Mathematical
Inequalities
& Applications
Volume 24, Number 4 (2021), 1001–1016 doi:10.7153/mia-2021-24-70

SOME HARDY–TYPE INEQUALITIES IN BANACH FUNCTION SPACES

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(*Communicated by S. Varošanec*)

Abstract. Some new inequalities of Hardy-type in Banach function space settings are proved and discussed. In particular, these results generalize and unify several classical Hardy-type inequalities. Some results are new also in the classical situation.

1. Introduction

Hardy's famous inequality from 1925 [5] reads:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \tag{1.1}$$

for $p > 1$ and where f is a positive measurable function on $(0,\infty)$.

In 1928, G. H. Hardy proved the following generalization of (1.1)

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha\,dx \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha\,dx, \tag{1.2}$$

whenever $p \geqslant 1$ and $\alpha < p-1$. The constant $\left(\frac{p}{p-1-\alpha}\right)^p$ is sharp, see [6]. After this, a lot of generalizations and complementary results have been published. These are called today Hardy-type inequalities. See e.g. the monographs [7], [13], [14], [15], [19] and the references given there. Most of these developments have been concerning inequalities between weighted Lebesgue spaces. However, some results concerning Hardy-type inequalities related to other function spaces are known. For example, Section 7.6 of [15], from 2017, was devoted to this subject. Such inequalities are known e.g. for Orlicz, Lorentz, rearrangement invariant, Morrey-type, Hölder-type and Lebesgue spaces with variable exponent. It was also mentioned in [15, p. 411] that very little is known in the general Banach function space. It was only referred to [3] and Theorem 18 in [14]. After that, another result of this type was published in [1, Theorem 3.2]. See also [17].

The aim of this paper is to continue this research by proving and discussing some new Hardy-type inequalities in this frame. We assume that f is a real valued function.

First we mention some classical and new results which have guided us.

Mathematics subject classification (2020): Primary 26D10; Secondary 26D15, 26D20, 47B38.
Keywords and phrases: Integral inequalities, Hardy's inequality, Pólya-Knopp's inequality, Hardy-Knopp's inequality, Banach function spaces.

REMARK 1.1. It was recently pointed out that (1.2) is not a genuine generalization of (1.1) because both are equivalent to the"fundamental" Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \leqslant \int_0^\infty g^p(x)\frac{dx}{x}, \tag{1.3}$$

see e.g. the new book [15] and the references given there. This equivalence follows by just doing the substitutions $f(x) = g(x^{\frac{p-1}{p}})x^{-\frac{1}{p}}$ and $f(x) = g(x^{\frac{p-\alpha-1}{p}})x^{-\frac{\alpha+1}{p}}$, respectively.

By just replacing $f(x)$ by $(f(x))^{1/p}$ in (1.1) and letting $p \to \infty$ we get the following limit inequality called Pólya-Knopp's inequality:

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx \leqslant e\int_0^\infty f(x)\,dx. \tag{1.4}$$

The constant e is sharp. Moreover, by making a similar limiting procedure in (1.2) we obtain the following weighted version of (1.4):

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) x^\alpha\,dx \leqslant e^{1+\alpha}\int_0^\infty f(x)x^\alpha\,dx, \tag{1.5}$$

for $\alpha > -1$.

REMARK 1.2. Sometimes the inequality (1.4) is referred to as the Knopp inequality with reference to the paper [9] from 1928. But, it is clear that it was known before, e.g. in his 1925 paper [5] G. H. Hardy informed that G. Pólya had pointed out this inequality to him, via the limit argument above. This is the main reason why the inequality has got this name.

REMARK 1.3. Note that the proof of (1.3) consists only of an application of Jensen's inequality and Fubini's theorem. The power weighted inequalities are more or less equivalent with the basic inequality. See [20] and [15, Theorem 7.10]. Moreover, in this way we see that all such inequalities are sharp since (1.3) is sharp, see again [15].

Hardy-Knopp's inequality

The simple proof of (1.3) presented in Remark 1.3 can be repeated for any convex function Φ. Precisely, if Φ is positive and convex on the range of f, then

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\frac{dx}{x}. \tag{1.6}$$

Sometimes, this inequality is called the Hardy-Knopp type inequality since it directly implies

(a) the inequality (1.3), and thus of (1.1) and (1.2), see Remark 1.1, when applied to $\Phi(u) = u^p$, $p \geqslant 1$.

(b) the inequality (1.4) when applied to $\Phi(u) = \exp(u)$ and replacing $f(x)$ by $\ln x f(x)$.

(c) the inequality (1.5) when applied to $\Phi(u) = \exp(u)$ and replacing $f(x)$ by $\ln x^{\alpha+1} f(x)$.

Hardy-type inequalities with kernel operators

We consider the following general Hardy-type kernel-operator A_k defined by

$$A_k f(x) := \frac{1}{K(x)} \int_0^x k(x,y) f(y)\,\mathrm{d}y, \tag{1.7}$$

where $k(x,y)$ is non-negative measurable function and

$$K(x) := \int_0^x k(x,y)\,\mathrm{d}y. \tag{1.8}$$

If $k(x,y) \equiv 1$, A_k coincides with the usual Hardy operator H defined by $Hf(x) := \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y$.

