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NUMERICAL RADIUS IN HILBERT C^*–MODULES

ALI ZAMANI

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Abstract. Utilizing the linking algebra of a Hilbert C^*-module $(\mathscr{V}, \|\cdot\|)$, we introduce $\Omega(x)$ as a definition of numerical radius for an element $x \in \mathscr{V}$ and then show that $\Omega(\cdot)$ is a norm on $\mathscr{V}$ such that $\frac{1}{2}\|x\| \leqslant \Omega(x) \leqslant \|x\|$. In addition, we obtain an equivalent condition for $\Omega(x) = \frac{1}{2}\|x\|$. Moreover, we present a refinement of the triangle inequality for the norm $\Omega(\cdot)$. Some other related results are also discussed.

1. Introduction

The notion of Hilbert C^*-module is a natural generalization of that of Hilbert space arising under replacement of the field of scalars $\mathbb{C}$ by a C^*-algebra. This concept plays a significant role in the theory of operator algebras, quantum groups, noncommutative geometry and K-theory; see [10, 11].

Let us give that some necessary background and set up our notation. An element a in a C^*-algebra $\mathscr{A}$ is called positive (we write $0 \leqslant a$) if $a = b^*b$ for some $b \in \mathscr{A}$. For an element a of $\mathscr{A}$, we denote by

$$\mathrm{Re}\, a = \frac{1}{2}(a + a^*), \quad \mathrm{Im}\, a = \frac{1}{2i}(a - a^*)$$

the real and the imaginary part of a. By $\mathscr{A}'$ we denote the dual space of $\mathscr{A}$. A positive linear functional of $\mathscr{A}$ is a map $\varphi \in \mathscr{A}'$ such that $0 \leqslant \varphi(a)$ whenever $0 \leqslant a$. The set of all states of $\mathscr{A}$, that is, the set of all positive linear functionals of $\mathscr{A}$ of norm 1, is denoted by $\mathscr{S}(\mathscr{A})$. An inner product module over $\mathscr{A}$ is a (left) $\mathscr{A}$-module $\mathscr{V}$ equipped with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle$, which is $\mathbb{C}$-linear and $\mathscr{A}$-linear in the first variable and has the properties $\langle x,y\rangle^* = \langle y,x\rangle$ as well as $0 \leqslant \langle x,x\rangle$ with equality if and only if $x = 0$. The $\mathscr{A}$-module $\mathscr{V}$ is called a Hilbert $\mathscr{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$. In a Hilbert $\mathscr{A}$-module $\mathscr{V}$ we have the following version of the Cauchy–Schwarz inequality:

$$\langle y,x\rangle\langle x,y\rangle \leqslant \|x\|^2\langle y,y\rangle, \qquad (x,y \in \mathscr{V}). \tag{1}$$

Every C^*-algebra $\mathscr{A}$ can be regarded as a Hilbert C^*-module over itself where the inner product is defined by $\langle a,b\rangle = a^*b$. Let $\mathscr{V}$ and $\mathscr{W}$ be two Hilbert $\mathscr{A}$-modules. A

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mapping $T:\mathscr{V}\longrightarrow\mathscr{W}$ is called adjointable if there exists a mapping $S:\mathscr{W}\longrightarrow\mathscr{V}$ such that $\langle Tx,y\rangle=\langle x,Sy\rangle$ for all $x\in\mathscr{V}, y\in\mathscr{W}$. The unique mapping S is denoted by T^* and is called the adjoint operator of T. The space $\mathbb{B}(\mathscr{V},\mathscr{W})$ of all adjointable maps between Hilbert $\mathscr{A}$-modules $\mathscr{V}$ and $\mathscr{W}$ is a Banach space, while $\mathbb{B}(\mathscr{V}):=\mathbb{B}(\mathscr{V},\mathscr{V})$ is a C^*-algebra. By $\mathbb{K}(\mathscr{V},\mathscr{W})$ we denote the closed linear subspace of $\mathbb{B}(\mathscr{V},\mathscr{W})$ spanned by $\{\theta_{x,y}: x\in\mathscr{W}, y\in\mathscr{V}\}$, where $\theta_{x,y}$ is defined by $\theta_{x,y}(z)=x\langle y,z\rangle$. Elements of $\mathbb{K}(\mathscr{V},\mathscr{W})$ are often referred to as "compact" operators. We write $\mathbb{K}(\mathscr{V})$ for $\mathbb{K}(\mathscr{V},\mathscr{V})$. Given a Hilbert $\mathscr{A}$-module $\mathscr{V}$, the linking algebra $\mathbb{L}(\mathscr{V})$ is defined as the matrix algebra of the form

$$\mathbb{L}(\mathscr{V})=\begin{bmatrix}\mathbb{K}(\mathscr{A}) & \mathbb{K}(\mathscr{V},\mathscr{A})\\ \mathbb{K}(\mathscr{A},\mathscr{V}) & \mathbb{K}(\mathscr{V})\end{bmatrix}.$$

