Mathematical
Inequalities
& Applications
Volume 24, Number 4 (2021), 993–999 doi:10.7153/mia-2021-24-69

NEW BOUNDS FOR GENERALIZED TAYLOR EXPANSIONS

JOSIPA BARIĆ, LJILJANKA KVESIĆ*, JOSIP PEČARIĆ
AND MIHAELA RIBIČIĆ PENAVA

(*Communicated by I. Perić*)

Abstract. We give inequalities for higher order convex functions involving harmonic sequence of polynomials. As a consequence, we obtain bounds for generalized Taylor expansions.

1. Introduction

We say that a sequence of polynomials $\{P_n\}_{n\geqslant 0}$ is a harmonic sequence of polynomials if it satisfies the following two properties:

$$P_n'(t) = P_{n-1}(t) \text{ for all } t\in\mathbb{R} \text{ and } n\in\mathbb{N}$$
$$P_0(t) = 1.$$

The following generalization of Taylor's formula is given in [2].

THEOREM 1. *Let $\{P_n\}_{n\geqslant 0}$ be a harmonic sequence of polynomials. Further, let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and $f: I\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous. Then for any $x\in I$*

$$f(x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)\right] + R_n(f;a,x) \tag{1}$$

where

$$R_n(f;a,x) = (-1)^n\int_a^x P_n(t)f^{(n+1)}(t)dt. \tag{2}$$

This is, indeed, a generalization because (1) reduces to Taylor's formula if Theorem 1 is applied for the harmonic sequence of polynomials

$$P_n(t) = \frac{(t-x)^n}{n!}.$$

We will give bounds for generalized Taylor's difference by using the weighted Hermite-Hadamard inequality given in the following theorem (see [4]).

Mathematics subject classification (2020): 26D15.
Keywords and phrases: Taylor expansion, harmonic sequence of polynomials, Hermite-Hadamard inequality.
The research of the third author has been supported by the RUDN University Strategic Academic Leadership Program.
* Corresponding author.

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THEOREM 2. *If $f:[a,x]\to\mathbb{R}$ is convex and $p:[a,x]\to\mathbb{R}$ is of a constant sign on $[a,x]$, then we have*

$$f(\lambda)\leqslant\frac{1}{P(x)}\int_a^x p(t)f(t)dt\leqslant\frac{x-\lambda}{x-a}f(a)+\frac{\lambda-a}{x-a}f(x),\tag{3}$$

where

$$P(s)=\int_a^s p(t)dt$$

and

$$\lambda=\frac{1}{P(x)}\int_a^x tp(t)dt.$$

2. New results

THEOREM 3. *Let $n\in\mathbb{N}$ be fixed and let $\{P_k\}_{k\geqslant 0}$ be a harmonic sequence of polynomials such that P_{n-1} has a constant sign on $[a,x]$. If $f:[a,x]\to\mathbb{R}$ is an $(n+2)$-convex function, then*

$$\begin{aligned}f^{(n)}(\lambda)&\leqslant\frac{1}{P(x)}\sum_{k=0}^{n-1}(-1)^k\left[P_k(x)f^{(k)}(x)-P_k(a)f^{(k)}(a)\right]\\&\leqslant\frac{x-\lambda}{x-a}f^{(n)}(a)+\frac{\lambda-a}{x-a}f^{(n)}(x),\end{aligned}\tag{4}$$

where

$$P(x)=(-1)^{n-1}\int_a^x P_{n-1}(t)dt\tag{5}$$

and

$$\lambda=\frac{(-1)^{n-1}}{P(x)}\int_a^x tP_{n-1}(t)dt.\tag{6}$$

Proof. If f is an $(n+2)$-convex function, then the function $f^{(n)}$ is convex. Furthermore, a convex function is Lipschitz continuous and, thus, absolutely continuous. Therefore, equality (1) from Theorem 1 yields

$$\sum_{k=0}^{n-1}(-1)^k\left[P_k(x)f^{(k)}(x)-P_k(a)f^{(k)}(a)\right]=(-1)^{n-1}\int_a^x P_{n-1}(t)f^{(n)}(t)dt.\tag{7}$$

Since $f^{(n)}$ is convex, applying Theorem 2 with $p(t)=(-1)^{n-1}P_{n-1}(t)$ one gets

$$f^{(n)}(\lambda)\leqslant\frac{(-1)^{n-1}}{P(x)}\int_a^x P_{n-1}(t)f^{(n)}(t)dt\leqslant\frac{x-\lambda}{x-a}f^{(n)}(a)+\frac{\lambda-a}{x-a}f^{(n)}(x),\tag{8}$$

where $P(x)$ and λ are as in (5) and (6). Using identity (7), inequalities (8) become the required inequalities (4). $\square$