With some restrictions on the kernels (e.g. so called Oinarov kernels, homogeneous kernels, product kernels) it is fairly much knowledge about Hardy-type inequalities even in this case, see Chapter 2 and Section 7.5 of the book [15] and the references given there. However, without such restrictions there remain many open questions, see [15, Section 7.5]. But also in this general case there exist some results e.g. the following (c.f. [8, Theorem 4.4]):

THEOREM 1.4. *Let $1 < p \leqslant q < \infty$, $0 < b \leqslant \infty$, $s \in (1,p)$, let Φ be a positive and convex function on (a,c), $-\infty \leqslant a < c \leqslant \infty$, and let A_k be the operator defined by* (1.7)*. Then the inequality*

$$\left(\int_0^b \Phi^q\big(A_k f(x)\big) u(x) \frac{\mathrm{d}x}{x}\right)^{\frac{1}{q}} \leqslant C \left(\int_0^b \Phi^p\big(f(x)\big) v(x) \frac{\mathrm{d}x}{x}\right)^{\frac{1}{p}} \tag{1.9}$$

holds for all functions $f(x)$, $a < f(x) < c$, $x \in [0,b]$, and some constant $C > 0$ if

$$A(s) := \sup_{0<t<b} \left(\int_t^b \left(\frac{k(x,t)}{K(x)}\right)^q u(x) V(x)^{\frac{q(p-s)}{p}} \frac{\mathrm{d}x}{x}\right)^{\frac{1}{q}} V(t)^{\frac{s-1}{p}} < \infty,$$

where $u(x)$ and $v(x)$ are general weight functions and $V(x) = \int\limits_0^x \frac{v^{1-p'}(t)}{t^{1-p'}}\mathrm{d}t$, $p' = p/(p-1)$. Moreover, if C is the sharp constant in (1.9)*, then*

$$C \leqslant \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s).$$

In this paper, we prove some new Hardy-type inequalities in a Banach function setting, which, in particular, generalize and unify all results mentioned above. For the presentations of these results we need some notations and definitions.

Let (Ω,Σ,μ) be a complete σ-finite measure space and $L^0(\mu)=L^0(\Omega,\Sigma,\mu)$ denote the space of (equivalence classes) of μ-measurable real-valued functions endowed with the topology of convergence in measure relative to each set of finite measure.

DEFINITION 1.5. A Banach space $E\subset L^0(\mu)$ is called a Banach function space on (Ω,Σ,μ) if there exists a $u\in E$ such that $u>0$ a.e. and E satisfies the following ideal property:

$$x\in L^0(\mu),\;\; y\in E,\;\; |x|\leqslant |y|\;\; \mu-a.e. \Rightarrow x\in E \text{ and } \|x\|_E\leqslant \|y\|_E.$$

The non-increasing rearrangement $f^*(t)$, $0<t<\infty$, of a μ-measurable function f on Ω is defined by

$$f^*(t) := \inf\{\lambda>0 : \mu_f(\lambda)\leqslant t\},$$

where $\mu_f(\lambda)$ is the distribution function defined by

$$\mu_f(\lambda) := \mu\left(\{t\in\Omega : |f(t)|>\lambda\}\right).$$

DEFINITION 1.6. The Lorentz spaces $L^{p,q}$, $0<p<\infty$ and $0<q\leqslant\infty$ are defined by the quasi-norm $\|f\|_{L^{p,q}}$ defined by

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}},$$

with the usual modification when $q=\infty$.

If we replace $t^{\frac{1}{p}}$ by a more general weight function $w(t)$ we arrive at the more general weighted Lorentz spaces $\Lambda^q(w)$. For the case $p=q$, the Lorentz space $L^{p,q}$ coincides with the usual Lebesgue space L^p with the norm

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

For the proofs of some main results we need the following Lemma:

LEMMA 1.7. *Assume that the Banach function space E has the Fatou property. Let $f(x,t)\geqslant 0$ on $\Omega\times T$ and let for almost every $t\in T$, $f(x,t)\in E$. If the function $\|f^r(x,t)\|_E^{\frac{1}{r}}$ is integrable on T, then, for $r\geqslant 1$,*

$$\left\| \left(\int_T f(x,t)\mathrm{d}t\right)^r \right\|_E^{\frac{1}{r}} \leqslant \int_T \|f^r(x,t)\|_E^{\frac{1}{r}} \mathrm{d}t.$$

The paper is organized as follows: In Section 2 we state and prove the main results in this frame involving the classical Hardy operator (see Theorems 2.1, 2.7 and 2.11). In particular, we then cover, unify and generalize all results mentioned above before Theorem 1.4. In Section 3 we introduce a new generalized Hardy operator covering the case with kernel operators described above. We state and prove some new Hardy-type inequalities involving these operators (see Theorems 3.1 and 3.5). In particular, Theorem 1.4 appears as a special case. Finally, Section 4 is reserved for some final remarks and examples.

CONVENTION. Throughout this paper we assume that f is a measurable function on the considered measure space.

2. The main results involving the classical Hardy operator

Our first main result reads:

THEOREM 2.1. *Let $0<b\leqslant\infty$, $-\infty\leqslant a<c\leqslant\infty$, let Φ be a positive and convex function on (a,c) and E be a Banach function space on $[0,b)$. If E has the Fatou property and $a<f(x)<c$, then*

$$\left\|\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E \leqslant \int_0^b \Phi(f(t))\left\|\frac{1}{x}\chi_{[t,b]}(x)\right\|_E \mathrm{d}t, \tag{2.1}$$

provided both sides have sense.