Then $\mathbb{L}(\mathscr{V})$ has a canonical embedding as a closed subalgebra of the adjointable operators on the Hilbert $\mathscr{A}$-module $\mathscr{A}\oplus\mathscr{V}$ via

$$\begin{bmatrix}X & Y\\ Z & W\end{bmatrix}\begin{bmatrix}a\\ x\end{bmatrix}=\begin{bmatrix}Xa+Yx\\ Za+Wx\end{bmatrix}$$

which makes $\mathbb{L}(\mathscr{V})$ a C^*-algebra (cf. [15], Lemma 2.32 and Corollary 3.21). Each $x\in\mathscr{V}$ induces the maps $r_x\in\mathbb{B}(\mathscr{A},\mathscr{V})$ and $l_x\in\mathbb{B}(\mathscr{V},\mathscr{A})$ given by $r_x(a)=xa$ and $l_x(y)=\langle x,y\rangle$, respectively, such that $r_x^*=l_x$. The map $x\mapsto r_x$ is an isometric linear isomorphism of $\mathscr{V}$ to $\mathbb{K}(\mathscr{A},\mathscr{V})$ and $x\mapsto l_x$ is an isometric conjugate linear isomorphism of $\mathscr{V}$ to $\mathbb{K}(\mathscr{V},\mathscr{A})$. Further, every $a\in\mathscr{A}$ induces the map $T_a\in\mathbb{K}(\mathscr{A})$ given by $T_a(b)=ab$. The map $a\mapsto T_a$ defines an isomorphism between C^*-algebras $\mathscr{A}$ and $\mathbb{K}(\mathscr{A})$. Therefore, we may write

$$\mathbb{L}(\mathscr{V})=\left\{\begin{bmatrix}T_a & l_y\\ r_x & T\end{bmatrix}: a\in\mathscr{A}, x,y\in\mathscr{V}, T\in\mathbb{K}(\mathscr{V})\right\},$$

and identify the C^*-subalgebras of compact operators with the corresponding corners in the linking algebra: $\mathbb{K}(\mathscr{A})=\mathbb{K}(\mathscr{A}\oplus 0)\subseteq\mathbb{K}(\mathscr{A}\oplus\mathscr{V})=\mathbb{L}(\mathscr{V})$ and $\mathbb{K}(\mathscr{V})=\mathbb{K}(0\oplus\mathscr{V})\subseteq\mathbb{K}(\mathscr{A}\oplus\mathscr{V})=\mathbb{L}(\mathscr{V})$. We refer the reader to [10, 11] for more information on Hilbert C^*-modules and linking algebras.

Now, let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with inner product $[\cdot,\cdot]$. The numerical range of an element $A\in\mathbb{B}(\mathscr{H})$ is defined

$$W(A):=\{[A\xi,\xi]: \xi\in\mathscr{H}, \|\xi\|=1\}.$$

It is known that $W(A)$ is a nonempty bounded convex subset of $\mathbb{C}$ (not necessarily closed). This concept is useful in studying linear operators and have attracted the attention of many authors in the last few decades (e.g., see [8], and references therein). The numerical radius of A is given by

$$w(A)=\sup\{|[A\xi,\xi]|: \xi\in\mathscr{H}, \|\xi\|=1\}.$$

It is known that $w(\cdot)$ is a norm on $\mathbb{B}(\mathscr{H})$ and satisfies

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|$$

for each $A \in \mathbb{B}(\mathscr{H})$. Some generalizations of the numerical radius $A \in \mathbb{B}(\mathscr{H})$ can be found in [2, 22].

In the next section, we first utilize the linking algebra $\mathbb{L}(\mathscr{V})$ of a Hilbert $\mathscr{A}$-module $\mathscr{V}$ to introduce $\Phi(x)$ as a definition of numerical range for an arbitrary element $x \in \mathscr{V}$. We then use this set to define numerical radius of x and denote it by $\Omega(x)$. In particular, we show that $\Omega(\cdot)$ is a norm on $\mathscr{V}$, which is equivalent to the norm $\|\cdot\|$ and the following inequalities hold for every $x \in \mathscr{V}$:

$$\frac{1}{2}\|x\| \leqslant \Omega(x) \leqslant \|x\|. \tag{2}$$

We also establish an inequality that refines the first inequality in (2). In addition, we prove that $\Omega(x) = \frac{1}{2}\|x\|$ if and only if $\|x\| = \left\|\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}\right\|$ for all complex unit λ. Furthermore, for $x \in \mathscr{V}$ and $a \in \mathscr{A}$ we prove that

$$\Omega(xa \pm xa^*) \leqslant 2\|a \pm a^*\|\Omega(x).$$

We finally present a refinement of the triangle inequality for the norm $\Omega(\cdot)$.

