\\
&|\partial_{x_2}^{\beta}\mathscr{K}(x_1,x_2)|\leqslant C\min\left\{|x_1|^{-n_1},|x_1|^{-n_1-\delta}\right\}(|x_1|+|x_2|)^{-n_2-|\beta|};\\
&|\partial_{x_1}^{\alpha}\partial_{x_2}^{\beta}\mathscr{K}(x_1,x_2,x_3)|\leqslant C|x_1|^{-n_1-|\alpha|}(|x_1|+|x_2|)^{-n_2-|\beta|}.
\end{aligned}\tag{7}
$$

(2) (Cancellation Condition)

(i) For every multi-index α and any given normalized bump function φ on $\mathbb{R}^m$ and any $r>0$, there exists a $\delta>0$ such that

$$
\Big|\int_{\mathbb{R}^{n_2}}\mathscr{K}(x_1,x_2)\varphi(rx_2)dx_2\Big|\leqslant C\min\left\{|x_1|^{-n_1},|x_1|^{-n_1-\delta}\right\}\tag{8}
$$

and

$$
\Big|\int_{\mathbb{R}^{n_2}}\partial_{x_1}^{\alpha}\mathscr{K}(x_1,x_2)\varphi(rx_2)dx_2\Big|\leqslant C|x_1|^{-n_1-|\alpha|}.\tag{9}
$$

(ii) For every multi-index β and any given normalized bump function φ on $\mathbb{R}^n$ and any $r>0$, there exists a $\delta>0$ such that

$$
\Big|\int_{\mathbb{R}^{n_1}}\mathscr{K}(x_1,x_2)\varphi(rx_1)dx_1\Big|\leqslant C\min\left\{|x_2|^{-n_2},|x_2|^{-n_2-\delta}\right\}\tag{10}
$$

and

$$
\Big|\int_{\mathbb{R}^{n_1}}\partial_{x_2}^{\beta}\mathscr{K}(x_1,x_2)\varphi(rx_1)dx_1\Big|\leqslant C|x_2|^{-n_2-|\beta|}.\tag{11}
$$

(iii) For any given normalized bump function φ on $\mathbb{R}^{n_1+n_2}$ and any $r_1,r_2>0$, we have

$$
\Big|\int_{\mathbb{R}^{n_1+n_2}}\mathscr{K}(x_1,x_2)\varphi(r_1x_1,r_2x_2)dx_1dx_2\Big|\leqslant C.\tag{12}
$$

As mentioned by Nagel, Ricci, and Stein in [31], the bump functions in Definitions 1 and 2 (2)-(iii) can be replaced by the tensor product of normalized bump functions on $\mathbb{R}^{n_1+n_2}$ and $\mathbb{R}^{n_2}$.

The following theorem is similar to Proposition 3.2 and Lemma 4.5 in [29], which reveals the relation between the inhomogeneous product kernel and the inhomogeneous flag kernel.

THEOREM 1. *Let $\mathscr{K}^\sharp$ be an integrable function on $\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}$ and which is a inhomogeneous product kernel as in Definition 1. Then the function*

$$\mathscr{K}(x_1,x_2)=\int_{\mathbb{R}^{n_2}}\mathscr{K}^\sharp(x_1,x_2-x_3,x_3)dx_3 \tag{13}$$

satisfies (7)–(12) with constants that depend only the constants in (1)–(6) and not on the L^1-norm of $\mathscr{K}^\sharp$.

Conversely, given $\mathscr{K}\in L^1(\mathbb{R}^{n_1+n_2})$ satisfies (7)–(12), define

$$\mathscr{K}^\sharp(x_1,x_2,x_3)=\frac{1}{|x_1|^{n_2}}\chi\left(\frac{x_2}{|x_1|}\right)\mathscr{K}(x_1,x_2+x_3),$$

where χ is a non-negative smooth function supported on $[1/2,1]^{n_2}$ such that $\int\chi=1$. Then $\mathscr{K}^\sharp$ is an integrable function on $\mathbb{R}^{n+m}\times\mathbb{R}^{n_2}$ such that (13) holds and satisfies (1)–(6).

Note that convolution with a inhomogeneous flag singular kernel is a special case of product singular kernel. As a consequence, the L^p, $1<p<\infty$, boundedness of inhomogeneous flag singular integrals follows directly from the product theory on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. A basic question arises: Can one establish the endpoint estimates of inhomogeneous flag singular integral operators on Lipschitz spaces? The goal of this note is address this question. To be more precise, we will establish a theory of the inhomogeneous flag Lipschitz spaces on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, that is, in a sense, intermediate between those of the classical Lipschitz spaces on $\mathbb{R}^{n_1+n_2}$ and the product Lipschitz spaces on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. For more about the classical Lipschitz spaces and multi-parameter Lipschitz spaces, see [5, 17, 23, 25, 37, 39].

We will characterize the inhomogeneous flag Lipschitz spaces via the Littlewood-Paley theory and prove that the inhomogeneous flag singular integral operators are bounded on these Lipschitz spaces.

Now we introduce the following notation:

$$\begin{aligned}
&\Delta^1_{(u,v)}f(x_1,x_2)=f(x_1-u,x_2-v)-f(x_1,x_2),\\
&\Delta^{1,Z}_{(u,v)}f(x_1,x_2)=f(x_1+u,x_2+v)-2f(x_1,x_2)+f(x_1-u,x_2-v),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta^2_w f(x_1,x_2)=f(x_1,x_2-w)-f(x_1,x_2),\\
&\Delta^{2,Z}_w f(x_1,x_2)=f(x_1,x_2+w)-2f(x_1,x_2)+f(x_1,x_2-w).
\end{aligned}$$

The inhomogeneous flag Lipschitz space is defined as follows.

DEFINITION 3. Let $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_1,\alpha_2>0$. The inhomogeneous flag Lipschitz space $\mathrm{Lip}^\alpha_{\mathrm{flag}}$ is defined to be the set of all bounded continuous functions f defined on $\mathbb{R}^{n_1+n_2}$ such that

(i) when $0<\alpha_1,\alpha_2<1$,

$$\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}:=\|f\|_\infty+\sup_{(u,v)\neq 0}\frac{|\Delta^1_{(u,v)}f|}{|(u,v)|^{\alpha_1}}+\sup_{w\neq 0}\frac{|\Delta^2_w f|}{|w|^{\alpha_2}}+\sup_{(u,v),w\neq 0}\frac{|\Delta^2_w\Delta^1_{(u,v)}f|}{|(u,v)|^{\alpha_1}|w|^{\alpha_2}}<\infty,$$

where $|(u,v)|^2=(|u|^2+|v|^2)^{\frac{1}{2}}$.