Proof. Let $D=\{(x,t):0\leqslant x\leqslant b, 0\leqslant t\leqslant x\}$. Then

$$\chi_D(x,t)=\chi_{[0,x]}(t)=\chi_{[t,b]}(x). \tag{2.2}$$

By using Jensen's inequality, the lattice property of E, Lemma 1.7 with $r=1$ and (2.2) we find that

$$\begin{aligned}
\left\|\Phi\left[\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right]\right\|_E &\leqslant \left\|\int_0^x \frac{\Phi(f(t))}{x}\mathrm{d}t\right\|_E = \left\|\int_0^b \frac{\Phi(f(t))}{x}\chi_{[0,x]}(t)\mathrm{d}t\right\|_E \\
&= \left\|\int_0^b \frac{\Phi(f(t))}{x}\chi_D(x,t)\mathrm{d}t\right\|_E \leqslant \int_0^b \left\|\frac{\Phi(f(t))}{x}\chi_D(x,t)\right\|_E \mathrm{d}t \\
&= \int_0^b \left\|\frac{\Phi(f(t))}{x}\chi_{[t,b]}(x)\right\|_E \mathrm{d}t = \int_0^b \Phi(f(t))\left\|\frac{1}{x}\chi_{[t,b]}(x)\right\|_E \mathrm{d}t.
\end{aligned}$$

The proof is complete. □

Next we state the following Pólya-Knopp type inequality for Lorentz spaces:

COROLLARY 2.2. *Let $0 < b \leqslant \infty$, $-\infty \leqslant a < c \leqslant \infty$, $1 \leqslant p < \infty$ and $0 < q < \infty$. Moreover, let Φ be a positive and convex function on (a,c) and $a < f(x) < c$. Then*

$$\begin{aligned} &\left[\int_0^b \left[\left(\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right)^*\right]^q x^{\frac{q}{p}-1}\mathrm{d}x\right]^{\frac{1}{q}} \\ &\leqslant \int_0^b \Phi(f(t))\left[\int_0^{b-t}\left(\frac{x^{\frac{1}{p}}}{x+t}\right)^q \frac{\mathrm{d}x}{x}\right]^{\frac{1}{q}}\mathrm{d}t. \end{aligned} \tag{2.3}$$

Proof. It is well-known that the Banach function spaces $E = L^{p,q}$ satisfy the Fatou property (see e.g. [2]). Denote $k(x) = \frac{1}{x}\chi_{[t,b]}(x)$. Then $k^*(x) = \frac{1}{x+t}$ for $x \in [0, b-t)$ and $k^*(x) = 0$ for $b - t \leqslant x \leqslant b$. Therefore

$$\left\|\frac{1}{x}\chi_{[t,b]}(x)\right\|_E = \left(\int_0^b \left(x^{\frac{1}{p}}k^*(x)\right)^q \frac{\mathrm{d}x}{x}\right)^{\frac{1}{q}} = \left(\int_0^{b-t}\left(\frac{x^{\frac{1}{p}}}{x+t}\right)^q \frac{\mathrm{d}x}{x}\right)^{\frac{1}{q}},$$

so (2.3) coincides with (2.1). The proof is complete. □

REMARK 2.3. In some cases the last integral on the right hand side of (2.3) can be calculated exactly so (2.3) can be written in a more explicit form. For example, if $p = q \geqslant 1$ i.e. $E = L^p\left((0,b), \frac{\mathrm{d}x}{x}\right)$, then we have

$$\left\|\frac{1}{x}\chi_{[t,b]}(x)\right\|_E = \left(\int_t^b \frac{1}{x^{p+1}}\mathrm{d}x\right)^{\frac{1}{p}} = \left(\frac{1}{p}\right)^{\frac{1}{p}} (t^{-p} - b^{-p})^{\frac{1}{p}},$$

so that

$$\left[\int_0^b \Phi^p\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\frac{\mathrm{d}x}{x}\right]^{\frac{1}{p}} \leqslant \left(\frac{1}{p}\right)^{\frac{1}{p}} \int_0^b \Phi(f(x))\left[1 - \left(\frac{x}{b}\right)^p\right]^{\frac{1}{p}}\frac{\mathrm{d}x}{x}, \quad p \geqslant 1. \tag{2.4}$$

For some more such examples see our Remark 4.1.

REMARK 2.4. For $p = 1$, (2.4) was proved in [4] (see also [8]). Moreover, by modifying the proof we see that (2.4) holds in the reversed direction if Φ instead is a positive and concave function.

EXAMPLE 2.5. By using the fact that $\Phi(u) = u^p$ is convex for the cases $p < 0$ and $p \geqslant 1$ and concave for $0 < p < 1$ and making the same substitutions as in Remark 1.1 we obtain the following fairly recent sharp generalization of (1.1) and (1.2) yielding also for finite intervals (see [15, Theorem 7.10] and the references therein):

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p x^\alpha \mathrm{d}x \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^b f^p(x)x^\alpha\left[1 - \left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right]\mathrm{d}x \tag{2.5}$$

for $0 < b \leqslant \infty$, $p \geqslant 1$, $\alpha < p-1$ or $p < 0$, $\alpha > p-1$. Moreover, (2.5) holds in the reversed direction if $0 < p < 1$. In all cases the inequalities are sharp.