2. Main results

We start our work with the following definition.

DEFINITION 1. Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. The numerical range of $x \in \mathscr{V}$ is defined as the set

$$\Phi(x) := \left\{\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) : \varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)\right\}.$$

Next, we present some properties of the numerical range in Hilbert C^*-modules.

THEOREM 1. *Let x and y be elements of a Hilbert $\mathscr{A}$-module $\mathscr{V}$ and let $\alpha \in \mathbb{C}$. Then*

(i) $\Phi(\alpha x) = \alpha\Phi(x)$ *(homogeneous).*

(ii) $\Phi(x+y) \subseteq \Phi(x) + \Phi(y)$ *(subadditive).*

(iii) $\Phi(x)$ *is a nonempty compact convex subset of* $\mathbb{C}$.

Proof. Let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. For every $a \in \mathscr{A}$, we have

$$r_{\alpha x}(a) = (\alpha x)a = \alpha(xa) = \big(\alpha r_x\big)(a)$$

and

$$r_{x+y}(a) = (x+y)a = xa + ya = \big(r_x + r_y\big)(a).$$

Hence $r_{\alpha x} = \alpha r_x$ and $r_{x+y} = r_x + r_y$. Thus (i) and (ii) follow easily from the definition.

We now prove (iii). Since the existence of states on $\mathbb{L}(\mathscr{V})$ is guaranteed by the Hahn–Banach theorem, we have $\Phi(x) \neq \emptyset$. The convexity of $\Phi(x)$ is an easy consequence of the fact that a convex combination of two states is also a state. As for the compactness, note that the set $\mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$ is a weak*-closed subset of the unit ball $\Big\{\varphi \in \mathbb{L}^{'}(\mathscr{V}) : \|\varphi\| \leqslant 1\Big\}$ of $\mathbb{L}^{'}(\mathscr{V})$. Since, by the Banach–Alaoglu theorem, the latter is weak*-compact, the same is true for $\mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$. Hence $\Phi(x)$, the image of the weak*-continuous mapping $\varphi \mapsto \varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)$ for $\varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$, is compact in $\mathbb{C}$. $\square$

REMARK 1. It is known that the set of all states of a unital C^*-algebra $\mathscr{A} \subseteq \mathbb{B}(\mathscr{H})$ is a weak*-closed convex hull of the set of all vector states of $\mathscr{A}$, i.e., the states of $\mathscr{A}$ of the form $A \to [A\xi, \xi]$ for some unit vector ξ in $\mathscr{H}$. Also, for the Hilbert module $\mathscr{V} = \mathbb{B}(\mathscr{H})$ over the C^*-algebra $\mathbb{B}(\mathscr{H})$ is well known to be valid $\mathbb{K}(\mathbb{B}(\mathscr{H})) = \mathbb{K}(\mathscr{V}, \mathbb{B}(\mathscr{H})) = \mathbb{K}(\mathbb{B}(\mathscr{H}), \mathscr{V}) = \mathbb{K}(\mathscr{V}) = \mathbb{B}(\mathscr{H})$ (see [5, Remark 1.13]), so all corners in the linking algebra $\mathbb{L}(\mathscr{V})$ are equal to $\mathbb{B}(\mathscr{H})$. Hence, for $A \in \mathbb{B}(\mathscr{H})$, we have $\Phi(A) = \overline{W(A)}$.

Now, we are in a position to introduce numerical radius for elements of a Hilbert C^*-module. Some other related topics can be found in [3, 6, 12, 16, 17, 19].

DEFINITION 2. Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. The numerical radius of an element $x \in \mathscr{V}$ is defined as

$$\Omega(x) := \sup\left\{\left|\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| : \varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)\right\}.$$

In the following theorem, we prove that $\Omega(\cdot)$ is a norm on Hilbert C^*-module $\mathscr{V}$, which is equivalent to the norm $\|\cdot\|$.

THEOREM 2. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module. Then $\Omega(\cdot)$ is a norm on $\mathscr{V}$ and the following inequalities hold for every $x \in \mathscr{V}$:*

$$\frac{1}{2}\|x\| \leqslant \Omega(x) \leqslant \|x\|.$$

Proof. Let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. Let $x \in \mathscr{V}$. Clearly, $\Omega(x) \geqslant 0$. Let us now suppose $\Omega(x) = 0$. Then, by Definition 2, $\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} = 0$. Since $\left\| \begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} \right\| = \|x\|$, we get $\|x\| = 0$ and therefore, $x = 0$. Further, by Theorem 1 (i)-(ii), for $y, z \in \mathscr{V}$ and $\alpha \in \mathbb{C}$ we have $\Omega(\alpha y) = |\alpha|\Omega(y)$ and $\Omega(y+z) \leqslant \Omega(y) + \Omega(z)$. Thus $\Omega(\cdot)$ is a norm on $\mathscr{V}$.