(ii) when $\alpha_1=1, 0<\alpha_2<1$,

$$\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}:=\|f\|_\infty+\sup_{(u,v)\neq 0}\frac{|\Delta^{1,Z}_{(u,v)}f|}{|(u,v)|}+\sup_{w\neq 0}\frac{|\Delta^2_w f|}{|w|^{\alpha_2}}+\sup_{(u,v),w\neq 0}\frac{|\Delta^2_w\Delta^{1,Z}_{(u,v)}f|}{|(u,v)||w|^{\alpha_2}}<\infty;$$

(iii) when $0<\alpha_1<1, \alpha_2=1$,

$$\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}:=\|f\|_\infty+\sup_{(u,v)\neq 0}\frac{|\Delta^1_{(u,v)}f|}{|(u,v)|^{\alpha_1}}+\sup_{w\neq 0}\frac{|\Delta^{2,Z}_w f|}{|w|}+\sup_{(u,v),w\neq 0}\frac{|\Delta^{2,Z}_w\Delta^1_{(u,v)}f|}{|(u,v)|^{\alpha_1}|w|};$$

(iv) when $\alpha_1=\alpha_2=1$,

$$\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}:=\|f\|_\infty+\sup_{(u,v)\neq 0}\frac{|\Delta^{1,Z}_{(u,v)}f|}{|(u,v)|}+\sup_{w\neq 0}\frac{|\Delta^{2,Z}_w f|}{|w|}+\sup_{(u,v),w\neq 0}\frac{|\Delta^{2,Z}_w\Delta^{1,Z}_{(u,v)}f|}{|(u,v)||w|}.$$

When $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_1,\alpha_2>1$, we write $\alpha_1=m_1+r_1$ and $\alpha_2=m_2+r_2$ where m_1,m_2 are integers and $0<r_1,r_2\leqslant 1$. Then $f\in\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ means that all partial derivatives $\partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}f(x_1,x_2)$ with $|\beta_1|=m_1$, $|\beta_2|=m_2$, such that $\partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}f$ belongs to $\mathrm{Lip}^{r}_{\mathrm{flag}}$ with $r=(r_1,r_2)$ and

$$\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}(\alpha_1,\alpha_2)}:=\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\|\partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}f\|_{\mathrm{Lip}^{r}_{\mathrm{flag}}}.$$

In order to establish the boundedness of inhomogeneous flag singular integral operators on the inhomogeneous flag Lipschitz space $\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$, we will characterize $\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ via the Littlewood-Paley theory. For this purpose, we adapt some notations. Given a function φ on $\mathbb{R}^n$, denote

$$M_\varphi=\max\{N\in\mathbb{N}:\int_{\mathbb{R}^n}\varphi(x)x^\alpha dx=0,|\alpha|\leqslant N\},$$

where $\mathbb{N}$ denotes the class of all natural integrals, that is, $\mathbb{N}=\{0,1,2,\cdots\}$. In what follows, we use $\mathscr{D}(\mathbb{R}^n)$ to denote the set of all smooth functions with compact support on $\mathbb{R}^n$. We begin by recalling the standard local Calderón reproducing formula on $\mathbb{R}^{n_1+n_2}$.

THEOREM A. ([36]) *Let a radial function* $\varphi^{(1)}_0\in\mathscr{D}(\mathbb{R}^{n_1+n_2})$ *satisfy* $\int\varphi^{(1)}_0=1$, *and let* $\varphi^{(1)}(x)=\varphi^{(1)}_0(x)-2^{-(n_1+n_2)}\varphi^{(1)}_0(\frac{x}{2})$. *Then for any given integer* $M\geqslant 0$ *there exist two real even functions* $\psi^{(1)}_0$, $\psi^{(1)}\in\mathscr{D}(\mathbb{R}^{n_1+n_2})$ *with* $M_{\psi^{(1)}}\geqslant M$, *such that*

$$f(x)=\sum_{j=0}^{\infty}\psi^{(1)}_j*\varphi^{(1)}_j*f(x), \tag{14}$$

where $\psi_j^{(1)}(x) = 2^{j(n_1+n_2)}\psi^{(1)}(2^j x)$, $\varphi_j^{(1)}(x) = 2^{j(n_1+n_2)}\varphi^{(1)}(2^j x)$ *for* $j \geqslant 1$, *and the series converges in* $L^2(\mathbb{R}^{n_1+n_2})$, $\mathscr{S}(\mathbb{R}^{n_1+n_2})$ *and* $\mathscr{S}'(\mathbb{R}^{n_1+n_2})$.

It was pointed by Rychkov [36] that for any positive integer m, the function $\varphi^{(1)}$ in Theorem A can be chosen so that $M_{\varphi^{(1)}} \geqslant m$. We now hope to extend this formula to encompass the flag structure. Let a radial function $\varphi_0^{(2)} \in \mathscr{D}(\mathbb{R}^{n_2})$ satisfy $\int \varphi_0^{(2)} = 1$ and let $\varphi^{(2)}(x_2) = \varphi_0^{(1)}(x_2) - 2^{-n_2}\varphi_0^{(2)}(\frac{x_2}{2})$. Then for any $M \geqslant 0$ there exist two even functions $\psi_0^{(2)}$, $\psi^{(2)} \in \mathscr{D}(\mathbb{R}^{n_2})$ such that $M_{\psi^{(2)}} \geqslant M$, and

$$\varphi_0^{(2)}(\xi_2)\psi_0^{(2)}(\xi_2) + \sum_{k=1}^{\infty} \varphi^{(2)}(2^{-k}\xi_2)\psi^{(2)}(2^{-k}\xi_2) = 1.$$

Thus, we have the following local Calderón reproducing formula on $L^2(\mathbb{R}^{n_1+n_2})$: for $f \in L^2(\mathbb{R}^{n_1+n_2})$,

$$f(x) = \sum_{j,k=0}^{\infty} \psi_{j,k} * \varphi_{j,k} * f(x),\ f \in L^2(\mathbb{R}^{n_1+n_2}),$$

where the functions $\psi_{j,k}$ are given by the partial convolution $*_2$ in the second variable only,

$$\psi_{j,k}(x_1,x_2) = \int_{\mathbb{R}^{n_2}} \psi_j^{(1)}(x_1,x_2-v)\psi_k^{(2)}(v)dv,$$

where ψ_j is ψ_0 if $j=0$, otherwise the dilations of ψ. $\varphi_{j,k}$ is constructed similarly. Observe that the series converges in $L^2(\mathbb{R}^{n_1+n_2})$. Indeed,

$$\begin{aligned}\psi_{j,k} * \varphi_{j,k} * f(x) &= (\psi_j^{(1)} *_2 \psi_k^{(2)}) * (\varphi_j^{(1)} *_2 \varphi_k^{(2)}) * f(x)\\ &= (\psi_j^{(1)} * \varphi_j^{(1)}) * \big((\psi_k^{(2)} * \varphi_k^{(2)}) *_2 f(x)\big)\end{aligned}$$

implies (15) upon invoking the standard local Calderón reproducing formula on $\mathbb{R}^{n_2}$ and then Theorem A on $\mathbb{R}^{n_1+n_2}$.

Noting that $\varphi_{j,k} = \varphi_j^{(1)} *_2 \varphi_k^{(2)} \in \mathscr{S}(\mathbb{R}^{n_1+n_2})$, we will prove that the local Calderón reproducing formula (15) also converges in both test function spaces $\mathscr{S}(\mathbb{R}^{n_1+n_2})$ and distribution space $\mathscr{S}'(\mathbb{R}^{n_1+n_2})$ as follows.

THEOREM 2. *Assume that the functions* $\psi_{j,k}$ *and* $\varphi_{j,k}$ *are defined above. Then*

$$f(x) = \sum_{j,k=0}^{\infty} \psi_{j,k} * \varphi_{j,k} * f(x), \tag{15}$$

where the series converges in $\mathscr{S}(\mathbb{R}^{n_1+n_2})$ *and* $\mathscr{S}'(\mathbb{R}^{n_1+n_2})$.

We characterize the inhomogeneous flag Lipschitz space by the following theorem.

