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ORLICZ DUAL LOGARITHMIC MINKOWKI INEQUALITY

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Abstract. In this paper, we establish an Orlicz dual logarithmic Minkowski inequality by introducing a new concept of Orlicz dual mixed volume measure, and using the newly established Orlicz dual Minkowski inequality. The Orlicz dual logarithmic Minkowski inequality in special case yields the dual logarithmic Minkowski inequality. The L_p-dual mixed volume measure and L_p-dual logarithmic Minkowski inequality are first derived here.

1. Introduction

In 2016, Stancu [18] established the following logarithmic Minkowski inequality.

The logarithmic Minkowski inequality. *If K and L are convex bodies in $\mathbb{R}^n$ that containing the origin in their interior, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{v}_1 \geqslant \frac{1}{n}\ln\left(\frac{V(K)}{V(L)}\right), \tag{1.1}$$

with equality if and only if K and L are homothetic, where dv_1 is the mixed volume measure $dv_1 = \frac{1}{n}h_K dS_L$, and $d\overline{v}_1 = \frac{1}{V_1(L,K)}dv_1$ is its normalization, and $V_1(L,K)$ denotes the usual mixed volume of L and K, is defined by (*see e.g.* [2])

$$V_1(L,K) = \frac{1}{n}\int_{S^{n-1}} h_K dS_L,$$

and the functions h_K, h_L are the support functions.

If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then ([17])

$$h_K = \max\{x\cdot y : y\in K\},$$

for $x\in\mathbb{R}^n$, defines the support function h_K of K.

In 2017, Wang, Xu and Zhou [22] proved the following dual logarithmic Minkowski inequality, which is a special case $p=1$ of L_p-dual logarithmic Minkowski inequality in [22].

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The dual logarithmic Minkowski inequality. *If K and L are star bodies about the origin in $\mathbb{R}^n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\bar{V}_{-1}(L,K) \leqslant \frac{1}{n}\ln\left(\frac{V(K)}{V(L)}\right), \tag{1.2}$$

with equality if and only if K and L are dilates, and where

$$dV_{-1}(L,K) = \frac{1}{n}\rho(K,u)^{-1}\rho(L,u)^{n+1}$$

is the mixed radial cone volume measure of K and L, and

$$d\bar{V}_{-1}(L,K) = \frac{1}{\tilde{V}_{-1}(L,K)} dV_{-1}(L,K)$$

is its normalization, and $\tilde{V}_{-1}(L,K)$ denotes the dual mixed volume of L and K, is defined by (*see* [10])

$$\tilde{V}_{-1}(L,K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{-1}\rho(L,u)^{n+1} dS(u).$$

Here, $\rho(K,\cdot)$ denotes the radial function of star body K. The radial function of star body K is defined by (see [2])

$$\rho(K,u) = \max\{c \geqslant 0 : cu \in K\},$$

for $u \in S^{n-1}$.

Recently, the logarithmic Minkowski inequality and its dual form have attracted extensive attention and research, and the recent research can be found in the references [1], [3], [4], [7], [8], [9], [14], [15], [16], [19], [21], [23], [24], [25] and [26]. In the paper, we generalize the dual logarithmic Minkowski inequality (1.2) to the Orlicz space, and establish the following Orlicz dual logarithmic Minkowski inequality.

Orlicz dual logarithmic Minkowski inequality. *Let $\phi : (0,\infty) \to (0,\infty)$ be a convex and decreasing function such that $\phi(0) = \infty$, $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to 0}\phi(t) = \infty$. If K and L are star bodies about the origin in $\mathbb{R}^n$, then*

$$\int_{S^{n-1}} \ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right) d\widetilde{V}_{\phi}(L,K) \geqslant \ln\left(\phi\left(\left(\frac{V(K)}{V(L)}\right)^{1/n}\right)\right). \tag{1.3}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates, where $d\widetilde{V}_{\phi}(L,K)$ denotes a new Orlicz dual mixed volume probability measure of star bodies L and K, is defined by (see Section 3)

$$d\widetilde{V}_{\phi}(L,K) = \frac{1}{n\widetilde{V}_{\phi}(L,K)}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\rho(L,u)^n dS(u), \tag{1.4}$$

and $\widetilde{V}_\phi(L,K)$ *denotes the Orlicz dual mixed volume, is defined by* (*see* [27])

$$\widetilde{V}_\phi(L,K)=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\rho(L,u)^n dS(u). \tag{1.5}$$

Obviously, when $\phi(t)=\frac{1}{t}$, (1.3) becomes the dual logarithmic Minkowski inequality (1.2). On the other hand, when $\phi(t)=t^{-p}$ and $p\geqslant 1$, (1.3) becomes the following L_p-dual logarithmic Minkowski inequality.