COROLLARY 1. *Let n, $\{P_k\}_{k\in\mathbb{N}_0}$ and f be as in Theorem 3. Then, inequalities (4) hold, where*

$$P(x)=\sum_{k=0}^{n-1}(-1)^k\left[P_k(x)\frac{x^{n-k}}{(n-k)!}-P_k(a)\frac{a^{n-k}}{(n-k)!}\right] \tag{9}$$

and

$$\lambda=\frac{1}{P(x)}\sum_{k=0}^{n-1}(-1)^k\left[P_k(x)\frac{x^{n+1-k}}{(n+1-k)!}-P_k(a)\frac{a^{n+1-k}}{(n+1-k)!}\right]. \tag{10}$$

Proof. The function $f(t)=\frac{t^n}{n!}$ satisfies $f^{(n)}(t)=1$, so applying identity (7) for this function we can calculate $P(x)$ and we obtain (9). To calculate λ we take the function $f(t)=\frac{t^{n+1}}{(n+1)!}$ since its n-th derivative is $f^{(n)}(t)=t$ and identity (7) then gives (10). □

COROLLARY 2. *Let n, $\{P_k\}_{k\in\mathbb{N}_0}$ and f be as in Theorem 3. Then, inequalities (4) hold, where*

$$P(x)=(-1)^{n-1}\left(P_n(x)-P_n(a)\right) \tag{11}$$

and

$$\lambda=\frac{(-1)^{n-1}}{P(x)}\left(xP_n(x)-aP_n(a)-P_{n+1}(x)+P_{n+1}(a)\right). \tag{12}$$

Proof. Expression (11) is a trivial consequence of $P_n'(t)=P_{n-1}(t)$ and (12) is a simple consequence of integration by parts. □

3. Applications

In this section we will apply the results from the previous section for some special harmonic sequences of polynomials. The polynomials

$$P_n(t)=\frac{(t-x)^n}{n!} \tag{13}$$

satisfy $P_n'(t)=P_{n-1}(t)$ and $P_0(t)=1$. For these polynomials the equality (1) becomes the Taylor expansion of the function f around the point a. The inequalities from Theorem 3 for these harmonic sequence of polynomials is stated in the following theorem.

THEOREM 4. *Let $f:[a,x]\to\mathbb{R}$ be an $(n+2)$-convex function. Then*

$$\begin{aligned}\frac{(x-a)^n}{n!}f^{(n)}\left(\frac{na+x}{n+1}\right)&\leqslant f(x)-f(a)-\sum_{k=1}^{n-1}\frac{(x-a)^k}{k!}f^{(k)}(a)\\&\leqslant\frac{(x-a)^n}{n!}\left[\frac{n}{n+1}f^{(n)}(a)+\frac{1}{n+1}f^{(n)}(x)\right]\end{aligned} \tag{14}$$

Proof. The polynomials given by (13) satisfy $(-1)^{n-1}P_{n-1}(t) \geqslant 0$ and $P(x)$ from (5) is positive. Therefore, multiplying inequalities (4) with $P(x)$ gives

$$\begin{aligned} P(x)f^{(n)}(\lambda) &\leqslant f(x)-f(a)+\sum_{k=1}^{n-1}(-1)^k\left[P_k(x)f^{(k)}(x)-P_k(a)f^{(k)}(a)\right] \\ &\leqslant P(x)\left[\frac{x-\lambda}{x-a}f^{(n)}(a)+\frac{\lambda-a}{x-a}f^{(n)}(x)\right] \end{aligned} \tag{15}$$

Formulas (11) and (12) for $P(x)$ and λ from Corollary 2 and simple calculations yield

$$P_k(x)f^{(k)}(x)-P_k(a)f^{(k)}(a)=(-1)^{k+1}\frac{(x-a)^k}{k!}f^{(k)}(a),$$

$P(x)=\frac{(x-a)^n}{n!}$ and $\lambda=\frac{na+x}{n+1}$. Inserting these expressions in (15) we obtain the required inequalities (14). □

The polynomials

$$P_n(t)=\frac{1}{n!}\left(t-\frac{a+x}{2}\right)^n \tag{16}$$

obviously satisfy $P_n'(t)=P_{n-1}(t)$ and $P_0(t)=1$. Applying our results to this sequence of harmonic polynomials we get the following result.