THEOREM 3. *$f \in \mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$ if and only if $f \in \mathscr{S}'(\mathbb{R}^{n_1+n_2})$ and*

$$\|\varphi_{j,k} * f\|_{L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \leqslant C2^{-j\alpha_1}2^{-k\alpha_2}.$$

Moreover,

$$\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \approx \sup_{j,k\geqslant 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k} * f\|_{L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

It was well known that in the classical one-parameter case, the space BMO, as the dual of H^1, can be characterized by the Carleson measure. Moreover, Chang and Fefferman in [1] proved that the dual of the product H^1 is characterized by the product Carleson measure. The generalized Carleson measure space CMO^p_{flag} associated with the flag singular integrals was first introduced by Han, Lu and Sawyer [22]. They showed that the dual of the flag Hardy space H^p_{flag} is CMO^p_{flag}. In particular, $CMO^1_{flag} = BMO^1_{flag}$. Following the idea employed in [22], we introduce the local flag Carleson measure space cmo^p_{flag} which is defined as the set of $f \in \mathscr{S}'(\mathbb{R}^{n_1+n_2})$ such that

$$\sup_{\Omega}\left\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{j,k\geqslant 0}\int_\Omega \sum_{I,J:I\times J\subseteq\Omega}|\varphi_{j,k} * f(x_1,x_2)|^2\chi_I(x_1)\chi_J(x_2)dx_1dx_2\right\}^{\frac{1}{2}} < \infty$$

for all open sets Ω in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with finite measures, and $I \subset \mathbb{R}^{n_1}$, $J \subset \mathbb{R}^{n_2}$, are dyadic cubes with side-length $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-(j\wedge k)}$ respectively, and where $\varphi_{j,k}$ are the same as Theorem 3. As mentioned in [22], we denote bmo_{flag} by the space cmo^1_{flag}. Observe that if $f \in \mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ with $\alpha = (\alpha_1, \alpha_2)$, by Theorem 3, we have $|\varphi_{j,k} * f| \leqslant C2^{-j\alpha_1}2^{-k\alpha_2}$. Then

$$\begin{aligned}
&\frac{1}{|\Omega|}\sum_{j,k\geqslant 0}\int_\Omega \sum_{I,J:I\times J\subseteq\Omega}|\varphi_{j,k} * f(x_1,x_2)|^2\chi_I(x_1)\chi_J(x_2)dx_1dx_2\\
&\leqslant C\frac{1}{|\Omega|}\sum_{j,k\geqslant 0}2^{-2j\alpha_1}2^{-2k\alpha_2}\int_\Omega \sum_{I,J:I\times J\subseteq\Omega}\chi_I(x_1)\chi_J(x_2)dx_1dx_2 < \infty,
\end{aligned}$$

which means that $f \in bmo_{flag}$.

Our last main result of this paper is the following theorem.

THEOREM 4. *The inhomogeneous flag singular integral operator T is bounded on $\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$. Furthermore,*

$$\|Tf\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.$$

This paper is organized as follows. In the next section, we will give the proof of Theorem 1. Theorem 2 will be proved in Section 3. Section 4 is devoted to the Littlewood-Paley characterization of inhomogeneous flag Lipschitz space. In the last section, as an application, we prove the boundedness of singular integral operators on these spaces.

Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables, but whose value may vary from line to line. We use the notation $A \approx B$ to denote that there exists a positive constant C such that $C^{-1}B \leqslant A \leqslant CB$. Let $j \wedge j'$ be the minimum of j and j'.

2. Proof of Theorem 1

The main purpose of this section is to show the relation the inhomogeneous product kernel and the inhomogeneous flag kernel. One one hand, we first show that if $K^\sharp$ is a inhomogeneous product kernel on $\mathbb{R}^{n_1\times n_2}\times\mathbb{R}^{n_3}$, then the function

$$K(x_1,x_2)=\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_2-x_3,x_3)dx_3$$

satisfies (7)–(12). To simplify the notation, we take $\alpha=\beta=0$. The extension to the general case does not present any difficulty.

We prove (7) first. We only verify it for $|x_1|>1$. If $|x_1|<1$, the proof below needs a slight modification, but is even more simple. Let φ be a normalized bump function on $\mathbb{R}^{n_3}$, supported on the unit ball and identically equal to 1 for $|x_3|\leqslant 1/2$. Then

$$\begin{aligned}
K(x_1,x_2) &= \int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_2-x_3,x_3)dx_3\\
&= \int_{\mathbb{R}^{n_2}} [K^\sharp(x_1,x_2-x_3,x_3)-K^\sharp(x_1,x_2,x_3)]\varphi(x_3)dx_3\\
&\quad+\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_2,x_3)\varphi(x_3)dx_3+\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_2-x_3,x_3)(1-\varphi(x_3))dx_3\\
&= I_1+I_2+I_3.
\end{aligned}$$

For I_1 we use the mean value theorem to obtain

$$\begin{aligned}
|I_1| &= |\int_{\mathbb{R}^{n_2}} [K^\sharp(x_1,x_2-x_3,x_3)-K^\sharp(x_1,x_2,x_3)]\varphi(x_3)dx_3|\\
&\leqslant C\int_{|x_3|\leqslant 1}(|x_1|+|x_2-\theta x_3|)^{-n_1-n_2-1}dx_3\\
&\leqslant C(|x_1|+|x_2|)^{-n_1-n_2-1}.
\end{aligned}$$

By (2), $|I_2|\leqslant C(|x_1|+|x_2|)^{-n_1-n_2-\delta}$. Finally,

$$\begin{aligned}
|I_3| &\leqslant |\int_{|x_3|\geqslant 1/2}|K^\sharp(x_1,x_2-x_3,x_3)|dx_3\\
&\leqslant C\int_{|x_3|\geqslant 1/2}(|x_1|+|x_2-x_3|)^{-n_1-n_2-\delta}|x_3|^{-n_2-\delta}dx_3\\
&\leqslant C|x_1|^{-n_1-\delta_1}(|x_1|+|x_2|)^{-n_2-\delta_2}
\end{aligned}$$

with $\delta_1+\delta_2=\delta$.

Next we prove (10). Then

$$\begin{aligned}\int_{\mathbb{R}^{n_2}} K(x_1,x_2)\varphi(rx_2)dx_2 &= \int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_2-x_3,x_3)\varphi(rx_2)dx_3dx_2\\ &= \int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_2,x_3)\varphi(r(x_2+x_3))dx_3dx_2.\end{aligned}$$

Integrating in du first, we observe that, for each fixed x_2, $\varphi(r(x_2+x_3))$ is the translate of a normalized bump function scaled by r. Hence

$$\begin{aligned}\left|\int_{\mathbb{R}^{n_2}} K(x_1,x_2)\varphi(rx_2)dx_2\right| &\leqslant C\int_{\mathbb{R}^{n_2}} \min\{(|x_1|+|x_2|)^{-n_1-n_2},(|x_1|+|x_2|)^{-n_1-n_2-\delta}\}dx_2\\ &\leqslant C\min\{|x_1|^{-n_1},|x_1|^{-n_1-\delta}\}.\end{aligned}$$

We prove now (11). It suffices for us to check the case where $|x_2|>1$. Since $|x_2|\leqslant 1$, it has been proved

$$\left|\int_{\mathbb{R}^{n_1}} K(x_1,x_2)\varphi(rx_1)dx_1\right| \leqslant C|x_2|^{-n_2}.$$

We take a normalized bump function η on $\mathbb{R}^{n_2}$, supported on $\{|x|:|x|\leqslant 1/2\}$, identically equal to 1 on $\{|x|:|x|\leqslant 1/4\}$, and write

$$\begin{aligned}&\int_{\mathbb{R}^{n_1}} K(x_1,x_2)\varphi(rx_1)dx_1\\ &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} [K^\sharp(x_1,x_3,x_2-x_3)-K^\sharp(x_1,x_3,x_2)]\varphi(rx_1)\eta\left(\frac{x_3}{|x_2|}\right)dx_1dx_3\\ &\quad+\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_3,x_2)\varphi(rx_1)\eta\left(\frac{x_3}{|x_2|}\right)dx_1dx_3\\ &\quad+\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_3,x_2-x_3)\varphi(rx_1)\left(1-\eta\left(\frac{x_3}{|x_2|}\right)\right)dx_1dx_3\\ &= I_1+I_2+I_3.\end{aligned}$$