REMARK 2.6. By replacing $f(x)$ by $(f(x))^{\frac{1}{p}}$ in (2.5) and performing the standard limiting procedure as $p \to \infty$ we obtain the following weighted sharp version of (1.5):

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(t)\mathrm{d}t\right) x^\alpha \mathrm{d}x \leqslant e^{1+\alpha}\int_0^b f(x)x^\alpha\left(1-\frac{x}{b}\right)\mathrm{d}x,\ \ \alpha > -1,$$

yielding also for finite intervals.

Next we state the following complement of Theorem 2.1 for $b = \infty$:

THEOREM 2.7. *Let $-\infty \leqslant a < c \leqslant \infty$, let Φ be a positive and convex function on (a,c) and $E = E[0,\infty)$ be a Banach function space with Fatou property. Then, for $p > 1$,*

$$\left\|\Phi^p\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E \leqslant \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}\Phi^p(f(t))\left\|\frac{1}{x^{2-\frac{1}{p}}}\chi_{[t,\infty)}(x)\right\|_E \mathrm{d}t. \quad (2.6)$$

Proof. For the proof we use the Jensen inequality, Hölder's inequality and Lemma 1.7 for the function $g(x,t) = \chi_D(x,t)$ and we apply (2.2) for $b = \infty$. Indeed, with $q = p/(p-1)$ we have that

$$\begin{aligned}
\left\|\Phi^p\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E &\leqslant \left\|\left(\frac{1}{x}\int_0^x \Phi(f(t))\,\mathrm{d}t\right)^p\right\|_E \\
&\leqslant \left\|x^{-p}\int_0^x t^{\frac{p-1}{p}}\Phi^p(f(t))\,\mathrm{d}t\cdot\left(\int_0^x t^{\frac{(1-p)q}{p^2}}\mathrm{d}t\right)^{\frac{p}{q}}\right\|_E \\
&= \left(\frac{p}{p-1}\right)^{p-1}\left\|\int_0^\infty x^{-2+\frac{1}{p}}\chi_{[0,x]}(t)t^{1-\frac{1}{p}}\Phi^p(f(t))\,\mathrm{d}t\right\|_E \\
&= \left(\frac{p}{p-1}\right)^{p-1}\left\|\int_0^\infty x^{-2+\frac{1}{p}}\chi_D(x,t)t^{1-\frac{1}{p}}\Phi^p(f(t))\,\mathrm{d}t\right\|_E \\
&\leqslant \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}\Phi^p(f(t))\left\|\frac{1}{x^{2-\frac{1}{p}}}\chi_D(x,t)\right\|_E \mathrm{d}t \\
&= \left(\frac{p}{p-1}\right)^{p-1}\int_0^\infty t^{1-\frac{1}{p}}\Phi^p(f(t))\left\|\frac{1}{x^{2-\frac{1}{p}}}\chi_{[t,\infty)}(x)\right\|_E \mathrm{d}t
\end{aligned}$$

The proof is complete. $\square$

REMARK 2.8. It is a continuity between Theorems 2.7 and 2.1. Indeed, since $\left(\frac{p}{p-1}\right)^{p-1} \to 1$ when $p \to 1^+$ we have that (2.6) tends to (2.1) with $b = \infty$ when $p \to 1^+$.

REMARK 2.9. By applying Theorem 2.7 for the case $E = L^{1,q}$ we can state another variant of Hardy-type inequality than that in Corollary 2.2.

REMARK 2.10. It is obvious that all results in this Section can be obtained in the "dual" situation when $\int_0^x f(t)\mathrm{d}t$ is replaced by $\int_x^\infty f(t)\mathrm{d}t$, $0 \leqslant b \leqslant x < \infty$. It turns out that it is convenient to just consider the Hardy type operator $H^* : H^*f(x) = x\int_x^\infty \frac{f(t)}{t^2}\mathrm{d}t$. As an example we formulate the following "dual" version of Theorem 2.1:

THEOREM 2.11. *Let $-\infty \leqslant a < c \leqslant \infty$, let Φ be a positive and convex function on (a,c) and E be a Banach function space on $[b,\infty)$, $b \geqslant 0$, with the Fatou property. Then, whenever $a < f(x) < c$,*

$$\left\|\Phi\left(x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t\right)\right\|_E \leqslant \int_b^\infty \frac{\Phi(f(t))}{t^2}\left\|x\chi_{[b,t)}(x)\right\|_E \mathrm{d}t.$$

Proof. Let $D = \{(x,t) : x \geqslant b, x \leqslant t < \infty\}$. Then

$$\chi_D(x,t) = \chi_{[x,\infty)}(t) = \chi_{[b,t)}(x). \tag{2.7}$$

By using (2.7) and the same arguments as in the proof of Theorem 2.1 we obtain that

$$\begin{aligned}
\left\|\Phi\left(x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t\right)\right\|_E &\leqslant \left\|\int_x^\infty \frac{x}{t^2}\Phi(f(t))\,\mathrm{d}t\right\|_E \\
&= \left\|\int_b^\infty \frac{x}{t^2}\Phi(f(t))\,\chi_{[x,\infty)}(t)\,\mathrm{d}t\right\|_E \\
&= \left\|\int_b^\infty \frac{x}{t^2}\Phi(f(t))\,\chi_D(x,t)\,\mathrm{d}t\right\|_E \\
&\leqslant \int_b^\infty \left\|\frac{x}{t^2}\Phi(f(t))\,\chi_D(x,t)\right\|_E \mathrm{d}t \\
&= \int_b^\infty \left\|\frac{x}{t^2}\Phi(f(t))\,\chi_{[b,t)}(x)\right\|_E \mathrm{d}t \\
&= \int_b^\infty \frac{\Phi(f(t))}{t^2}\left\|x\chi_{[b,t)}(x)\right\|_E \mathrm{d}t.
\end{aligned}$$