On the other hands, for every $\varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$, we have

$$\left|\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| \leqslant \left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\| = \|x\|.$$

So, by taking the supremum over $\varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$ in the above inequality, we deduce that

$$\Omega(x) \leqslant \|x\|. \tag{3}$$

Now let $\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix} = \mathrm{Re}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) + i\,\mathrm{Im}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)$ be the Cartesian decomposition of $\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}$. By [13, Theorem 3.3.6], there exist $\varphi_1, \varphi_2 \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$ such that

$$\left|\varphi_1\left(\mathrm{Re}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| = \left\|\mathrm{Re}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right\| \tag{4}$$

and

$$\left|\varphi_2\left(\mathrm{Im}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| = \left\|\mathrm{Im}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right\|. \tag{5}$$

Therefore, by (4) and (5), we have

$$\begin{aligned}
\frac{1}{2}\|x\| &= \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\| \\
&\leqslant \frac{1}{2}\left\|\mathrm{Re}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right\| + \frac{1}{2}\left\|\mathrm{Im}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left|\varphi_1\left(\mathrm{Re}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| + \frac{1}{2}\left|\varphi_2\left(\mathrm{Im}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| \\
&= \frac{1}{4}\left|\varphi_1\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) + \overline{\varphi_1}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| + \frac{1}{4}\left|\varphi_2\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) - \overline{\varphi_2}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| \\
&\leqslant \frac{1}{2}\left|\varphi_1\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| + \frac{1}{2}\left|\varphi_2\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| \leqslant \frac{1}{2}\Omega(x) + \frac{1}{2}\Omega(x) = \Omega(x),
\end{aligned}$$

whence

$$\frac{1}{2}\|x\| \leqslant \Omega(x). \tag{6}$$

From (3) and (6), we deduce the desired result. $\square$

For $A \in \mathbb{B}(\mathscr{H})$, we note that (see [20]) $w(A) = \sup_{\lambda\in\mathbb{T}} \|\mathrm{Re}(\lambda A)\|$. Here, as usual, $\mathbb{T}$ is the unit circle of the complex plane $\mathbb{C}$. This motivates the following result.

THEOREM 3. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. Then*

$$\Omega(x) = \frac{1}{2}\sup_{\lambda\in\mathbb{T}} \left\| \begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix} \right\|,$$

for every $x \in \mathscr{V}$.

Proof. Let $x \in \mathscr{V}$. First, we show that

$$\sup_{\lambda\in\mathbb{T}} \left| \mathrm{Re}\left(\lambda\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| = \left|\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| \tag{7}$$

for every $\varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$.

Let $\varphi \in \mathscr{S}\big(\mathbb{L}(\mathscr{V})\big)$. We may assume that $\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) \neq 0$, otherwise (7) trivially holds. Put

$$\lambda_0 = \frac{\overline{\varphi}\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)}{\left|\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right|}.$$

Then we have

$$\begin{aligned} \left|\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| &= \left|\mathrm{Re}\left(\lambda_0\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| \\ &\leqslant \sup_{\lambda\in\mathbb{T}} \left|\mathrm{Re}\left(\lambda\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right)\right| \\ &\leqslant \sup_{\lambda\in\mathbb{T}} \left|\lambda\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right| = \left|\varphi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\right|, \end{aligned}$$

and hence (7) holds.

Now, since $\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}$ is self adjoint for any $\lambda \in \mathbb{T}$, by [13, Theorem 3.3.6], we obtain

$$\left\| \begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix} \right\| = \sup_{\varphi\in\mathscr{S}(\mathbb{L}(\mathscr{V}))} \left|\varphi\left(\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}\right)\right|. \tag{8}$$

Therefore,

$$\begin{aligned}
\sup_{\lambda\in\mathbb{T}}\left\|\begin{bmatrix}0 & \overline{\lambda}l_x\\ \lambda r_x & 0\end{bmatrix}\right\| &\overset{(8)}{=} \sup_{\lambda\in\mathbb{T}}\sup_{\varphi\in\mathscr{S}(\mathbb{L}(\mathscr{V}))}\left|\varphi\left(\begin{bmatrix}0 & \overline{\lambda}l_x\\ \lambda r_x & 0\end{bmatrix}\right)\right|\\
&= 2\sup_{\lambda\in\mathbb{T}}\sup_{\varphi\in\mathscr{S}(\mathbb{L}(\mathscr{V}))}\left|\varphi\left(\mathrm{Re}\left(\lambda\begin{bmatrix}0 & 0\\ r_x & 0\end{bmatrix}\right)\right)\right|\\
&= 2\sup_{\lambda\in\mathbb{T}}\sup_{\varphi\in\mathscr{S}(\mathbb{L}(\mathscr{V}))}\left|\mathrm{Re}\left(\lambda\varphi\left(\begin{bmatrix}0 & 0\\ r_x & 0\end{bmatrix}\right)\right)\right|\\
&= 2\sup_{\varphi\in\mathscr{S}(\mathbb{L}(\mathscr{V}))}\sup_{\lambda\in\mathbb{T}}\left|\mathrm{Re}\left(\lambda\varphi\left(\begin{bmatrix}0 & 0\\ r_x & 0\end{bmatrix}\right)\right)\right|\\
&\overset{(7)}{=} 2\sup_{\varphi\in\mathscr{S}(\mathbb{L}(\mathscr{V}))}\left|\varphi\left(\begin{bmatrix}0 & 0\\ r_x & 0\end{bmatrix}\right)\right| = 2\Omega(x).
\end{aligned}$$