Since $|x_3|\leqslant 1/2|x_2|$ in I_1, we have

$$\begin{aligned}|I_1| &\leqslant C\int_{|x_3|\leqslant\frac{1}{2}|x_2|}\int_{\mathbb{R}^{n_1}} \min\{(|x_1|+|x_3|)^{-n_1-n_2},(|x_1|+|x_3|)^{-n_1-n_2-\delta}\}|x_2|^{-n_2-1}|x_3|dx_1dx_3\\ &\leqslant C\min\{|x_2|^{-n_2},|x_2|^{-n_2-\delta'}\}\ \text{ for some }\ \delta'>0.\end{aligned}$$

For I_2 we distinguish between the two cases $r\leqslant 1$ and $r>1$. If $r>1$, we take a normalized bump function $\tilde\eta$ on $\mathbb{R}^{n_2}$, supported on $\{|x|:|x|\leqslant 1\}$, identically equal to 1 on $\{|x|:|x|\leqslant 1/2\}$. Then

$$\begin{aligned}I_2 &= \int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_3,x_2)\varphi(rx_1)\eta\left(\frac{x_3}{|x_2|}\right)\tilde\eta(x_3)dx_1dx_3\\ &\quad+\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_3,x_2)\varphi(rx_1)\eta\left(\frac{x_3}{|x_2|}\right)(1-\tilde\eta(x_3))dx_1dx_3\\ &= I_{21}+I_{22}.\end{aligned}$$

Since $\varphi(rx_1)\eta(\frac{x_3}{|x_2|})\tilde{\eta}(x_3)$ is a normalized bump function, (4) implies that

$$|I_{21}| \leqslant C\min\{|x_2|^{-n_2}, |x_2|^{-n_2-\delta}\}.$$

Also,

$$\begin{aligned}|I_{22}| &\leqslant C\int_{|x_3|\geqslant 1/2}\int_{|x_1|\leqslant 1/r}\frac{1}{(|x_1|+|x_3|)^{n_1+n_2+\delta}}\min\{|x_2|^{-n_2}, |x_2|^{-n_2-\delta}\}dx_1dx_3\\ &\leqslant C\min\{|x_2|^{-n_2}, |x_2|^{-n_2-\delta}\}.\end{aligned}$$

If $r \leqslant 1$, we take another normalized bump function $\tilde{\varphi}$ on $\mathbb{R}^{n_1}$, supported on $\{|x| : |x| \leqslant 1\}$, identically equal to 1 on $\{|x| : |x| \leqslant 1/2\}$, and write

$$\begin{aligned}I_2 = &\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_3,x_2)\varphi(rx_1)\tilde{\varphi}(x_1/|x_2|)\eta\left(\frac{x_3}{|x_2|}\right)dx_1dx_3\\ &+\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} K^\sharp(x_1,x_3,x_2)\varphi(rx_1)(1-\tilde{\varphi}(x_1/|x_2|))\eta\left(\frac{x_3}{|x_2|}\right)dx_1dx_3.\end{aligned}$$

Then I_2 can be estimated as in the previous case.

In order to deal with I_3, we need a normalized bump function λ on $\mathbb{R}^{n_1}$, supported on $\{|x| : |x| \leqslant 1/2\}$, identically equal to 1 on $\{|x| : |x| \leqslant 1/4\}$. Then

$$\begin{aligned}I_3 = &\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}[K^\sharp(x_1,x_3,x_2-x_3) - K^\sharp(x_1,x_2,x_2-x_3)]\varphi(rx_1)\\ &\times\left(1-\eta\left(\frac{x_3}{|x_2|}\right)\right)\lambda\left(\frac{x_2-x_3}{|x_2|}\right)dx_1dx_3\\ &+\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}K^\sharp(x_1,x_2,x_2-x_3)\varphi(rx_1)\left(1-\eta\left(\frac{x_3}{|x_2|}\right)\right)\lambda\left(\frac{x_2-x_3}{|x_2|}\right)dx_1dx_3\\ &+\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}K^\sharp(x_1,x_3,x_2-x_3)\varphi(rx_1)\\ &\times\left(1-\eta\left(\frac{x_3}{|x_2|}\right)\right)\left(1-\lambda\left(\frac{x_2-x_3}{|x_2|}\right)\right)dx_1dx_3\\ =\ &I_{31}+I_{32}+I_{33}.\end{aligned}$$

Then

$$\begin{aligned}|I_{31}| &\leqslant C\int_{|x_2-x_3|\leqslant\frac{|x_2|}{2}}\int_{\mathbb{R}^{n_1}}\frac{1}{(|x_1|+|x_2|)^{n_1+n_2+1}}|x_2-x_3|\\ &\qquad\times\min\left\{\frac{1}{|x_2-x_3|^{n_2}}, \frac{1}{|x_2-x_3|^{n_2+\delta}}\right\}dx_1dx_3\\ &\leqslant C\min\{|x_2|^{-n_2}, |x_2|^{-n_2-\delta'}\}\end{aligned}$$

for some $\delta > 0$. Since $(1-\eta(x_3))\lambda(x_2-x_3)$ is the translate of a normalized bump function for every x_2, we obtain from (2)

$$\begin{aligned}|I_{32}| &\leqslant C\int_{\mathbb{R}^{n_1}}\min\left\{\frac{1}{(|x_1|+|x_2|)^{n_1+n_2}}, \frac{1}{(|x_1|+|x_2|)^{n_1+n_2+\delta}}\right\}dx_1\\ &\leqslant C\min\{|x_2|^{-n_2}, |x_2|^{-n_2-\delta}\}.\end{aligned}$$

Finally,

$$
\begin{aligned}
|I_{33}| &\leqslant C\int_{|x_3|\geqslant\frac{|x_2|}{4},|x_2-x_3|\geqslant\frac{|x_2|}{4}} \min\left\{\frac{1}{|x_3|^{n_2}},\frac{1}{|x_3|^{n_2+\delta}}\right\}\min\left\{\frac{1}{|x_2-x_3|^{n_2}},\frac{1}{|x_2-x_3|^{n_2+\delta}}\right\}\\
&\leqslant C\min\{|x_2|^{-n_2},|x_2|^{-n_2-\delta}\}.
\end{aligned}
$$

The cancellation condition of (12) follows from Proposition 3.2 in [29] on Heisenberg group.

We now prove the converse implication of Theorem 1. Suppose that $\mathscr{K}\in L^1(\mathbb{R}^{n+m})$ satisfies (7)–(12). To begin with,

$$
\begin{aligned}
\int_{\mathbb{R}^{n_2}}\mathscr{K}^\sharp(x_1,x_2-x_3,x_3)dx_3 &= \int_{\mathbb{R}^{n_2}}\frac{1}{|x_1|^{n_2}}\chi\left(\frac{x_2-x_3}{|x_1|}\right)\mathscr{K}(x_1,x_2)dx_3\\
&=\mathscr{K}(x_1,x_2).
\end{aligned}
$$

Noting the support of $K^\sharp$ and using (7), we have

$$
\begin{aligned}
&|\mathscr{K}^\sharp(x_1,x_2,x_3)|\\
&\leqslant C|x_1|^{-n_2}\min\left\{\frac{1}{|x_1|^{n_1}},\frac{1}{|x_1|^{n_1+\delta}}\right\}\min\left\{\frac{1}{(|x_1|+|x_2+x_3|)^{n_2}},\frac{1}{(|x_1|+|x_2+x_3|)^{n_2+\delta}}\right\}\\
&\leqslant C\min\left\{\frac{1}{(|x_1|+|x_2|)^{n_1+n_2}},\frac{1}{(|x_1|+|x_2|)^{n_1+n_2+\delta}}\right\}\min\left\{\frac{1}{|x_3|^{n_2}},\frac{1}{|x_3|^{n_2+\delta}}\right\},
\end{aligned}
$$

since $|x_3|\leqslant|x_2|+|x_2+x_3|\leqslant C(|x_1|+|x_2+x_3|)$. The other estimates in (1) can be obtained similarly. The proof of the cancellation conditions (2)–(6) is very similar to Lemma 4.5 in [29]. We leave the details to the reader. Hence, the proof of Theorem 1 is concluded.