The proof is complete. □

REMARK 2.12. It is possible to write a similar Corollary for Lorentz spaces as that in Corollary 2.2. In particular, by applying Theorem 2.11 with $E = L_1\left((b,\infty),\frac{\mathrm{d}x}{x}\right)$, $b \geqslant 0$, and the function $\Phi(u) = u^p$, $p < 0$ or $p \geqslant 1$ we obtain the following "dual" version of the well-known sharp inequality (see [15])

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \frac{\mathrm{d}x}{x} \leqslant 1\int_0^b f^p(x)\left(1-\frac{x}{b}\right)\frac{\mathrm{d}x}{x} \tag{2.8}$$

when $0 < b \leqslant \infty$, $p < 0$ or $p \geqslant 1$ (for the case $b = \infty$ this is just (1.3) and for $0 < p < 1$ (2.8) holds in the reversed direction).

COROLLARY 2.13. *Let* $0 \leqslant b < \infty$. *Then*

$$\int_b^\infty \left(x \int_x^\infty \frac{f(t)}{t^2} \mathrm{d}t \right)^p \frac{\mathrm{d}x}{x} \leqslant 1 \int_b^\infty f^p(x) \left(1 - \frac{b}{x}\right) \frac{\mathrm{d}x}{x} \tag{2.9}$$

whenever $p < 0$ *or* $p \geqslant 1$. *Moreover,* (2.9) *holds in the reversed direction when* $0 < p < 1$ *(for the case* $p < 0$ *we as usual assume that* $f(x) > 0$*). The inequality* (2.9) *is sharp.*

Proof. Let $p < 0$ or $p \geqslant 1$. Apply Theorem 2.11 with $E = L_1\left((b,\infty), \frac{\mathrm{d}x}{x}\right)$ and $\Phi(u) = u^p$. Note that

$$\|x\chi_{[b,t)}(x)\|_E = \int_b^t \mathrm{d}x = t - b.$$

Hence,

$$\int_b^\infty \frac{f^p(t)}{t^2} \|x\chi_{[b,t)}(x)\|_E \, \mathrm{d}t = \int_b^\infty f^p(t) \left(1 - \frac{b}{t}\right) \frac{\mathrm{d}t}{t}.$$

Moreover, in this case

$$\left\| \Phi\left(x \int_x^\infty \frac{f^p(t)}{t^2} \mathrm{d}t \right) \right\|_E = \int_b^\infty \left(x \int_x^\infty \frac{f(t)}{t^2} \mathrm{d}t \right)^p \frac{\mathrm{d}x}{x}$$

and (2.9) follows from Theorem 2.11. The case $0 < p < 1$ can be proved by using reversed Jensen inequality for this special case in the proof of Theorem 2.11. The sharpness of the inequality (2.9) is more or less obvious but can be done in detail by repeating the arguments of the proof of the sharpness of (2.8) (see [15, proof of Lemma 7.8]). □

As an application of Corollary 2.13, by making substitutions similar to those in Remark 1.1 we obtain the following (see [15, Theorem 7.10]).

EXAMPLE 2.14. Let f be a positive function on $[b,\infty)$, $b \geqslant 0$. Then the sharp inequality

$$\int_b^\infty \left(\frac{1}{x} \int_x^\infty f(t) \mathrm{d}t \right)^p x^\alpha \mathrm{d}x$$
$$\leqslant \left(\frac{p}{\alpha + 1 - p} \right)^p \int_b^\infty f^p(x) x^\alpha \left[1 - \left(\frac{b}{x} \right)^{\frac{\alpha+1-p}{p}} \right] \mathrm{d}x \tag{2.10}$$

holds whenever $p \geqslant 1$, $\alpha > p-1$ or $p < 0$, $\alpha < p-1$. Moreover, (2.10) holds in the reversed direction if $0 < p < 1$, $\alpha > p-1$.

3. The main results involving some generalized Hardy operators

In order to be able to cover also situations involving more general Hardy-type operators like that in Theorem 1.4, we consider the following more general situation: Let σ be a positive measure on the measure space S and let σ_x denote a σ-finite positive measure on S such that $\sigma_x(S) < \infty$. Moreover, we suppose that σ_x is absolutely continuous with respect to σ. We define the general Hardy type operator T as follows:

$$Tf(x) = \frac{1}{\sigma_x(S)} \int_S f(t) \mathrm{d}\sigma_x(t),$$

where f is defined on S with values in (a,b), $-\infty \leqslant a < b \leqslant \infty$, and $\sigma_x(S) = \int_S \mathrm{d}\sigma_x(t)$.

Our first main result in this section reads:

THEOREM 3.1. *Let $-\infty \leqslant a < b \leqslant \infty$ and let Φ be a positive and convex function on (a,b), where $\Phi(f)$ is measurable on S. Moreover, let E be a Banach function space on S with Fatou property. Then*

$$\|\Phi(Tf(\cdot))\|_E \leqslant \int_S \Phi(f(y)) \left\| \frac{1}{\sigma_x(S)} \frac{\mathrm{d}\sigma_x(y)}{\mathrm{d}\sigma(y)} \right\|_E \mathrm{d}\sigma(y) \tag{3.1}$$

for any f defined on S with values in (a,b) such that the right hand side is finite.