Thus

$$\frac{1}{2}\sup_{\lambda\in\mathbb{T}}\left\|\begin{bmatrix}0 & \overline{\lambda}l_x\\ \lambda r_x & 0\end{bmatrix}\right\| = \Omega(x). \quad \square$$

We can obtain a refinement of inequality (6) as follows.

THEOREM 4. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. For $x\in\mathscr{V}$ the following inequality holds:*

$$\frac{1}{8}\left(4\|x\|+2|\Gamma-\Gamma'|+\Delta+\Delta'\right)\leqslant\Omega(x),$$

where $\Gamma=\max\left\{\|x\|,\left\|\begin{bmatrix}0 & l_x\\ r_x & 0\end{bmatrix}\right\|\right\}$, $\Gamma'=\max\left\{\|x\|,\left\|\begin{bmatrix}0 & -l_x\\ r_x & 0\end{bmatrix}\right\|\right\}$, $\Delta=\left|\|x\|-\left\|\begin{bmatrix}0 & l_x\\ r_x & 0\end{bmatrix}\right\|\right|$ *and* $\Delta'=\left|\|x\|-\left\|\begin{bmatrix}0 & -l_x\\ r_x & 0\end{bmatrix}\right\|\right|$.

Proof. Since $\Omega(x)=\frac{1}{2}\sup_{\lambda\in\mathbb{T}}\left\|\begin{bmatrix}0 & \overline{\lambda}l_x\\ \lambda r_x & 0\end{bmatrix}\right\|$, by taking $\lambda=1$ and $\lambda=i$, we have

$$\Omega(x)\geqslant\frac{1}{2}\left\|\begin{bmatrix}0 & l_x\\ r_x & 0\end{bmatrix}\right\| \quad\text{and}\quad \Omega(x)\geqslant\frac{1}{2}\left\|\begin{bmatrix}0 & -l_x\\ r_x & 0\end{bmatrix}\right\|. \tag{9}$$

So, by (6) and (9) we have $\Omega(x)\geqslant\frac{1}{2}\max\{\Gamma,\Gamma'\}$. Therefore,

$$\begin{aligned}
\Omega(x) &\geqslant \frac{\Gamma+\Gamma'}{4}+\frac{|\Gamma-\Gamma'|}{4}\\
&= \frac{1}{4}\left(\frac{1}{2}\left(\|x\|+\left\|\begin{bmatrix}0 & l_x\\ r_x & 0\end{bmatrix}\right\|\right)+\frac{1}{2}\Delta\right)\\
&\qquad+\frac{1}{4}\left(\frac{1}{2}\left(\|x\|+\left\|\begin{bmatrix}0 & -l_x\\ r_x & 0\end{bmatrix}\right\|\right)+\frac{1}{2}\Delta'\right)+\frac{|\Gamma-\Gamma'|}{4}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left(\left\|\begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & -l_x \\ r_x & 0 \end{bmatrix}\right\|\right) + \frac{1}{4}\|x\| + \frac{\Delta+\Delta'}{8} + \frac{|\Gamma-\Gamma'|}{4} \\
&\geqslant \frac{1}{8}\left\|\begin{bmatrix} 0 & l_x \\ r_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -l_x \\ r_x & 0 \end{bmatrix}\right\| + \frac{1}{4}\|x\| + \frac{\Delta+\Delta'}{8} + \frac{|\Gamma-\Gamma'|}{4} \\
&= \frac{1}{4}\left\|\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right\| + \frac{1}{4}\|x\| + \frac{\Delta+\Delta'}{8} + \frac{|\Gamma-\Gamma'|}{4} \\
&= \frac{1}{4}\|x\| + \frac{1}{4}\|x\| + \frac{\Delta+\Delta'}{8} + \frac{|\Gamma-\Gamma'|}{4} \\
&= \frac{1}{2}\|x\| + \frac{\Delta+\Delta'}{8} + \frac{|\Gamma-\Gamma'|}{4}.
\end{aligned}
$$

Thus

$$\frac{1}{2}\|x\| + \frac{\Delta+\Delta'}{8} + \frac{|\Gamma-\Gamma'|}{4} \leqslant \Omega(x). \quad \square$$

In the following result, we state a necessary and sufficient condition for the equality case in the inequality (6).