3. Proof of Theorem 2

In order to prove that the local Calderón reproducing formula (15) converges in both test function space and distribution space, we recall the following well-known one-parameter almost orthogonality estimates (see, for example, [12, 13]). From now on, we use the notation $j\wedge k=\min\{j,k\}$. For some positive integer $N\geqslant 1$, set

$$
\mathscr{S}_N(\mathbb{R}^n)=\{\phi\in\mathscr{S}(\mathbb{R}^n):\int_{\mathbb{R}^n}\phi(x)x^\alpha dx=0,|\alpha|\leqslant N-1\}.
$$

LEMMA 1. *Let ψ, $\varphi\in\mathscr{S}(\mathbb{R}^n)$ and $j,k\in\mathbb{Z}$, $j\leqslant k$. If $\varphi\in\mathscr{S}_N(\mathbb{R}^n)$, then for any given positive integer L, there exists a constant C depending only ψ,φ,n,N and L so that*

$$
|\psi_j*\varphi_k(x)|\leqslant C2^{-(k-j)N}\frac{2^{-jL}}{(2^{-j}+|x|)^{n+L}}.
$$

To show Theorem 2, we only need to prove that the series in (15) in $\mathscr{S}(\mathbb{R}^{n_1+n_2})$ if $f\in\mathscr{S}(\mathbb{R}^{n_1+n_2})$. The convergence in $\mathscr{S}'(\mathbb{R}^{n_1+n_2})$ then follows from a standard duality argument. The key for doing this is the almost orthogonal estimates: for any given positive integers L,N and $f\in\mathscr{S}(\mathbb{R}^{n_1+n_2})$, there exists a constant $C>0$ independent of j and k such that

$$|\varphi_{j,k}*f(x)|\leqslant C2^{-jN}2^{-kN}\frac{1}{(1+|x|)^L}. \tag{16}$$

Assume that (16) holds for the moment. Given any positive integers M_1, M_2 and denoting $E=\{j,k\in\mathbb{N}:0\leqslant j\leqslant M_1, 0\leqslant k\leqslant M_2\}$, by (16),

$$\begin{aligned}
&\sum_{j,k\in E^c}\Big|\int\varphi_{j,k}*f(y)(D^\alpha\psi_{j,k})(x-y)dy\Big|\\
&\leqslant C\sum_{j,k\in E^c}2^{-jN'}2^{-kN'}\int\frac{1}{(1+|y_1|+|y_2|)^L}\frac{1}{(1+|x_1-y_2|+|x_2-y_2|)^L}dy_1dy_2\\
&\leqslant C\sum_{j,k\in E^c}2^{-jN'}2^{-kN'}\frac{1}{(1+|x|)^L}\quad\text{for some } N'>0,
\end{aligned}$$

since N can be chosen arbitrarily large, which further implies that the local Calderón reproducing formula (15) holds in $\mathscr{S}(\mathbb{R}^{n_1+n_2})$.

It remains to verify (16). Note that $\varphi_{j,k}*f=(\varphi_j^{(1)}*f)*_2\varphi_k^{(2)}$. Thus by the almost orthogonality estimate on $\mathbb{R}^{n_1+n_2}$,

$$|\varphi_j^{(1)}*f(x)|\leqslant C2^{-jN}\frac{1}{(1+|x|)^{n_1+n_2+L}},$$

which implies

$$|\varphi_{j,k}*f(x)|\leqslant C2^{-jN}\frac{1}{(1+|x|)^{n_1+n_2+L}}. \tag{17}$$

On the other hand, $\varphi_{j,k}*f=\varphi_j^{(1)}*(f*_2\varphi_k^{(2)})$. Arguing as above, we have

$$|\varphi_{j,k}*f(x)|\leqslant C2^{-kN}\frac{1}{(1+|x|)^{n_1+n_2+L}}. \tag{18}$$

By choosing a sufficiently large N in (17)–(18) and taking the geometric mean, (16) follows.

4. Proof of Theorem 3

We first show that if $f\in\mathrm{Lip}^\alpha_{\mathrm{flag}}$ with $\alpha=(\alpha_1,\alpha_2)$, $0<\alpha_1,\alpha_2<1$, then $f\in\mathscr{S}'(\mathbb{R}^{n_1+n_2})$. To do this, for each $g\in\mathscr{S}(\mathbb{R}^{n_1+n_2})$, by the local Calderón reproducing formula (15), we have

$$g(x)=\sum_{j,k\geqslant0}\psi_{j,k}*\varphi_{j,k}*g(x),$$

where the series converges in $\mathscr{S}(\mathbb{R}^{n_1+n_2})$. Therefore, for $f \in \mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ with $0<\alpha_1,\alpha_2<1$, it suffices to show that $\sum_{j,k\geqslant 0}\langle \varphi_{j,k}*f, \psi_{j,k}*g\rangle$ is well defined for $g\in\mathscr{S}(\mathbb{R}^{n_1+n_2})$. To this end, we estimate $\langle \varphi_{j,k}*f, \psi_{j,k}*g\rangle$ as follows.

Case 1: $j=k=0$.

$$\begin{aligned}|\varphi_{0,0}*f(x_1,x_2)| &= |\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\varphi_0^{(1)}(u,v)\varphi_0^{(2)}(w)f(x_1-u,x_2-v-w)dwdudv| \\ &\leqslant C\|f\|_\infty \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.\end{aligned}$$

This implies that

$$|\langle \varphi_{0,0}*f,\psi_{0,0}*g\rangle| \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\|g\|_{\mathscr{S}}.$$

Case 2: $j\geqslant 1$, $k=0$.

By the cancellation condition on $\varphi_j^{(1)}$, we have

$$\begin{aligned}\varphi_{j,0}*f(x_1,x_2) &= \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\varphi_j^{(1)}(u,v)\varphi_0^{(2)}(w)f(x_1-u,x_2-v-w)dudvdw \\ &= \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\varphi_j^{(1)}(u,v)\varphi_0^{(2)}(w)\Delta^1_{(u,v)}f(x_1,x_2)dudvdw.\end{aligned}$$

The size condition of $\varphi_j^{(1)}$ and $\varphi_0^{(2)}$ and the fact that $f\in\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ give us that

$$\begin{aligned}&\left|\varphi_{j,0}*f(x_1,x_2)\right| \\ &\leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}|(u,v)|^{\alpha_1}\frac{2^{-j}}{(2^{-j}+|(u,v)|)^{n_1+n_2+1}}|\varphi_0^{(2)}(w)|dudvdw \\ &\leqslant C2^{-j\alpha_1}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.\end{aligned}$$

Therefore, we obtain that

$$|\langle \varphi_{j,0}*f,\psi_{j,0}*g\rangle| \leqslant C2^{-j\alpha_1}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\|\psi_{j,0}*g\|_{L^1(\mathbb{R}^{n_1+n_2})} \leqslant C2^{-j\alpha_1}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\|g\|_{\mathscr{S}}.$$

Case 3: $j=0$, $k\geqslant 1$.

Repeating the similar argument as the Case 3, we get

$$|\langle \varphi_{0,k}*f,\psi_{0,k}*g\rangle| \leqslant C2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\|g\|_{\mathscr{S}}.$$

Case 4: $j\geqslant 1$, $k\geqslant 1$.