THEOREM 5. *Let n be odd and $f:[a,x]\to\mathbb{R}$ be an $(n+2)$-convex function. Then the following inequalities hold*

$$\begin{aligned} &\frac{(x-a)^n}{2^{n-1}n!}f^{(n)}\left(\frac{a+x}{2}\right) \\ &\leqslant f(x)-f(a)-\sum_{k=1}^{n-1}\frac{(x-a)^k}{2^k k!}\left(f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right) \\ &\leqslant \frac{(x-a)^n}{2^n n!}\left(f^{(n)}(a)+f^{(n)}(x)\right). \end{aligned} \tag{17}$$

Proof. For polynomials P_n given by (16) we have

$$P_k(x)f^{(k)}(x)-P_k(a)f^{(k)}(a)=\frac{(x-a)^k}{2^k k!}[f^{(k)}(x)-(-1)^k f^{(k)}(a)] \tag{18}$$

Since n is odd, we have $P_{n-1}(t)\geqslant 0$ for $t\in[a,x]$ and $P(x)$ from (5) is positive. Therefore, multiplying inequalities (4) with $P(x)$ and taking into account (18) we get

$$\begin{aligned} P(x)f^{(n)}(\lambda) &\leqslant f(x)-f(a)-\sum_{k=1}^{n-1}\frac{(x-a)^k}{2^k k!}\left(f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right) \\ &\leqslant P(x)\left[\frac{x-\lambda}{x-a}f^{(n)}(a)+\frac{\lambda-a}{x-a}f^{(n)}(x)\right]. \end{aligned} \tag{19}$$

Corollary 2 and simple calculations yield $P(x) = \frac{(x-a)^n}{2^{n-1}n!}$ and $\lambda = \frac{x+a}{2}$ and inserting these in (19) gives the required inequalities (17). $\square$

For the next example we will first recall some definitions and basic properties of the Euler polynomials. All the results and properties of the Euler polynomials stated here can be found in Chapter 23 of [1]. The Euler polynomials can be defined by the series expansion

$$\frac{2e^{tx}}{e^x+1} = \sum_{n=0}^{\infty} \frac{E_n(t)}{n!} x^n, \quad |x| < \pi, t \in \mathbb{R}.$$

The first few Euler polynomials are

$$E_0(t) = 1,\ E_1(t) = t - \frac{1}{2},\ E_2(t) = t^2 - t,\ E_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{4},\ \ldots$$

The Euler polynomials are uniquely determined by the following two properties ([1, 23.1.5 and 23.1.6])

$$E_n'(t) = nE_{n-1}(t), \quad \text{for } n \in \mathbb{N};\ E_0(t) = 1, \tag{20}$$

$$E_n(t+1) + E_n(t) = 2t^n, \quad \text{for } n \in \mathbb{N}_0. \tag{21}$$

The values of the Euler polynomials at 0 and 1 satisfy ([1, 23.1.20])

$$E_n(1) = -E_n(0) = \frac{2}{n+1}(2^{n+1}-1)B_{n+1}, \qquad \text{for } n \in \mathbb{N}, \tag{22}$$

where B_n are the Bernoulli numbers. Since $B_{2n+1} = 0$, the values in (22) for even n are equal to zero, i. e.

$$E_{2n}(1) = E_{2n}(0) = 0, \qquad \text{for } n \in \mathbb{N}.$$

The Euler polynomials satisfy ([1, 23.1.8, 23.1.13 and 23.1.14])

$$E_n(1-t) = (-1)^n E_n(t), \qquad \text{for } n \in \mathbb{N}_0, \tag{23}$$

$$(-1)^n E_{2n}(t) > 0, \qquad \text{for } 0 < t < \frac{1}{2},\ n \in \mathbb{N}, \tag{24}$$

$$(-1)^n E_{2n-1}(t) > 0, \qquad \text{for } 0 < t < \frac{1}{2},\ n \in \mathbb{N}. \tag{25}$$

In particular, properties (23) and (24) yield

$$E_{4m}(t) \geqslant 0 \quad \text{and} \quad E_{4m+2}(t) \leqslant 0, \qquad \text{for } 0 \leqslant t \leqslant 1,\ m \in \mathbb{N}_0 \tag{26}$$

Due to (20), the polynomials

$$P_n(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right) \tag{27}$$

satisfy $P_n'(t) = P_{n-1}(t)$ and $P_0(t) = 1$, so they form a harmonic sequence of polynomials.

THEOREM 6. *Let $n=2m+1$, $m\in\mathbb{N}_0$, and $f:[a,x]\to\mathbb{R}$ be an $(n+2)$-convex function. If m is even, then the following inequalities hold*

$$\begin{aligned}
&\frac{2(x-a)^n}{n!}E_n(1)f^{(n)}\left(\frac{a+x}{2}\right)\\
&\leqslant f(x)-f(a)-\sum_{k=1}^{m}\frac{(x-a)^{2k-1}}{(2k-1)!}E_{2k-1}(1)\left(f^{(2k-1)}(x)+f^{(2k-1)}(a)\right)\\
&\leqslant \frac{(x-a)^n}{n!}E_n(1)\left(f^{(n)}(a)+f^{(n)}(x)\right), \qquad (28)
\end{aligned}$$

while if m is odd, the reversed inequalities in (28) hold.