COROLLARY 1. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. Let $x \in \mathscr{V}$. Then $\Omega(x) = \frac{1}{2}\|x\|$ if and only if $\|x\| = \left\|\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}\right\|$ for all $\lambda \in \mathbb{T}$.*

Proof. Let us first suppose that $\Omega(x) = \frac{1}{2}\|x\|$. For every $\lambda \in \mathbb{T}$ then we have $\Omega(\lambda x) = \frac{1}{2}\|\lambda x\|$. Therefore, by Theorem 4, we obtain

$$\Delta = \left|\|\lambda x\| - \left\|\begin{bmatrix} 0 & l_{\lambda x} \\ r_{\lambda x} & 0 \end{bmatrix}\right\|\right| = 0.$$

From this it follows that $\|x\| = \left\|\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}\right\|$.

Conversely, if $\|x\| = \left\|\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}\right\|$ for all $\lambda \in \mathbb{T}$, then

$$\frac{1}{2}\sup_{\lambda \in \mathbb{T}}\left\|\begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}\right\| = \frac{1}{2}\|x\|,$$

and so, by Theorem 3, $\Omega(x) = \frac{1}{2}\|x\|$. $\square$

For every $a \in \mathscr{A}$ and $x \in \mathscr{V}$, by the inequalities (3) and (6), we have

$$\Omega(xa + xa^*) \leqslant \|xa + xa^*\| \leqslant 2\|a\|\|x\| \leqslant 4\|a\|\Omega(x),$$

and hence

$$\Omega(xa + xa^*) \leqslant 4\|a\|\Omega(x). \tag{10}$$

In the following theorem, we improve the inequality (10).

THEOREM 5. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module. Let $a \in \mathscr{A}$ and $x \in \mathscr{V}$. Then*

$$\Omega(xa + xa^*) \leqslant 2\|a + a^*\|\Omega(x).$$

Proof. Let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. For every $b \in \mathscr{A}$ and $y \in \mathscr{V}$, we have

$$r_{xa}(b) = (xa)b = x(ab) = x(T_a(b)) = r_x T_a(b)$$

and

$$l_{xa}(y) = \langle xa, y\rangle = a^*\langle x, y\rangle = a^*(l_x(y)) = T_{a^*} l_x(y).$$

Hence $r_{xa} = r_x T_a$ and $l_{xa} = T_{a^*} l_x$. Now, let $\lambda \in \mathbb{T}$. Therefore,

$$\begin{aligned}
\left\| \begin{bmatrix} 0 & \overline{\lambda} l_{(xa+xa^*)} \\ \lambda r_{(xa+xa^*)} & 0 \end{bmatrix} \right\| &= \left\| \begin{bmatrix} 0 & \overline{\lambda}(T_{a^*} l_x + T_a l_x) \\ \lambda(r_x T_a + r_x T_{a^*}) & 0 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 0 & \overline{\lambda} T_{a+a^*} l_x \\ \lambda r_x T_{a+a^*} & 0 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix} \begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix} \right\| \\
&\leqslant 2 \left\| \begin{bmatrix} T_{a+a^*} & 0 \\ 0 & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix} \right\| \\
&\leqslant 4\|a + a^*\|\Omega(x),
\end{aligned}$$

and so

$$\frac{1}{2} \left\| \begin{bmatrix} 0 & \overline{\lambda} l_{(xa+xa^*)} \\ \lambda r_{(xa+xa^*)} & 0 \end{bmatrix} \right\| \leqslant 2\|a + a^*\|\Omega(x).$$

Taking the supremum over $\lambda \in \mathbb{T}$ in the above inequality, we deduce that

$$\Omega(xa + xa^*) \leqslant 2\|a + a^*\|\Omega(x). \quad \square$$

As an immediate consequence of Theorem 5, we have the following result.

COROLLARY 2. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $a \in \mathscr{A}$ and $x \in \mathscr{V}$. If $xa = xa^*$, then*

$$\Omega(xa) \leqslant \|a + a^*\|\Omega(x).$$

REMARK 2. Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $a \in \mathscr{A}$ and $x \in \mathscr{V}$. Replace a by ia in Theorem 5, to obtain $\Omega(xa - xa^*) \leqslant 2\|a - a^*\|\Omega(x)$. Thus

$$\Omega(xa \pm xa^*) \leqslant 2\|a \pm a^*\|\Omega(x).$$

In what follows, $r(a)$ stands for the spectral radius of an arbitrary element a in a C^*-algebra $\mathscr{A}$. It is well known that for every $a \in \mathscr{A}$, we have $r(a) \leqslant \|a\|$ and that equality holds in this inequality if a is normal. The following lemma gives us a spectral radius inequality for sums of elements in C^*-algebras.

