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EXTENSIONS OF LEMOS–SOARES TYPE LOG–MAJORIZATION

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Abstract. In this paper, we shall obtain extensions of Lemos-Soares log-majorization via Furuta inequality.

1. Introduction

A capital letter, such as T, stands for an $n \times n$ complex matrix.

$T > 0$ means that T is a positive definite matrix and $T \geqslant 0$ means that T is a positive semidefinite matrix, respectively. $||T||$ stands for the spectral norm of T. $A \geqslant B$ means that $A - B \geqslant 0$.

F. Kubo and T. Ando, in [6], introduce the α-power mean of A and B as follows,

$$A\sharp_\alpha B = \begin{cases} A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, & A,B > 0,\ \alpha \in [0,1]; \\ \lim\limits_{\varepsilon \to 0^+} (A+\varepsilon I)\sharp_\alpha (B+\varepsilon I). & A,B \geqslant 0,\ \alpha \in [0,1]. \end{cases}$$

There are many perfect properties on α-power mean. Here we list three of them.

(P1) Monotonicity. If $0 \leqslant B \leqslant A$ and $0 \leqslant D \leqslant C$, then $B\sharp_\alpha D \leqslant A\sharp_\alpha C$.

(P2) Continuity. If $A^{(k)} \to A$ and $B^{(k)} \to B$ as $k \to \infty$, then $A^{(k)}\sharp_\alpha B^{(k)} \to A\sharp_\alpha B$ as $k \to \infty$.

(P3) Determinantial identity. $\det A\sharp_\alpha B = (\det A)^{1-\alpha}(\det B)^\alpha$.

T. Ando and F. Hiai, in [1], introduce the relationship between two positive semidefinite matrices A and B, called log-majorization, denoted by $A \underset{(\log)}{\succ} B$, if

$$\prod_{i=1}^{k} \lambda_i(A) \geqslant \prod_{i=1}^{k} \lambda_i(B) \quad (k = 1,2,\cdots,n-1)$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B) \quad (i.e. \quad detA = detB)$$

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hold, where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$ and $\lambda_1(B) \geqslant \lambda_2(B) \geqslant \cdots \geqslant \lambda_n(B)$ are the eigenvalues of A and B, respectively, arranged in decreasing order.

There are many perfect theorems on α-power mean and log-majorization, such as [3, 7, 8, 10].

Very recently, R. Lemos and G. Soares developed the related theory and obtained the following result.

THEOREM 1. ([7, 8], Lemos-Soares log-majorization) *If $A,B \geqslant 0$, then*

$$A^{\frac{1}{2}}(A\sharp B)B(A\sharp B)A^{\frac{1}{2}} \underset{(\log)}{\prec} (A^{\frac{1}{2}}BA^{\frac{1}{2}})^2$$

holds.

Theorem 1 was first shown in [7, Theorem 3.3]. It readily implies the corresponding result with AB^2A in the right hand considered in [7, 8] according to the famous Araki-Lieb-Thirring inequality [2].

In this paper, we shall extend Theorem 1 via Furuta inequality.

In order to prove the main results, we first list some useful lemmas such as Furuta inequality.

LEMMA 1. ([4], Furuta inequality) *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$ and $p \geqslant 1$,*

$$A^{1+r} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

and

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant B^{1+r}$$

hold.

LEMMA 2. ([5, 9], Löwner-Heinz inequality) *Let $A \geqslant B \geqslant 0$, then for every $0 \leqslant \alpha \leqslant 1$, $A^\alpha \geqslant B^\alpha$ holds.*

2. Extensions of Lemos-Soares log-majorization

In this section, we shall show extensions of Lemos-Soares log-majorization derived from Furuta inequality.

THEOREM 2. *If $A,B \geqslant 0$, then*

$$A^{\frac{t}{2}}(A^r\sharp_\alpha B^p)B^r(A^r\sharp_\alpha B^p)A^{\frac{t}{2}} \underset{(\log)}{\prec} (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{2\alpha p+r}$$

holds for $0 \leqslant r \leqslant 1$, $p \geqslant 1$, $0 \leqslant 2\alpha \leqslant \frac{1+r}{p+r}$, where $t = 2\alpha(p+r)-r$.

Proof. Without loss of generality, we may suppose $A,\ B > 0$.

By the famous antisymmetric tensor power technique, we only need to prove that $A^{\frac{1}{2}}BA^{\frac{1}{2}} \leqslant I$ ensures that

$$A^{\frac{t}{2}}(A^r\sharp_\alpha B^p)B^r(A^r\sharp_\alpha B^p)A^{\frac{t}{2}} \leqslant I.$$

It is clear that $A^{\frac{1}{2}}BA^{\frac{1}{2}} \leqslant I$ is equivalent to $B \leqslant A^{-1}$.