Applying the cancellation conditions on both $\varphi_j^{(1)}$ and $\varphi_k^{(2)}$, we have

$$\begin{aligned}&|\varphi_{j,k}*f(x_1,x_2)| \\ &= |\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\varphi_j^{(1)}(u,v)\varphi_k^{(2)}(w)f(x_1-u,x_2-v-w)dudvdw| \\ &= |\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\varphi_j^{(1)}(u,v)\varphi_k^{(2)}(v)\Delta^2_w\Delta^1_{(u,v)}(f)(x_1,x_2)dudvdw| \\ &\leqslant C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}|\varphi_j^{(1)}(u,v)\varphi_k^{(2)}(w)||(u,v)|^{\alpha_1}|w|^{\alpha_2}dudvdw \\ &\leqslant C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}},\end{aligned}$$

which yields

$$|\langle \varphi_{j,k} * f, \psi_{j,k} * g\rangle| \leqslant C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\|g\|_{\mathscr{S}}.$$

Combing these four cases, we obtain that

$$|\langle \varphi_{j,k} * f, \psi_{j,k} * g\rangle| \leqslant C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\|g\|_{\mathscr{S}}$$

and thus, $\langle f,g\rangle$ is well defined. In addition, we also obtain $2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k} * f\|_{L^\infty} \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}$ for any $j,k \geqslant 0$.

When $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = 1$, $0 < \alpha_2 < 1$, we only need to consider the cases where $j \geqslant 1$, $k = 0$ and $j,k \geqslant 1$ since the other two cases are similar. Indeed, if $j \geqslant 1$, $k = 0$, noting first that $\varphi^{(1)}_j$ is a radial function and then applying the cancellation conditions on $\varphi^{(1)}_j$, we have

$$\begin{aligned}
&|\varphi_{j,0} * f(x_1,x_2)|\\
&= \frac{1}{2}\Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} \varphi^{(1)}_j(u,v)\varphi^{(2)}_0(w)[f(x_1-u,x_2-v-w)\\
&\qquad + f(x_1+u,x_2+v-w)]dudvdw\\
&= \frac{1}{2}\Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} \varphi^{(1)}_j(u,v)\varphi^{(2)}_0(w)\Delta^{1,Z}_{(u,v)}f(x_1,x_2)dudv\Big|\\
&\leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} |(u,v)|\frac{2^{-2j}}{(2^{-j}+|(u,v)|)^{n_1+n_2+2}}|\varphi^{(2)}_0(w)|dudvdw\\
&\leqslant C2^{-j}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.
\end{aligned}$$

If $j,k \geqslant 1$, then

$$\begin{aligned}
&\varphi_{j,k} * f(x_1,x_2)\\
&= \frac{1}{2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} \varphi^{(1)}_j(u,v)\psi^{(2)}_k(w)[f(x_1-u,x_2-v-w)+f(x_1+u,x_2+v-w)]dudv\\
&= \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} \varphi^{(1)}_j(u,v)\varphi^{(2)}_k(w)\Delta_w\Delta^{1,Z}_{(u,v)}f(x_1,x_2)dudvdw
\end{aligned}$$

The last equality follows the cancellations conditions on $\varphi^{(1)}_j$ and $\varphi^{(2)}_k$. Hence,

$$\begin{aligned}
&|\varphi_{j,k} * f(x_1,x_2)|\\
&\leqslant C2^{-j}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}} |\varphi^{(1)}_j(u,v)\varphi^{(2)}_k(w)||(u,v)||w|^{\alpha_2}dudvdw\\
&\leqslant C2^{-j}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.
\end{aligned}$$

Thus, $\langle f,g\rangle$ is well defined and

$$\sup_{j,k\geqslant 0} 2^{j}2^{k\alpha_2}\|\psi_{j,k} * f\|_{L^\infty} \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.$$

All other cases $\alpha=(\alpha_1,\alpha_2)$ where $0<\alpha_1<1$, $\alpha_2=1$ or $\alpha_1=\alpha_2=1$ can be handled similarly. For the case where $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_1,\alpha_2>1$ and $\alpha_1=m_1+r_1$ and $\alpha_2=m_2+r_2$, $0<r_1,r_2\leqslant 1$. Set $\widehat{\tilde{\varphi}_j^{(1)}}(\xi_1,\xi_2)=\frac{\widehat{\varphi_j^{(1)}}(\xi_1,\xi_2)}{(-2\pi i\xi_1)^{\beta_1}}$ and $\widehat{\tilde{\varphi}_k^{(2)}}(\xi_2)=\frac{\widehat{\psi_k^{(2)}}(\xi_2)}{(-2\pi i\xi_2)^{\beta_2}}$ for $j,k\geqslant 0$, where $|\beta_1|=m_1$ and $|\beta_2|=m_2$. Then

$$\varphi_{j,k}*f(x_1,x_2)=\partial^{\beta_1}\partial^{\beta_2}(\tilde{\varphi}_{j,k}*f)(x_1,x_2)=(-1)^{m_1+m_2}\tilde{\varphi}_{j,k}*\partial^{\beta_1}\partial^{\beta_2}f(x_1,x_2),$$

where $\tilde{\varphi}_{j,k}=\tilde{\varphi}_j^{(1)}*_2\tilde{\varphi}_k^{(2)}$. Note that $2^{jm_1}2^{km_2}\tilde{\varphi}_{j,k}$ satisfy the similar smoothness, size and cancellation properties as $\varphi_{j,k}$. Therefore, repeating the same proof gives that

$$\begin{aligned}|\varphi_{j,k}*f|&=|2^{-jm_1}2^{-km_2}(2^{jm_1}2^{km_2}\tilde{\varphi}_{j,k})*\partial^{\beta_1}\partial^{\beta_2}f|\\&\leqslant C2^{-jm_1}2^{-km_2}2^{-jr_1}2^{-kr_2}\|\partial^{\beta_1}\partial^{\beta_2}f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}\\&=C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.\end{aligned}$$

Therefore, this case can be also handled similarly.

We now prove the converse implication of Theorem 3. Suppose that $f\in\mathscr{S}'(\mathbb{R}^{n_1+n_2})$ satisfying

$$\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}\leqslant C$$

with $\alpha_1,\alpha_2>0$. We first show that f coincides with a continuous function. As mentioned,

$$f(x_1,x_2)=\sum_{j,k\geqslant 0}\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2)\text{ in }\mathscr{S}'(\mathbb{R}^{n_1+n_2}).$$

Then

$$|\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2)|\leqslant\|\varphi_{j,k}*f\|_{L^\infty}\|\psi_{j,k}\|_{L^1}\leqslant C2^{-j\alpha_1}2^{-k\alpha_2}\big(\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}\big).$$

Thus, the series $\sum_{j,k\geqslant 0}\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2)$ converges uniformly in x,y. Since $\psi_{j,k}*\varphi_{j,k}*f$ is continuous in $\mathbb{R}^{n_1+n_2}$, then the sum function f is also continuous in $\mathbb{R}^{n_1+n_2}$. Moreover,

$$\|f\|_{L^\infty}\leqslant C\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}. \tag{19}$$

Now we show that $f\in\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$. First, if $\alpha=(\alpha_1,\alpha_2)$ with $0<\alpha_1,\alpha_2<1$, we then prove that

$$|\Delta^1_{(u,v)}f(x_1,x_2)|\leqslant C|(u,v)|^{\alpha_1}\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

From (19), it suffices to consider $|(u,v)|<1$. Then

$$\begin{aligned}
&\Delta^1_{(u,v)}f(x_1,x_2)\\
&=\sum_{j,k\geqslant 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}[\psi_{j,k}(x_1-u-u',x_2-v-v')-\psi_{j,k}(x_1-u',x_2-v')]\varphi_{j,k}*f(u',v')du'dv'\\
&=\sum_{j,k\geqslant 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\Delta^1_{(u,v)}\psi^{(1)}_j(x_1-u',x_2-v'-w)\psi^{(2)}_k(w)\varphi_{j,k}*f(u',v')du'dv'dw.
\end{aligned}$$

We now choose a nonnegative integer m_1 such that $2^{-m_1-1}\leqslant|(u,v)|<2^{-m_1}$, and we split

$$\begin{aligned}
&|\Delta^1_{(u,v)}f(x_1,x_2)|\\
&\leqslant A\Big(\sum_{j=0}^{m_1}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}|\Delta^1_{(u,v)}\psi^{(1)}_j(x_1-u',x_2-v'-w)\psi^{(2)}_k(w)|du'dv'dw\\
&\quad+\sum_{j=m_1}^{\infty}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}|\Delta^1_{(u,v)}\psi^{(1)}_j(x_1-u',x_2-v'-w)\psi^{(2)}_k(w)|du'dv'dw\Big)\\
&:=I+II,
\end{aligned}$$

where $A=\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}$.