LEMMA 1. [21, Lemma 3.5] *Let $\mathscr{A}$ be a C^*-algebra and let $a,b \in \mathscr{A}$. Then*

$$r(a+b) \leqslant \left\| \begin{bmatrix} \|a\| & \|ab\|^{1/2} \\ \|ab\|^{1/2} & \|b\| \end{bmatrix} \right\|.$$

Now, we present a refinement of the triangle inequality for the numerical radius in Hilbert C^*-modules. We use some ideas of [1, Theorem 3.4]. We refer the reader to [4, 7, 14, 18] for more information on the triangle inequality.

THEOREM 6. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module and let $\mathbb{L}(\mathscr{V})$ be the linking algebra of $\mathscr{V}$. Let $x,y \in \mathscr{V}$. Then*

$$\Omega(x+y) \leqslant \left\| \begin{bmatrix} \Omega(x) & \frac{1}{2}\left\| \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix} \right\|^{1/2} \\ \frac{1}{2}\left\| \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix} \right\|^{1/2} & \Omega(y) \end{bmatrix} \right\| \leqslant \Omega(x)+\Omega(y).$$

Proof. Let $\lambda \in \mathbb{T}$. Put $a = \begin{bmatrix} 0 & \overline{\lambda} l_x \\ \lambda r_x & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & \overline{\lambda} l_y \\ \lambda r_y & 0 \end{bmatrix}$. Then

$$\|a\| \leqslant 2\Omega(x) \qquad \text{and} \qquad \|b\| \leqslant 2\Omega(y).$$

Also, for every $c \in \mathscr{A}$ and $z \in \mathscr{V}$, we have

$$l_x r_y(c) = l_x(yc) = \langle x, yc\rangle = \langle x,y\rangle c = T_{\langle x,y\rangle}(c)$$

and

$$r_x l_y(z) = r_x(\langle y,z\rangle) = x\langle y,z\rangle = \theta_{x,y}(z).$$

Thus $l_x r_y = T_{\langle x,y\rangle}$ and $r_x l_y = \theta_{x,y}$. Therefore, $ab = \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}$ and hence,

$$\left\| \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix} \right\| = \|ab\| \leqslant \|a\|\|b\| \leqslant 4\Omega(x)\Omega(y). \tag{11}$$

Since $\begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix}$ is a self adjoint element of C^*-algebra $\mathbb{L}(\mathscr{V})$, we have

$$\left\| \begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix} \right\| = r\left(\begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix} \right).$$

Therefore, by Lemma 1, we obtain

$$\left\|\begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix}\right\| = r\left(\begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix}\right) = r(a+b) \leqslant \left\|\begin{bmatrix} \|a\| & \|ab\|^{1/2} \\ \|ab\|^{1/2} & \|b\| \end{bmatrix}\right\|.$$

So, by the norm monotonicity of matrices with nonnegative entries (see, e.g., [9, p. 491]), we get

$$\left\|\begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix}\right\| \leqslant \left\|\begin{bmatrix} \sup\limits_{\lambda\in\mathbb{T}}\|a\| & \sup\limits_{\lambda\in\mathbb{T}}\|ab\|^{1/2} \\ \sup\limits_{\lambda\in\mathbb{T}}\|ab\|^{1/2} & \sup\limits_{\lambda\in\mathbb{T}}\|b\| \end{bmatrix}\right\| = \left\|\begin{bmatrix} 2\Omega(x) & \left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} \\ \left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} & 2\Omega(y) \end{bmatrix}\right\|.$$

Therefore, for every $\lambda \in \mathbb{T}$ we have

$$\frac{1}{2}\left\|\begin{bmatrix} 0 & \overline{\lambda} l_{(x+y)} \\ \lambda r_{(x+y)} & 0 \end{bmatrix}\right\| \leqslant \left\|\begin{bmatrix} \Omega(x) & \frac{1}{2}\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} \\ \frac{1}{2}\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} & \Omega(y) \end{bmatrix}\right\|,$$

whence

$$\Omega(x+y) \leqslant \left\|\begin{bmatrix} \Omega(x) & \frac{1}{2}\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} \\ \frac{1}{2}\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} & \Omega(y) \end{bmatrix}\right\|. \tag{12}$$

On the other hand, by (11), we have

$$\begin{aligned} &\left\|\begin{bmatrix} \Omega(x) & \frac{1}{2}\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} \\ \frac{1}{2}\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|^{1/2} & \Omega(y) \end{bmatrix}\right\| \\ &= \frac{1}{2}\left(\Omega(x)+\Omega(y)+\sqrt{(\Omega(x)-\Omega(y))^2+\left\|\begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix}\right\|}\right) \qquad (13) \\ &\leqslant \frac{1}{2}\left(\Omega(x)+\Omega(y)+\sqrt{(\Omega(x)-\Omega(y))^2+4\Omega(x)\Omega(y)}\right) = \Omega(x)+\Omega(y). \end{aligned}$$

Thus

$$\left\| \begin{bmatrix} \Omega(x) & \frac{1}{2}\left\| \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix} \right\|^{1/2} \\ \frac{1}{2}\left\| \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix} \right\|^{1/2} & \Omega(y) \end{bmatrix} \right\| \leqslant \Omega(x)+\Omega(y),$$

and the proof is completed. □

As a consequence of Theorem 6, we have the following result.