Proof. First we use Jensen's inequality and the lattice property of the norm and get that

$$\begin{aligned} \|\Phi(Tf(\cdot))\|_E &= \left\| \Phi\left(\frac{1}{\sigma_x(S)} \int_S f(y) \mathrm{d}\sigma_x(y) \right) \right\|_E \\ &\leqslant \left\| \int_S \frac{1}{\sigma_x(S)} \Phi(f(y)) \mathrm{d}\sigma_x(y) \right\|_E. \end{aligned} \tag{3.2}$$

Since σ_x is absolutely continuous with respect to σ, for any x, we have, by the Radon-Nikodym theorem, that

$$\int_S \frac{1}{\sigma_x(S)} \Phi(f) \mathrm{d}\sigma_x = \int_S \frac{1}{\sigma_x(S)} \Phi(f) \frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} \mathrm{d}\sigma.$$

Hence, by using this equality and Lemma 1.7 with $r = 1$, we find that

$$\left\| \int_S \frac{1}{\sigma_x(S)} \Phi(f) \mathrm{d}\sigma_x \right\|_E \leqslant \int_S \Phi(f) \left\| \frac{1}{\sigma_x(S)} \frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} \right\|_E \mathrm{d}\sigma.$$

The proof of (3.1) follows by just combining the last inequality with (3.2). □

Next, we point out the following illustrative application (c.f. Theorem 4.1 in [8]):

COROLLARY 3.2. *Let u be a weight function on $(0,b)$, $0<b\leqslant\infty$ and let $k(x,y)\geqslant 0$ be a measurable function on $(0,b)\times(0,b)$. Assume that $k(x,y)$ is locally integrable on $(0,b)$ for every fixed $y\in(0,b)$ and define v by*

$$v(y)=y\int_y^b \frac{k(x,y)}{K(x)}u(x)\frac{dx}{x}<\infty,\quad y\in(0,b).$$

If Φ is a positive and convex function on (a,c), $-\infty<a<c<\infty$, then

$$\int_0^b \Phi\left(A_k f(x)\right)u(x)\frac{\mathrm{d}x}{x}\leqslant\int_0^b \Phi\left(f(y)\right)v(y)\frac{\mathrm{d}y}{y},$$

for all f with $a<f(x)<c$, $0\leqslant x\leqslant b$. ($A_k(\cdot)$ and $K(x)$ are defined by (1.7) and (1.8), respectively).

Proof. Just apply Theorem 3.1 with $E=L^1\left((0,b);u(x)\frac{\mathrm{d}x}{x}\right)$, $S=(0,b)$, $\mathrm{d}\sigma=\mathrm{d}y$ and $\mathrm{d}\sigma_x(y)=\chi_{[0,x]}(y)k(x,y)\mathrm{d}y$ and make some standard calculations. □

EXAMPLE 3.3. By applying Corollary 3.2 with $\Phi(u)=e^u$ and replacing $f(x)$ by $\ln(f(x))^p$, $p>0$, we obtain the following kernel Pólya-Knopp inequality

$$\begin{aligned}&\int_0^b\left[\exp\left(\frac{1}{K(x)}\int_0^x k(x,y)\ln f(y)\mathrm{d}y\right)\right]^p u(x)\frac{\mathrm{d}x}{x}\\&\leqslant\int_0^b f^p(x)v(x)\frac{\mathrm{d}x}{x},\quad p>0,\end{aligned}\tag{3.3}$$

where $k(x,y)$, $K(x)$, $u(x)$ and $v(x)$ are defined as in Corollary 3.2.

REMARK 3.4. In particular, by applying (3.3) with $p=1$, $u(x)=1$, $k(x,y)=1$, so that $K(x)=x$ and making some obvious calculations we rediscover (1.4).

Our next aim is to derive a generalization of Theorem 1.4 to a partly Banach function setting (we keep the same notations $k(x,y)$, $K(x)$ and A_k as in Theorem 1.4).

THEOREM 3.5. *Let $1<p\leqslant q<\infty$, $0<b\leqslant\infty$ and $s\in(1,p)$. Let E be a Banach function space on $[0,b)$, which has Fatou property and let Φ be a positive and convex function on (a,c), $-\infty\leqslant a<c\leqslant\infty$. Then*

$$\left\|\Phi^q\left(A_k f(x)\right)u(x)\right\|_E^{\frac{1}{q}}\leqslant C\left(\int_0^b \Phi^p\left(f(x)\right)v(x)\frac{\mathrm{d}x}{x}\right)^{\frac{1}{p}}\tag{3.4}$$

holds for all functions $f(x)$, $a<f(x)<c$, and some constant $C>0$ if

$$\overline{A}(s):=\sup_{0<t<b}\left\|\left(\frac{k(x,t)}{K(x)}\right)^q u(x)V(x)^{\frac{q(p-s)}{p}}\chi_{[t,b]}(x)\right\|_E^{\frac{1}{q}}V(t)^{\frac{s-1}{p}}<\infty,\tag{3.5}$$

where $u(x)$ and $v(x)$ are weight functions and $V(t)=\int_0^t \frac{v^{1-p'}(x)}{x^{1-p'}}\mathrm{d}x$. Moreover, if C is the best possible constant in the above inequality, then

$$C \leqslant \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\overline{A}(s). \tag{3.6}$$