To estimate I, applying the smoothness condition on $\psi^{(1)}_j$ and the size condition on $\psi^{(2)}_k$ implies

$$I\leqslant CA\sum_{j=0}^{m_1}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\frac{|(u,v)|}{2^{-j}}\leqslant CA2^{m_1(1-\alpha_1)}|u|\leqslant CA|u|^{\alpha_1}.$$

To deal with II, the size conditions on both $\psi^{(1)}_j$ and $\psi^{(2)}_k$ yields

$$I\leqslant CA\sum_{j=m_1}^{\infty}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\leqslant CA2^{-m_1\alpha_1}|u|\leqslant CA|u|^{\alpha_1}.$$

Thus, we obtain that for any $(u,v)\neq 0,(x_1,x_2)\in\mathbb{R}^{n_1+n_2}$,

$$\frac{\Delta^1_{(u,v)}f(x_1,x_2)}{|(u,v)|^{\alpha_1}}\leqslant C\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

Similarly, for any $w\neq 0,(x_1,x_2)\in\mathbb{R}^{n_1+n_2}$, there holds

$$\frac{\Delta^2_w f(x_1,x_2)}{|w|^{\alpha_2}}\leqslant C\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

Finally, we prove that

$$|\Delta^2_w\Delta^1_{(u,v)}f(x_1,x_2)|\leqslant C|(u,v)|^{\alpha_1}|w|^{\alpha_2}\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

We only need consider $|(u,v)|<1$ and $|w|<1$. Let m_1, m_2 be the unique nonnegative integer such that $2^{-m_1-1}\leqslant|(u,v)|<2^{-m_1}$ and $2^{-m_2-1}\leqslant|w|<2^{-m_2}$. Observe that

$$\begin{aligned}&\Delta_w^2\Delta_{(u,v)}^1 f(x_1,x_2)\\&=\sum_{j,k\geqslant 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\Delta_{(u,v)}^1\psi_j^{(1)}(x_1-u',x_2-v'-w')\Delta_w^2\psi_k^{(2)}(w')\varphi_{j,k}*f(u',v')du'dv'dw'.\end{aligned}$$

Now we split the above series by

$$\begin{aligned}&\Big(\sum_{j=m_1}^{\infty}\sum_{k=m_2}^{\infty}+\sum_{j=0}^{m_1}\sum_{k=m_2}^{\infty}+\sum_{j=m_1}^{\infty}\sum_{k=0}^{m_2}+\sum_{j=0}^{m_1}\sum_{k=0}^{m_2}\Big)\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\Delta_{(u,v)}^1\psi_j^{(1)}(x_1-u',x_2-v'-w')\\&\quad\times\Delta_w^2\psi_k^{(2)}(w')\varphi_{j,k}*f(u',v')du'dv'dw'\\&:=B_1+B_2+B_3+B_4.\end{aligned}$$

To deal with the first series B_1 , applying the the size conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$ yields that

$$|B_1|\leqslant\sum_{j=m_1}^{\infty}\sum_{k=m_2}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\leqslant CA2^{-n_1\alpha_1}2^{-n_2\alpha_2}\leqslant CA|(u,v)|^{\alpha_1}|w|^{\alpha_2}.$$

To estimate the second series B_2, applying the smooth condition on $\psi_j^{(1)}$ and the size condition on $\psi_k^{(2)}$ implies that

$$|B_2|\leqslant CA\sum_{j=0}^{m_1}\sum_{k=m_2}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\frac{|(u,v)|}{2^{-j}}\leqslant CA2^{n_1(1-\alpha_1)}2^{-n_2\alpha_2}|u|\leqslant CA|(u,v)|^{\alpha_1}|w|^{\alpha_2}.$$

The estimate for third series B_3 is similar to the estimate for B_2. Finally, to handle with the last series B_4, applying the smoothness conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$, we get that

$$\begin{aligned}|B_4|&\leqslant C\sum_{j=0}^{m_1}\sum_{k=0}^{m_2}2^{-j\alpha_1}2^{-k\alpha_2}\frac{|(u,v)|}{2^{-j}}\frac{|w|}{2^{-k}}\\&\leqslant CA2^{n_1(1-\alpha_1)}2^{n_2(1-\alpha_2)}|(u,v)||w|\leqslant CA|(u,v)|^{\alpha_1}|w|^{\alpha_2}.\end{aligned}$$

When $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_1=\alpha_2=1$, observe that

$$\begin{aligned}&\Delta_w^{2,Z}\Delta_{(u,v)}^{1,Z} f(x_1,x_2)\\&=\sum_{j,k\geqslant 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_2}}\Delta_{(u,v)}^{1,Z}\psi_j^{(1)}(x_1-u',x_2-v'-w')\Delta_w^{2,Z}\psi_k^{(2)}(w')\varphi_{j,k}*f(u',v')du'dv'dw'.\end{aligned}$$

Repeating a similar calculation gives the desired result for this case. The other two cases, where $\alpha_1=1$, $0<\alpha_2<1$ and $0<\alpha_1<1$, $\alpha_2=1$, can be handled similarly.

Finally, when $1 < \alpha_1 = m_1 + r_1$, $1 < \alpha_2 = m_2 + r_2$ with $0 < r_1, r_2 \leqslant 1$, note that

$$\Delta_w^2 \Delta_{(u,v)}^1 \partial^{\beta_1} \partial^{\beta_2} f(x_1, x_2) = \sum_{j,k \geqslant 0} \iint_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} \Delta_{(u,v)}^1 \partial^{\beta_1} \psi_j^{(1)}(x_1 - u', x_2 - v' - w') \times \Delta_w^2 \partial^{\beta_2} \psi_k^{(2)}(w') \varphi_{j,k} * f(u', v') du' dv' dw$$

for $|\beta_1| = m_1$ and $|\beta_2| = m_2$. Again observe that the properties of $\partial^{\beta_1} \psi_j^{(1)}$ and $\partial^{\beta_2} \psi_k^{(2)}$ are similar to $2^{jm_1} \psi_j^{(1)}$ and $2^{km_2} \psi_k^{(2)}$, respectively, and hence the estimate for this case is the same as the proof for the case where $0 < \alpha_1, \alpha_2 \leqslant 1$. Therefore, the proof of Theorem 3 is completed.

5. Proof of Theorem 4

To prove Theorem 4, we need the following

LEMMA 2. *For any $f \in \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$ with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$, there exists a sequence $\{f_n\}$ such that $f_n \in L^2 \cap \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$ and f_n converges to f in the distribution sense. Furthermore,*

$$\|f_n\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}} \leqslant C \|f\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}},$$

where the constant C is independent of f_n and f.