By Furuta inequality, we have

$$(A^{-\frac{r}{2}}B^pA^{-\frac{r}{2}})^{\frac{1+r}{p+r}} \leqslant A^{-1-r}.$$

By Löwner-Heinz inequality, it follows that

$$(A^{-\frac{r}{2}}B^pA^{-\frac{r}{2}})^{2\alpha} \leqslant A^{-2\alpha(p+r)} \tag{2.1}$$

due to the fact that $0 \leqslant \frac{2\alpha(p+r)}{1+r} \leqslant 1$.

Thus,

$$\begin{aligned} A^{\frac{t}{2}}(A^r\sharp_\alpha B^p)B^r(A^r\sharp_\alpha B^p)A^{\frac{t}{2}} \leqslant & A^{\frac{t}{2}}(A^r\sharp_\alpha B^p)A^{-r}(A^r\sharp_\alpha B^p)A^{\frac{t}{2}} \\ = & A^{\frac{t+r}{2}}(A^{-\frac{r}{2}}B^pA^{-\frac{r}{2}})^{2\alpha}A^{\frac{t+r}{2}} \\ \leqslant & A^{t+r-2\alpha(p+r)} \\ = & I. \end{aligned}$$

The first inequality is from Löwner-Heinz inequality and the second inequality is from (2.1). This completes the proof. $\square$

COROLLARY 1. *If $A,B \geqslant 0$, then*

$$A^{\frac{t}{2}}(A^r\sharp_\alpha B^{\frac{1}{r}})B^r(A^r\sharp_\alpha B^{\frac{1}{r}})A^{\frac{t}{2}} \underset{(\log)}{\prec} (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{2\alpha}{r}+r}$$

holds for $0 \leqslant r \leqslant 1,\ 0 \leqslant 2\alpha \leqslant \frac{r+r^2}{1+r^2}$, *where* $t = 2\alpha(\frac{1}{r}+r)-r$.

Proof. Put $p = \frac{1}{r}$ in Theorem 2, we can obtain the above result. $\square$

COROLLARY 2. *If $A,B \geqslant 0$, then*

$$A^{\frac{4\alpha-1}{2}}(A\sharp_\alpha B)B(A\sharp_\alpha B)A^{\frac{4\alpha-1}{2}} \underset{(\log)}{\prec} (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{2\alpha+1}$$

holds for $0 \leqslant \alpha \leqslant \frac{1}{2}$.

Proof. Put $r = 1$ in Corollary 1, we can obtain the above result. $\square$

REMARK 1. If we put $\alpha = \frac{1}{2}$, Corollary 2 is just Lemos-Soares log-majorization.

REMARK 2. It doesn't always hold for $\frac{1}{2} < \alpha < 1$ in Corollary 2. Next, we give a counterexample. Put $A = \begin{bmatrix} 10 & 6 & 3 \\ 6 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 & 5 \\ 3 & 4 & 6 \\ 5 & 6 & 10 \end{bmatrix}$ and $\alpha = 0.9$. Then we have

$$\lambda_1\left(A^{\frac{4\alpha-1}{2}}(A\sharp_\alpha B)B(A\sharp_\alpha B)A^{\frac{4\alpha-1}{2}}\right) = 1.312835109283645 \times 10^6$$
$$> \lambda_1\left((A^{\frac{1}{2}}BA^{\frac{1}{2}})^{2\alpha+1}\right) = 1.227389551752334 \times 10^6.$$

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REFERENCES

[1] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., 197 (1994), 113–131.
[2] H. ARAKI, *On an inequality of Lieb and Thirring*, Lett Math Phys., 19 (1990), 167–170.
[3] M. FUJII AND E. KAMEI, *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl., 416 (2006), 541–545.
[4] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., 101 (1987), 85–88.
[5] E. HEINZ, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., 123 (1951), 415–438.
[6] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., 246 (1980), 205–224.
[7] R. LEMOS AND G. SOARES, *Some log-majorizations and an extension of a determinantal inequality*, Linear Algebra Appl., 547 (2018), 19–31.
[8] R. LEMOS AND G. SOARES, *Spectral inequalities for Kubo-Ando operator means*, Linear Algebra Appl., 607 (2020), 29–44.
[9] C. LÖWNER, *Über monotone Matrixfunktionen*, Math. Z., 38 (1934), 177–216.
[10] Y. SEO AND M. TOMINAGA, *A complement of the Ando-Hiai inequality*, Linear Algebra Appl., 429 (2008) 1546–1554.

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