COROLLARY 3. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module, and $x,y\in\mathscr{V}$. If $\Omega(x+y)=\Omega(x)+\Omega(y)$, then*

$$\Omega(x)\Omega(y)=\frac{1}{4}\left\| \begin{bmatrix} T_{\langle x,y\rangle} & 0 \\ 0 & \theta_{x,y} \end{bmatrix} \right\|.$$

In particular, $\Omega(x)=\frac{1}{2}\left\| \begin{bmatrix} T_{\langle x,x\rangle} & 0 \\ 0 & \theta_{x,x} \end{bmatrix} \right\|^{1/2}$.

The following lemma must be known to specialists. For the sake of completeness we include the proof.

LEMMA 2. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module, and $x,y\in\mathscr{V}$. Then*

$$\|\theta_{x,y}\|=\left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|.$$

Proof. We may assume that $x,y\neq 0$ otherwise the identity trivially holds. We have

$$\begin{aligned}\left\|\theta_{x,y}\left(\frac{y\langle x,x\rangle^{1/2}}{\left\|y\langle x,x\rangle^{1/2}\right\|}\right)\right\|^2 &= \frac{\left\|x\langle y,y\rangle\langle x,x\rangle^{1/2}\right\|^2}{\left\|y\langle x,x\rangle^{1/2}\right\|^2} \\ &= \frac{\left\|\langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle\langle y,y\rangle\langle x,x\rangle^{1/2}\right\|}{\left\|\langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle^{1/2}\right\|} \\ &= \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle^{1/2}\right\| = \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|^2,\end{aligned}$$

and so

$$\left\|\theta_{x,y}\left(\frac{y\langle x,x\rangle^{1/2}}{\left\|y\langle x,x\rangle^{1/2}\right\|}\right)\right\| = \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|.$$

Hence

$$\left\|\theta_{x,y}\right\| \geqslant \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|. \tag{14}$$

On the other hand, let $z\in\mathscr{V}$ with $\|z\|=1$. By (1) we have $\langle y,z\rangle\langle z,y\rangle \leqslant \langle y,y\rangle$ and hence by Theorem 2.2.5(2) of [13] it follows that

$$\langle x,x\rangle^{1/2}\langle y,z\rangle\langle z,y\rangle\langle x,x\rangle^{1/2} \leqslant \langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle^{1/2}.$$

So, [13, Theorem 2.2.5(3)] implies

$$\left\|\langle x,x\rangle^{1/2}\langle y,z\rangle\langle z,y\rangle\langle x,x\rangle^{1/2}\right\| \leqslant \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle^{1/2}\right\|. \tag{15}$$

Therefore,

$$\begin{aligned}
\left\|\theta_{x,y}(z)\right\| &= \|x\langle y,z\rangle\| \\
&= \|\langle z,y\rangle\langle x,x\rangle\langle y,z\rangle\|^{1/2} \\
&= \left\|\langle x,x\rangle^{1/2}\langle y,z\rangle\langle z,y\rangle\langle x,x\rangle^{1/2}\right\|^{1/2} \\
&\overset{(15)}{\leqslant} \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle\langle x,x\rangle^{1/2}\right\|^{1/2} = \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|,
\end{aligned}$$

whence

$$\left\|\theta_{x,y}\right\| \leqslant \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|. \tag{16}$$

Utilizing (14) and (16), we conclude that $\left\|\theta_{x,y}\right\| = \left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|$. $\square$

We close this paper with the following result.

COROLLARY 4. *Let $\mathscr{V}$ be a Hilbert $\mathscr{A}$-module, and $x,y\in\mathscr{V}$. If $\langle x,y\rangle=0$, then*

$$\begin{aligned}
\Omega(x+y) &\leqslant \frac{1}{2}\left(\Omega(x)+\Omega(y)+\sqrt{(\Omega(x)-\Omega(y))^2+\left\|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right\|}\right) \\
&\leqslant \Omega(x)+\Omega(y).
\end{aligned}$$

Proof. Since $\langle x,y\rangle=0$, we have $T_{\langle x,y\rangle}=0$. Hence from (12), (13) and Lemma 2 we deduce the desired result. $\square$

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Ali Zamani
School of Mathematics and Computer Sciences
Damghan University
Damghan, P.O. BOX 36715-364, Iran
e-mail: zamani.ali85@yahoo.com