The L_p-dual logarithmic Minkowski inequality. *If K and L are star bodies about the origin in $\mathbb{R}^n$ and $p\geqslant 1$, then*

$$\int_{S^{n-1}}\ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right)d\widetilde{V}_{-p}(L,K)\leqslant\frac{1}{n}\ln\left(\frac{V(K)}{V(L)}\right). \tag{1.6}$$

with equality if and only if K and L are dilates, where $d\widetilde{V}_{-p}(L,K)$ denotes the L_p-dual mixed volume probability measure of K and L, is defined by (see Section 3)

$$d\widetilde{V}_{-p}(L,K)=\frac{1}{n\widetilde{V}_{-p}(L,K)}\rho(K,u)^{-p}\rho(L,u)^{n+p}dS(u), \tag{1.7}$$

where $\widetilde{V}_{-p}(L,K)$ denotes the well-known L_p-dual mixed volume, is defined by ([10])

$$\widetilde{V}_{-p}(L,K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{-p}\rho(L,u)^{n+p}dS(u).$$

Apparently, when $p=1$, (1.6) becomes the dual logarithmic Minkowski inequality (1.2).

2. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K\subset\mathbb{R}^n$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. The unit ball in $\mathbb{R}^n$ and its surface are denoted by B and S^{n-1}, respectively. Let $\mathscr{K}^n$ denote the class of nonempty compact convex subsets containing the origin in their interiors in $\mathbb{R}^n$. Let $\phi:(0,\infty)\to(0,\infty)$ be a convex and decreasing function such that $\phi(0)=\infty$, $\lim_{t\to\infty}\phi(t)=0$ and $\lim_{t\to 0}\phi(t)=\infty$ and let $\mathscr{C}$ denote the class of the convex and decreasing functions ϕ. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K,\cdot)$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathscr{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. Two star bodies K and L are dilates if $\rho(K,u)/\rho(L,u)$ is independent of $u\in S^{n-1}$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K,L\in\mathscr{S}^n$, then

$$\tilde{\delta}(K,L)=|\rho(K,u)-\rho(L,u)|_\infty.$$

The formula for the volume of a compact star shaped set in $\mathbb{R}^n$ in hyperspherical coordinates is

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u).$$

2.1. The operations between star bodies

The Orlicz harmonic radial addition $K\widehat{+}_\phi L$ of two star bodies K and L, is defined by (see [27])

$$\rho(K\widehat{+}_\phi L,u)) = \sup\left\{\lambda > 0 : \phi\left(\frac{\rho(K,u)}{\lambda}\right) + \phi\left(\frac{\rho(L,u)}{\lambda}\right) \leqslant \phi(1)\right\}, \tag{2.1}$$

where $u \in S^{n-1}$, and $\phi : (0,\infty) \to (0,\infty)$ is a convex and decreasing function such that $\phi(0) = \infty$, $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to 0}\phi(t) = \infty$. Let $\mathscr{C}$ denote the class of the convex and decreasing functions ϕ with $\phi(0) = \infty$, $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to 0}\phi(t) = \infty$.

If $\phi(t) = t^{-p}$ and $p \geqslant 1$, then the Orlicz harmonic radial addition $+_\phi$ becomes the following p-harmonic radial addition. If K,L are star bodies, the p-harmonic radial addition, is defined by (see [13])

$$\rho(K\widehat{+}_p L,x)^{-p} = \rho(K,x)^{-p} + \rho(L,x)^{-p}, \tag{2.2}$$

for $p \geqslant 1$ and $x \in \mathbb{R}^n$. When $\phi(t) = \frac{1}{t}$, the Orlicz harmonic radial addition $+_\phi$ becomes the classical harmonic radial addition, is defined by (see [12])

$$\rho(K\widehat{+}L,x)^{-1} = \rho(K,x)^{-1} + \rho(L,x)^{-1}, \tag{2.3}$$

for $x \in \mathbb{R}^n$.