Proof. To do this, note that, by Theorem 2, for each $f \in \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$,

$$f(x) = \sum_{j,k=0}^{\infty} \psi_{j,k} * \varphi_{j,k} * f(x),$$

in the distribution sense. For any fixed $n > 0$, denote

$$E = \{(j,k) : 0 \leqslant j \leqslant n, 0 \leqslant k \leqslant n\},$$

and

$$f_n(x) = \sum_{(j,k) \in E} \psi_{j,k} * \varphi_{j,k} * f(x).$$

Obviously, $f_n \in L^2$ and converges to f in the distribution sense. To see that $f_n \in \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$, by Theorem 3,

$$\|f_n\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}} \leqslant C \sup_{j,k \geqslant 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\varphi_{j,k} * f_n\|_{L^\infty}.$$

Observe that

$$\varphi_{j,k} * f_n(x) = \sum_{(j',k') \in E} \varphi_{j,k} * \psi_{j',k'} * \varphi_{j',k'} * f(x).$$

We claim that for any given positive integer N and L, there exists a constant $C>0$ such that

$$|\varphi_{j,k}*\psi_{j',k'}(x)| \leqslant C2^{-|j-j'|N}2^{-|k-k'|N}\frac{2^{-(j\wedge j')L}}{(2^{-(j\wedge j')}+|x_1|)^{n_1+L}}\frac{2^{-(j\wedge j'\wedge k\wedge k')L}}{(2^{-(j\wedge j'\wedge k\wedge k')}+|x_2|)^{n_2+L}}. \tag{20}$$

Assuming the claim for the moment and applying Theorem 2 again, it follows that if $N>\alpha_1\vee\alpha_2$,

$$2^{j\alpha_1}2^{k\alpha_2}|\varphi_{j,k}*f_n(x)| \leqslant C\sup_{j',k'\geqslant 0}2^{j'\alpha_1}2^{k'\alpha_2}\|\varphi_{j',k'}*f\|_{L^\infty} \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}},$$

which yields that

$$\|f_n\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.$$

It remains to verify (20). Note that

$$\varphi_{j,k}*\psi_{j',k'} = (\varphi_j^{(1)}*\psi_{j'}^{(2)})*_2(\varphi_k^{(2)}*\psi_{k'}^{(2)}).$$

By Lemma 1, for any given positive integer N and L, there exists a constant $C>0$ such that

$$\begin{aligned}&|\varphi_{j,k}*\psi_{j',k'}(x)| \\ &\leqslant C2^{-|j-j'|N}2^{-|k-k'|N}\int_{\mathbb{R}^{n_2}}\frac{2^{-(j\wedge j')L}}{(2^{-(j\wedge j')}+|x_1|+|x_2-v|)^{n_1+n_2+L}}\frac{2^{-(k\wedge k')L}}{(2^{-(k\wedge k')}+|v|)^{n_2+L}}dv.\end{aligned}$$

By an estimate given in Lemma 52 in [22], (20) follows.

We are now ready to prove Theorem 4.

Proof of Theorem 4. First we claim that if $f\in L^2$ and $T=K*f$ is a inhomogeneous flag singular integral operator on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with a inhomogeneous flag kernel as given in Definition 2, then

$$\|T(f)\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \leqslant C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}. \tag{21}$$

Indeed, by Theorem 2,

$$\|T(f)\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \leqslant C\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*T(f)\|_{L^\infty}.$$

Observe that T is bounded on $L^2(\mathbb{R}^{n_1+n_2})$, and hence

$$\varphi_{j,k}*T(f)(x) = \sum_{j',k'\geqslant 0}\varphi_{j,k}*\mathscr{K}*\psi_{j',k'}*\varphi_{j',k'}*f(x).$$

By Theorem 1, $\mathscr{K}(x_1,x_2)=\int_{\mathscr{R}^{n_2}}\mathscr{K}^\sharp(x_1,x_2-x_3,x_3)dx_3$, where $\mathscr{K}^\sharp$ is a inhomogeneous product singular integral kernel on $\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}$. Note that

$$\varphi_{j,k}*\mathscr{K}*\psi_{j',k'}(x_1,x_2) = \int_{\mathbb{R}^{n_2}}\Phi_{j,k}*\mathscr{K}^\sharp*\Psi_{j',k'}(x_1,x_2-x_3,x_3)dx_3,$$

where $\Phi_{j,k}(x_1,x_2,x_3)=\varphi_j^{(1)}(x_1,x_2)\varphi_k^{(2)}(x_3)$ and $\Psi_{j',k'}(x_1,x_2,x_3)=\psi_{j'}^{(1)}(x_1,x_2)\psi_{k'}^{(2)}(x_3)$. Applying the classical almost orthogonal estimates with $\Phi_{j,k}$, $\mathscr{K}^\sharp$ and $\Psi_{j',k'}$ on $\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_2}$ (see Lemmas 2.6 and 2.7 in [28]), we have that for any given positive integer M,

$$\begin{aligned}&|\Phi_{j,k}*\mathscr{K}^\sharp*\Psi_{j',k'}(x_1,x_2,x_3)|\\&\leqslant C2^{-|j-j'|M}2^{-|k-k'|M}\frac{2^{-(j\wedge j')\sigma}}{(2^{-(j\wedge j')}+|x_1|+|x_2|)^{n_1+n_2+\sigma}}\frac{2^{-(k\wedge k')\sigma}}{(2^{-(k\wedge k')}+|x_2|)^{n_2+\sigma}},\end{aligned}$$

where $\sigma=\delta$ when $j=0,\ j'>0$ or $j>0,\ j'=0$, otherwise σ can be sufficiently large. Thus, we obtain that

$$\begin{aligned}&|\varphi_{j,k}*\mathscr{K}*\psi_{j',k'}(x_1,x_2)|\\&\leqslant C2^{-|j-j'|M}2^{-|k-k'|M}\frac{2^{-(j\wedge j')\sigma}}{(2^{-(j\wedge j')}+|x_1|)^{n_1+\sigma}}\frac{2^{-(j\wedge j'\wedge k\wedge k')\sigma}}{(2^{-(j\wedge j'\wedge k\wedge k')}+|x_2|)^{n_2+\sigma}}.\end{aligned}$$

Repeating the same proof as in Lemma 2 gives that

$$\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*T(f)\|_{L^\infty}\leqslant C\sup_{j,k\geqslant 0}2^{j'\alpha_1}2^{k'\alpha_2}\|\varphi_{j',k'}*f\|_{L^\infty}\leqslant C\|f\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}.$$

if $M>\alpha_1\vee\alpha_2$, which yields the claim (21).

We now extend T to $\mathrm{Lip}^\alpha_{\mathrm{flag}}$ as follows. First, if $f\in\mathrm{Lip}^\alpha_{\mathrm{flag}}$, then, as mentioned in Lemma 1, there exists a sequence $\{f_n\}\in L^2\cap\mathrm{Lip}^\alpha_{\mathrm{flag}}$ such that f_n converges to f in the distribution sense and $\|f_n\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}\leqslant C\|f\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}$. It follows from the claim (21) that

$$\|T(f_n)-T(f_m)\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}\leqslant C\|f_n-f_m\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}$$

and hence $T(f_n)$ converges in the distribution sense. We define

$$T(f)=\lim_{n\to\infty}T(f_n)$$

in the distribution sense. We obtain, by Theorem 3 and the above claim in (21),

$$\begin{aligned}\|Tf\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}&\leqslant C\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*T(f)\|_{L^\infty}\\&\leqslant C\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\lim_{n\to\infty}\varphi_{j,k}*T(f_n)\|_{L^\infty}\\&\leqslant C\liminf_{n\to\infty}\sup_{j,k\geqslant 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*T(f_n)\|_{L^\infty}\\&\leqslant C\liminf_{n\to\infty}\|f_n\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}\leqslant C\|f\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}.\end{aligned}$$

The proof of Theorem 4 is finished. $\square$

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