Proof. When $m=2k$ is even, then $n=2m+1=4k+1$, so the property (26) yields $E_{n-1}(t)=E_{4k}(t)\geqslant 0$ for $t\in[0,1]$. Therefore, the polynomials given by (27) satisfy

$$P_{n-1}(t)=\frac{(x-a)^{n-1}}{(n-1)!}E_{4k}\left(\frac{t-a}{x-a}\right)\geqslant 0, \qquad \text{for } a\leqslant t\leqslant x.$$

Similarly, when $m=2k+1$ is odd, then $n=2m+1=4k+3$ and $E_{n-1}(t)=E_{4k+2}(t)\leqslant 0$ for $t\in[0,1]$, so

$$P_{n-1}(t)=\frac{(x-a)^{n-1}}{(n-1)!}E_{4k+2}\left(\frac{t-a}{x-a}\right)\leqslant 0, \qquad \text{for } a\leqslant t\leqslant x.$$

Further, since $E_k(0)=-E_k(1)$, we have

$$\begin{aligned}
P_k(x)f^{(k)}(x)-P_k(a)f^{(k)}(a)&=\frac{(x-a)^k}{k!}E_k(1)f^{(k)}(x)-\frac{(x-a)^k}{k!}E_k(0)f^{(k)}(a)\\
&=\frac{(x-a)^k}{k!}E_k(1)\left[f^{(k)}(x)+f^{(k)}(a)\right].
\end{aligned}$$

We can apply Theorem 3 and, since $E_{2k}(1)=0$, inequalities (4) for even m become

$$\begin{aligned}
P(x)f^{(n)}(\lambda)&\leqslant f(x)-f(a)-\sum_{k=1}^{2m}\frac{(x-a)^{2k-1}}{(2k-1)!}E_{2k-1}(1)\left(f^{(2k-1)}(x)+f^{(2k-1)}(a)\right)\\
&\leqslant P(x)\left[\frac{x-\lambda}{x-a}f^{(n)}(a)+\frac{\lambda-a}{x-a}f^{(n)}(x)\right], \qquad (29)
\end{aligned}$$

while for odd m the reverse inequalities hold. By Corollary 2 we have

$$\begin{aligned}
P(x)&=(-1)^{n-1}\left(P_n(x)-P_n(a)\right)\\
&=\frac{(x-a)^n}{n!}E_n(1)-\frac{(x-a)^n}{n!}E_n(0)=\frac{2(x-a)^n}{n!}E_n(1) \qquad (30)
\end{aligned}$$

Next, since $P_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!}E_{2m+2}(1) = 0$ and $P_{n+1}(a) = \frac{(x-a)^{n+1}}{(n+1)!}E_{2m+2}(0) = 0$, we have

$$\begin{aligned}
& xP_n(x) - aP_n(a) - P_{n+1}(x) + P_{n+1}(a) \\
&= xP_n(x) - aP_n(a) = x\frac{(x-a)^n}{n!}E_n(1) - a\frac{(x-a)^n}{n!}E_n(0) \\
&= \frac{(x-a)^n}{n!}E_n(1)(x+a).
\end{aligned}$$

We can now calculate λ from (12) and get

$$\lambda = \frac{xP_n(x) - aP_n(a) - P_{n+1}(x) + P_{n+1}(a)}{P(x)} = \frac{x+a}{2} \tag{31}$$

Finally, inserting $P(x)$ and λ from (30) and (31) in (29) we obtain the required inequality (28). $\square$

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(Received January 8, 2021)

Josipa Barić
University of Split
Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture
Rudjera Boškovića 32, 21 000 Split, Croatia
e-mail: jbaric@fesb.hr

Ljiljanka Kvesić
Faculty of Science and Education
University of Mostar
Matice hrvatske bb, 88 000 Mostar, Bosnia and Herzegovina
e-mail: ljiljanka.kvesic@fpmoz.sum.ba

Josip Pečarić
RUDN University
Miklukho-Maklaya str. 6, 117198 Moscow, Russia
e-mail: pecaric@element.hr

Mihaela Ribičić Penava
Department of Mathematics
University of Osijek
Trg Ljudevita Gaja 6, 31 000 Osijek, Croatia
e-mail: mihaela@mathos.hr