For any $p \neq 0$, the p-radial addition $K\widetilde{+}_p L$ is defined by (see [6])

$$\rho(K\widetilde{+}_p L,x)^p = \rho(K,x)^p + \rho(L,x)^p, \tag{2.4}$$

for $x \in \mathbb{R}^n$ and $K,L \in \mathscr{S}^n$. Obviously, when $p = 1$, the p-radial addition $\widetilde{+}_p$ becomes the well-known radial addition $\widetilde{+}$, is defined by (see [11])

$$\rho(K\widetilde{+}L,x) = \rho(K,x) + \rho(L,x). \tag{2.5}$$

2.2. The dual mixed volumes

The Orlicz dual mixed volume with respect to the Orlicz harmonic radial addition is denoted by $\widetilde{V}_\phi(K,L)$, is defined by (see [27])

$$\begin{aligned}\widetilde{V}_\phi(K,L) &:= \frac{\phi_r'(1)}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_\phi \varepsilon\cdot L) - V(K)}{\varepsilon}\\ &= \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^n dS(u),\end{aligned} \tag{2.6}$$

where the right derivative of a real-valued function ϕ is denoted by ϕ_r' and $K+_\phi \varepsilon\cdot L$ is the Orlicz linear combination of K and L (see [27]).

When $\phi(t)=t^{-p}$ and $p\geqslant 1$, $\widetilde{V}_\phi(K,L)$ becomes the L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ with respect to the p-harmonic radial addition, is defined by (see [5])

$$\begin{aligned}\widetilde{V}_{-p}(K,L)&=-\frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_p\varepsilon\cdot L)-V(K)}{\varepsilon}\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n+p}\rho(L,u)^{-p}dS(u).\end{aligned}\tag{2.7}$$

where $K,L\in\mathscr{S}^n$ and $p\geqslant 1$. When $\phi(t)=\frac{1}{t}$, $\widetilde{V}_\phi(K,L)$ becomes the dual mixed volume $\widetilde{V}_{-1}(K,L)$ with respect to the harmonic radial addition, is defined by (see [12])

$$\begin{aligned}\widetilde{V}_{-1}(K,L)&=\lim_{\varepsilon\to 0^+}\frac{V(K)-V(K\widehat{+}\varepsilon\cdot L)}{\varepsilon}\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n+1}\rho(L,u)^{-1}dS(u),\end{aligned}\tag{2.8}$$

where $\widehat{+}$ is the harmonic radial addition. Obviously, when $K=L$, $\widetilde{V}_{-1}(K,L)$ becomes $V(K)$.

On the other hand, the first dual mixed volume with respect to the radial addition $\widetilde{V}_1(K,L)$, is defined by (see [11])

$$\begin{aligned}\widetilde{V}_1(K,L)&=\frac{1}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}\varepsilon\cdot L)-V(K)}{\varepsilon}\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-1}\rho(L,u)dS(u),\end{aligned}\tag{2.9}$$

where $K,L\in\mathscr{S}^n$. The dual mixed quermassintegral of star bodies K and L, $\widetilde{W}_i(K,L)$, is defined by

$$\begin{aligned}\widetilde{W}_i(K,L)&=\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i-1}\rho(L,u)dS(u),\end{aligned}\tag{2.10}$$

where $K,L\in\mathscr{S}^n$ and $0\leqslant i<n$. Obviously, when $K=L$, $\widetilde{V}_1(K,L)$ becomes $V(K)$.

2.3. The dual Minkowski inequalities

Orlicz dual Minkowski inequality for the Orlicz dual volumes is the following: If $K,L\in\mathscr{S}^n$ and $\phi\in\mathscr{C}$, then (see [27])

$$\widetilde{V}_\phi(K,L)\geqslant V(K)\cdot\phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right),\tag{2.11}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Obviously, when $\phi(t) = t^{-p}$ and $p \geqslant 1$, (2.11) becomes the L_p-dual Minkowski inequality (see [10]): If $K, L \in \mathscr{S}^n$ and $p \geqslant 1$, then

$$\widetilde{V}_{-p}(K,L)^n \geqslant V(K)^{n+p} V(L)^{-p}, \tag{2.12}$$

with equality if and only if K and L are dilates. When $\phi(t) = \frac{1}{t}$, (2.11) becomes the dual Minkowski inequality (see [12]):

$$\widetilde{V}_{-1}(K,L)^n \geqslant V(K)^{n+1} V(L)^{-1}, \tag{2.13}$$

with equality if and only if K and L are dilates.