Proof. For simplicity we introduce the notation $\mathrm{d}\sigma_x=\mathrm{d}\sigma_x(t)=\chi_{[0,x)}(t)k(x,t)\mathrm{d}t$. First we use Jensen's inequality and the lattice property of the norm and get that

$$\begin{aligned}\|\Phi^q(A_k f(\cdot))u\|_E^{\frac{1}{q}} &= \left\|\left(\Phi\left(\frac{1}{K(x)}\int_S f(t)\mathrm{d}\sigma_x(t)\right)\right)^q u(x)\right\|_E^{\frac{1}{q}}\\ &\leqslant \left\|\left(\frac{1}{K(x)}\int_S \Phi(f(t))\mathrm{d}\sigma_x(t)\right)^q u(x)\right\|_E^{\frac{1}{q}} := B.\end{aligned} \tag{3.7}$$

Let $\Phi^p(f(t))\frac{v(t)}{t}=\Phi(g(t))$. Then B can be written as

$$B=\left\|\left(\int_S \Phi^{\frac{1}{p}}(g(t))V(t)^{\frac{s-1}{p}}V(t)^{-\frac{s-1}{p}}v(t)^{-\frac{1}{p}}t^{\frac{1}{p}}\frac{\mathrm{d}\sigma_x(t)}{\mathrm{d}\sigma(t)}\mathrm{d}\sigma(t)\right)^q \frac{u(x)}{K(x)^q}\right\|_E^{\frac{1}{q}}. \tag{3.8}$$

Next we apply the Hölder inequality with power $p>1$ and obtain that

$$\begin{aligned}B &\leqslant \left\|\left(\int_0^x k^p(x,t)\Phi(g(t))V(t)^{s-1}\mathrm{d}t\right)^{\frac{q}{p}}\right.\\ &\qquad\left.\times\left(\int_0^x V(t)^{-\frac{p'(s-1)}{p}}v(t)^{1-p'}t^{p'-1}\mathrm{d}t\right)^{\frac{q}{p'}}\frac{u(x)}{K(x)^q}\right\|_E^{\frac{1}{q}}\\ &= \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\left\|\left(\int_0^x k^p(x,t)\Phi(g(t))V(t)^{s-1}\mathrm{d}t\right)^{\frac{q}{p}}V(x)^{\frac{q(p-s)}{p}}\frac{u(x)}{K(x)^q}\right\|_E^{\frac{1}{q}}\\ &= \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\left\|\left(\int_0^b k^p(x,t)\Phi(g(t))V(t)^{s-1}V(x)^{(p-s)}\frac{u(x)^{\frac{p}{q}}}{K(x)^p}\chi_{(0,x)}(t)\mathrm{d}t\right)^{\frac{q}{p}}\right\|_E^{\frac{1}{q}}.\end{aligned}$$

Since $\frac{q}{p}>1$, also the space $E^{\frac{q}{p}}$ has the Fatou property and we can apply Lemma 1.7 with $r=\frac{q}{p}$ and use (2.2) to find that

$$\begin{aligned}B &\leqslant \left[\frac{p-1}{p-s}\right]^{\frac{1}{p'}}\left\{\int_0^b\left\|\left[\frac{k(x,t)}{K(x)}\right]^q\Phi^{\frac{q}{p}}(g(t))V(t)^{\frac{(s-1)q}{p}}V(x)^{\frac{q(p-s)}{p}}u(x)\chi_{(t,b)}(x)\right\|_E^{\frac{p}{q}}\mathrm{d}t\right\}^{\frac{1}{p}}\\ &\leqslant \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\overline{A}(s)\left[\int_0^b\Phi(g(t))\mathrm{d}t\right]^{\frac{1}{p}}\\ &= \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\overline{A}(s)\left[\int_0^b\Phi^p(f(t))\frac{v(t)}{t}\mathrm{d}t\right]^{\frac{1}{p}}.\end{aligned} \tag{3.9}$$

By just combining (3.5) with (3.7)–(3.9) we conclude that (3.4) and also the estimate (3.6) hold. The proof is complete. □

REMARK 3.6. Theorem 1.4 is just the special case of Theorem 3.5 when $E = L^1\left((0,b); \frac{dx}{x}\right)$.

REMARK 3.7. Since the results in this Section can be applied for kernel operators the results in this Section may be seen as complements and further generalizations of some results in [8], [12](c.f. also [11]) and [18]. We just give one such example of application of Theorem 3.1 (see [8, Theorem 2.1] and c.f. also [18, Proposition 2.1]):

EXAMPLE 3.8. Let $0 < b_i \leqslant \infty$, $i = 1,2,\ldots,n$ $(n \in \mathbb{N})$, $-\infty \leqslant a < c \leqslant \infty$ and if Φ is a positive and convex function on (a,c), then

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1,\ldots,t_n)\, dt_1 \cdots dt_n\right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$
$$\leqslant \int_0^{b_1} \cdots \int_0^{b_n} \Phi\big(f(t_1,\ldots,t_n)\big) \left(1 - \frac{t_1}{b_1}\right) \cdots \left(1 - \frac{t_n}{b_n}\right) \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}.$$

4. Concluding examples and remarks

REMARK 4.1. (c.f. Remark 2.3). Some more cases when the last integral on the right hand side of (2.3) can be calculated exactly:

(a) $p = q = 2$. Then

$$\left[\int_0^{b-t} \frac{1}{(x+t)^2}\, dx\right]^{\frac{1}{2}} = \left(\frac{b-t}{tb}\right)^{\frac{1}{2}} \to \frac{1}{\sqrt{t}} \text{ as } b \to \infty.$$

(b) $p = 2$, $q = 1$. Then

$$\int_0^{b-t} \frac{1}{(x+t)\sqrt{x}}\, dx = \frac{2}{\sqrt{t}} \arctan\sqrt{\frac{b-t}{t}} \to \frac{\pi}{\sqrt{t}} \text{ as } b \to \infty.$$

(c) $p = \frac{17}{5}$, $q = \frac{17}{6}$. Then

$$\left(\int_0^{b-t} \frac{1}{x^{\frac{1}{6}}(x+t)^{\frac{17}{6}}}\, dx\right)^{\frac{6}{17}} = \left(\frac{6(b-t)^{\frac{5}{6}}(6b+5t)}{55t^2 b^{\frac{11}{6}}}\right)^{\frac{6}{17}}$$
$$\to \left(\frac{36}{55t^2}\right)^{\frac{6}{17}} \text{ as } b \to \infty.$$

In each of these cases the inequality (2.3) can be stated more explicitly like

EXAMPLE 4.2. Let Φ be a positive and convex function. Consider the following integrals

$$A=\int_0^\infty \Phi(f(x))\,\frac{\mathrm{d}x}{\sqrt{x}},\quad B=\int_0^\infty \frac{1}{x}\left(\int_0^x \Phi(f(t))\,\mathrm{d}t\right)\frac{\mathrm{d}x}{\sqrt{x}}$$

$$C=\int_0^\infty \frac{1}{\sqrt{x}}\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\mathrm{d}x \text{ and } D=\int_0^\infty \frac{1}{\sqrt{x}}\left(\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right)^*\mathrm{d}x.$$

By using (b) of Remark 4.1 we get that

$$D\leqslant \pi A$$

By using the Hardy-Littlewood inequality and Jensen's inequality, we find that

$$C\leqslant D \text{ and } C\leqslant B,$$

respectively. Moreover, by using (1.2) with $p=1$, $\alpha=-\frac{1}{2}$ and f replaced by $\Phi(f)$ we obtain the sharp inequality

$$B\leqslant 2A.$$

Hence, we have the following inequalities

$$C\leqslant D\leqslant \pi A \text{ and } C\leqslant B\leqslant 2A. \tag{4.1}$$

REMARK 4.3. By assuming that $f(x)$ is non-increasing and Φ is non-decreasing (or $f(x)$ is non-decreasing and Φ is non-increasing) we have $A\leqslant C$ and then from (4.1) we get the following:

$$A\leqslant C\leqslant B\leqslant 2A\leqslant 2C\leqslant 2D\leqslant 2\pi A,$$

so indeed the integrals A,B,C and D are equivalent in this case.

EXAMPLE 4.4. Another example of a Banach function space with Fatou property is the space $E=L_1(0,\infty)+L_\infty(0,\infty)$, which is an important space in particular in real interpolation theory (see e.g. [2] or [10]), with the usual norm

$$\|g\|_E=\int_0^1 g^*(t)\,\mathrm{d}t.$$

Then

$$\left\|\frac{1}{x}\chi_{[t,\infty)}(x)\right\|_E=\int_0^1 \frac{1}{(x+t)}\mathrm{d}x=\ln\frac{t+1}{t}.$$

Therefore, from (2.1), we have the following new inequality in this condition:

$$\int_0^1\left(\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right)^*\mathrm{d}x\leqslant \int_0^\infty \Phi(f(t))\ln\left(\frac{t+1}{t}\right)\mathrm{d}t,$$

yielding for any positive and convex function Φ and where $f(x)$ is in the definition set of Φ.

REMARK 4.5. All main results in this paper have been developed for Banach function spaces over a set with general measure $d\mu$. However, all applications have been given in the continuous case with Lebesgue measure. But similar applications can be given in the discrete case with counting measure $d\delta = \sum_{n=0}^{\infty} \delta_n$ implying the corresponding discrete inequalities.

In particular, by applying Theorem 2.1 with $E = L^p(d\delta) = l^p$ we get the following discrete inequality:

COROLLARY 4.6. *Let* $1 \leqslant k \leqslant \infty$ *and for* $p > 1$. *Then*

$$\left(\sum_{n=1}^{k} \Phi^p\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\right)^{\frac{1}{p}} \leqslant \sum_{i=1}^{k} \Phi(a_i)\left(\sum_{n=i}^{k} \frac{1}{n^p}\right)^{\frac{1}{p}},$$

whenever $\{a_i\}_1^k$ *is a non-negative sequence and* Φ *is a positive and convex function on this sequence.*

REMARK 4.7. (Concerning sharpness) As seen all our results in Section 2 gives the sharp constants in the special cases we have pointed out. Hence, we can claim that our main results in Section 2 are in this sense sharp. Also Theorem 3.1 is sharp in the same sense (see Example 3.3 and Remark 3.4). However, Theorem 3.5 depends on a not necessarily sharp constant C.

Acknowledgement. The forth named author thanks Addis Ababa University, Ethiopia, and Karlstad University, Sweden, for being accepted as student in the international PhD program between these universities. We also thank International Science Program in Uppsala, Sweden, for financial and moral support, which made this collaboration possible. Furthermore, we all thank the referee for some good suggestions which have improved the paper.

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(Received January 14, 2021)

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