The fundamental inequality for dual mixed quermassintegral stated that: If $K, L \in \mathscr{S}^n$ and $0 \leqslant i < n$, then

$$\widetilde{W}_i(K,L)^{n-i} \leqslant \widetilde{W}_i(K)^{n-1-i} \widetilde{W}_i(L), \tag{2.14}$$

with equality if and only if K and L are dilates. When $i = 0$, (2.14) becomes the Minkowski inequality for first dual mixed volume is the following: If $K, L \in \mathscr{S}^n$, then (see [11])

$$\widetilde{V}_1(K,L)^n \leqslant V(K)^{n-1} V(L), \tag{2.15}$$

with equality if and only if K and L are dilates.

3. Orlicz dual logarithmic Minkowski inequality

In the section, in order to derive the Orlicz dual logarithmic Minkowski inequality, we need to define some new mixed volume measures. From the definition of (2.8), we introduce the following dual mixed volume measure of star bodies L and K.

DEFINITION 1. (Dual mixed volume measure) For $L, K \in \mathscr{S}^n$, the dual mixed volume measure of L and K, is denoted by $d\tilde{v}_{-1}(L,K)$, is defined by

$$d\tilde{v}_{-1}(L,K) = \frac{1}{n} \rho(L,u)^{n+1} \rho(K,u)^{-1} dS(u). \tag{3.1}$$

When $K = L$, $d\tilde{v}_{-1}(L,K)$ becomes the dual cone volume measure $d\tilde{v}_L$, is defined by

$$d\tilde{v}_L = \frac{1}{n} \rho(L,u)^n dS(u).$$

From Definition 1, we find the following mixed volume probability measure.

$$d\widetilde{V}_{-1}(L,K) = \frac{1}{\widetilde{V}_{-1}(L,K)} d\tilde{v}_{-1}(L,K). \tag{3.2}$$

From the definition of (2.6), we introduce the following Orlicz dual mixed volume measure of star bodies L and K.

DEFINITION 2. (Orlicz dual mixed volume measure) For $L,K \in \mathscr{S}^n$ and $\phi \in \mathscr{C}$, the Orlicz dual mixed volume measure of L and K, is denoted by $d\tilde{v}_\phi(L,K)$, is defined by

$$d\tilde{v}_\phi(L,K) = \frac{1}{n}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\rho(L,u)^n dS(u). \tag{3.3}$$

From Definition 2, Orlicz dual mixed volume probability measure is defined by

$$d\widetilde{V}_\phi(L,K) = \frac{1}{\widetilde{V}_\phi(L,K)} d\tilde{v}_\phi(L,K). \tag{3.4}$$

Obviously, when $\phi(t) = \frac{1}{t}$, (3.3) and (3.4) become (3.1) and (3.2), respectively. When $\phi(t) = t^{-p}$ and $p \geqslant 1$, (3.4) becomes (1.7) stated in introduction.

THEOREM 1. (Orlicz dual logarithmic Minkowski inequality) *If $L,K \in \mathscr{S}^n$ and $\phi \in \mathscr{C}$, then*

$$\int_{S^{n-1}} \ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right) d\widetilde{V}_\phi(L,K) \geqslant \ln\left(\frac{\widetilde{V}_\phi(L,K)}{V(L)}\right) \geqslant \ln\left(\phi\left(\left(\frac{V(K)}{V(L)}\right)^{1/n}\right)\right). \tag{3.5}$$

If ϕ is strictly convex, each equality holds if and only if L and K are dilates.

Proof. From (3.1) and (3.3), we have

$$\int_{S^{n-1}} \phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right) \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\tilde{v}_L = \int_{S^{n-1}} \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\tilde{v}_\phi(L,K). \tag{3.6}$$

Note the following formula

$$\widetilde{V}_\phi(L,K) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\rho(L,u)^n dS(u).$$

From Lebesgue's dominated convergence theorem, we obtain

$$\int_{S^{n-1}} \phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}} d\tilde{v}_L \to \widetilde{V}_\phi(L,K)$$

as $q \to \infty$, and

$$\int_{S^{n-1}} \phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}} \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\tilde{v}_L \to \int_{S^{n-1}} \ln\left(\frac{\rho(K,u)}{\rho(L,u)}\right) d\tilde{v}_\phi(L,K)$$

as $q \to \infty$.

Let the function $g_{L,K}(q) : [1,\infty] \to \mathbb{R}$ be defined by

$$g_{L,K}(q) = \frac{1}{\widetilde{V}_\phi(L,K)} \int_{S^{n-1}} \phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}} d\tilde{v}_L. \tag{3.7}$$

By calculating the derivative and limit, we obtain

$$\frac{(q+n)^2}{n}\cdot\frac{dg_{L,K}(q)}{dq}=\frac{1}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}}\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\tilde{v}_L. \quad (3.8)$$

and

$$\lim_{q\to\infty} g_{L,K}(q)=1. \quad (3.9)$$

From (3.7), (3.8) and (3.9), and by using L'Hôpital's rule, we have

$$\begin{aligned}\lim_{q\to\infty}\ln\left(g_{L,K}(q)\right)^{q+n} &= -(q+n)^2\lim_{q\to\infty}\frac{1}{g_{L,K}(q)}\frac{dg_{L,K}(q)}{dq}\\ &= -\frac{n}{\widetilde{V}_\phi(L,K)}\lim_{q\to\infty}\frac{\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}}\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\tilde{v}_L}{g_{L,K}(q)}\\ &= -\frac{n}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\tilde{v}_L.\end{aligned}$$

Hence

$$\begin{aligned}&\exp\left(-\frac{n}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\tilde{v}_L\right)\\ &\quad=\lim_{q\to\infty}\left(g_{L,K}\right)^{q+n}\\ &\quad=\lim_{q\to\infty}\left(\frac{1}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}}d\tilde{v}_L\right)^{q+n}. \quad (3.10)\end{aligned}$$

Moreover, from Hölder's inequality

$$\begin{aligned}\left(\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}}d\tilde{v}_L\right)^{(q+n)/q}\left(\int_{S^{n-1}}d\tilde{v}_L\right)^{-n/q} &\leqslant \int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)d\tilde{v}_L\\ &= \widetilde{V}_\phi(L,K). \quad (3.11)\end{aligned}$$

From the equality of Hölder's inequality, it follows the equality in (3.11) holds if and only if $\rho(K,u)$ and $\rho(L,u)$ are proportional. This yields equality in (3.11) holds if and only if K and L are dilates, if ϕ is strictly convex. Namely

$$\left(\frac{1}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^{\frac{q}{q+n}}d\tilde{v}_L\right)^{q+n}\leqslant\left(\frac{V(L)}{\widetilde{V}_\phi(L,K)}\right)^n.$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. Hence

$$\exp\left(-\frac{n}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\tilde{v}_L\right)\leqslant\left(\frac{V(L)}{\widetilde{V}_\phi(L,K)}\right)^n.$$

Therefore

$$\frac{1}{\widetilde{V}_\phi(L,K)}\int_{S^{n-1}}\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\tilde{v}_L \geqslant \ln\left(\frac{\widetilde{V}_\phi(L,K)}{V(L)}\right).$$

That is

$$\int_{S^{n-1}}\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\widetilde{V}_\phi(L,K) \geqslant \ln\left(\frac{\widetilde{V}_\phi(L,K)}{V(L)}\right). \tag{3.12}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. The completes proof of the first inequality in (3.5).

Further, by using the Orlicz dual Minkowski inequality in (2.11), we obtain

$$\int_{S^{n-1}}\ln\left(\phi\left(\frac{\rho(K,u)}{\rho(L,u)}\right)\right)d\widetilde{V}_\phi(L,K) \geqslant \ln\left(\phi\left(\left(\frac{V(K)}{V(L)}\right)^{1/n}\right)\right).$$

If ϕ is strictly convex, equality holds if and only if L and K are dilates.

This completes the proof. □

When $\phi(t)=\frac{1}{t}$, (3.5) becomes the following logarithmic dual Minkowski inequality.

COROLLARY 1. (The logarithmic dual Minkowski inequality) *If $L,K\in\mathscr{S}^n$, then*

$$\int_{S^{n-1}}\ln\left(\frac{\rho(L,u)}{\rho(K,u)}\right)d\widetilde{V}_{-1}(L,K) \geqslant \ln\left(\frac{\widetilde{V}_{-1}(L,K)}{V(L)}\right) \geqslant \frac{1}{n}\ln\left(\frac{V(L)}{V(K)}\right),$$

each equality holds if and only if L and K are dilates.

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