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REFINEMENTS OF KY FAN'S EIGENVALUE INEQUALITY FOR SIMPLE EUCLIDEAN JORDAN ALGEBRAS BY USING GRADIENTS OF K–INCREASING FUNCTIONS

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Abstract. In this paper, by using some analytical methods based on gradients of K-increasing functions, we show refinements of the following Ky Fan like inequality: $\lambda(x+y) \prec \lambda(x)+\lambda(y)$ with elements x and y of a simple Euclidean Jordan algebra, the eigenvalue operator $\lambda(\cdot)$ and Schur's majorization $\prec$.

1. Introduction and summary

We begin with some notation.

A nonempty subset C of a real linear space W is said to be a *convex cone*, if **(i)** $a,b \in C$ implies $a+b \in C$, and **(ii)** $a \in C$ and $0 \leqslant t \in \mathbf{R}$ imply $ta \in C$.

We use the symbol $\leqslant_C$ to denote the *cone preorder* on W, induced by a convex cone $C \subset W$, and defined as follows: for $x,y \in W$,

$$y \leqslant_C x \quad \text{iff} \quad x-y \in C. \tag{1}$$

For a vector $z=(z_1,z_2,\dots,z_n) \in \mathbf{R}^n$ the symbols $z_{[1]},z_{[2]},\dots,z_{[n]}$ stands for the entries of z decreasingly ordered, i.e., $z_{[1]} \geqslant z_{[2]} \geqslant \dots \geqslant z_{[n]}$.

We denote $z_{\downarrow}=(z_{[1]},z_{[2]},\dots,z_{[n]})$.

We say that a vector $x=(x_1,x_2,\dots,x_n) \in \mathbf{R}^n$ *majorizes* a vector $y=(y_1,y_2,\dots,y_n) \in \mathbf{R}^n$ (written as $y \prec x$), if the sum of k largest entries of y does not exceed the sum of k largest entries of x for all $k=1,2,\dots,n$ with equality for $k=n$, that is

$$\sum_{i=1}^{k} y_{[i]} \leqslant \sum_{i=1}^{k} x_{[i]} \quad \text{for all } k=1,2,\dots,n, \quad \text{and} \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$$

(see [8, p. 8]).

It is known that for $x,y \in \mathbf{R}^n$,

$$y \prec x \quad \text{iff} \quad y_{\downarrow} \prec x_{\downarrow} \quad \text{iff} \quad y \in \operatorname{conv} \mathbf{P}_n x, \tag{2}$$

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where the symbol $\mathbf{P}_n$ represents the set of all n-by-n permutation matrices, and $\operatorname{conv}\mathbf{P}_n x$ is the convex hull of the orbit $\mathbf{P}_n x = \{px : p \in \mathbf{P}_n\}$ (see [8, p. 10]).

We equip $\mathbf{R}^n$ with the standard inner product $\langle \cdot,\cdot \rangle$.

We introduce the sets D and $\operatorname{dual} D$ by

$$D = \mathbf{R}^n_{\downarrow} = \{x = (x_1,\dots,x_n) \in \mathbf{R}^n : x_1 \geqslant \dots \geqslant x_n\}, \tag{3}$$

$$\operatorname{dual} D = \{v \in \mathbf{R}^n : \langle v,x \rangle \geqslant 0 \text{ for all } x \in D\}.$$

Observe that D and $\operatorname{dual} D$ are closed convex cones in $\mathbf{R}^n$.

The majorization preorder $\prec$ restricted to D is characterized, as follows. For $x,y \in D$,

$$y \prec x \quad \text{iff} \quad y \leqslant_{\operatorname{dual} D} x \tag{4}$$

(see [8, p. 596]). In general, for $x,y \in \mathbf{R}^n$,

$$y \prec x \quad \text{iff} \quad y_{\downarrow} \leqslant_{\operatorname{dual} D} x_{\downarrow} \quad \text{iff} \quad \langle d, y_{\downarrow} \rangle \leqslant \langle d, x_{\downarrow} \rangle \text{ for all } d \in D. \tag{5}$$

Furthermore, the following rearrangement inequality is satisfied:

$$\langle d,x \rangle \leqslant \langle d, x_{\downarrow} \rangle \text{ for all } x \in \mathbf{R}^n \text{ and } d \in D \tag{6}$$

(see [8, Proposition A.3, p. 207]).

A real function $\Phi : \mathbf{R}^n \to \mathbf{R}$ is said to be *Schur-convex* if for $x,y \in \mathbf{R}^n$,

$$y \prec x \quad \text{implies} \quad \Phi(y) \leqslant \Phi(x).$$

The Hardy-Littlewood-Pólya-Karamata Theorem [8, p. 92, p. 156] says that if $f : \mathbf{R} \to \mathbf{R}$ is a convex function and $x = (x_1,x_2,\dots,x_n) \in \mathbf{R}^n$, $y = (y_1,y_2,\dots,y_n) \in \mathbf{R}^n$, then

$$y \prec x \quad \text{implies} \quad \sum_{i=1}^n f(y_i) \leqslant \sum_{i=1}^n f(x_i). \tag{7}$$

By Schur-Ostrowski's Theorem (see [8, Theorem A.4.]), a differentiable function $\Phi : \mathbf{R}^n \to \mathbf{R}$ is Schur-convex if and only if

$$(x_i - x_j)\left(\frac{\partial \Phi(x)}{\partial x_i} - \frac{\partial \Phi(x)}{\partial x_j}\right) \geqslant 0 \text{ for } x = (x_1,\dots,x_n) \in \mathbf{R}^n,\ i,j = 1,\dots,n.$$

As a result, the gradient

$$\nabla \Phi(\cdot) = \left(\frac{\partial \Phi(\cdot)}{\partial x_1},\dots,\frac{\partial \Phi(\cdot)}{\partial x_n}\right)$$

of a differentiable Schur-convex function $\Phi : \mathbf{R}^n \to \mathbf{R}$ sends the interior of D into D. In consequence, for any permutation $p \in \mathbf{P}_n$, the gradient $\nabla \Phi(\cdot)$ sends the interior of pD into pD. Under the additional assumption that the gradient $\nabla \Phi(\cdot)$ is continuous, it holds that

$$\nabla \Phi(D) \subset D \quad \text{and} \quad \nabla \Phi(pD) \subset pD \quad \text{for any } p \in \mathbf{P}_n.$$

Many interesting results on Hermitian matrices can be expressed as majorization inequalities for their eigenvalues $\lambda(\cdot)$. One of the most important is the following Ky Fan's inequality:

$$\lambda(x+y) \prec \lambda(x)+\lambda(y) \tag{8}$$

for any Hermitian matrices x and y of the same size (see [2], [8, Theorem G.1, p. 329]).

In this paper, by developing a method used in [7, 11, 12, 13], our purpose is to refine the above inequality (8) in the context of simple Euclidean Jordan algebras.

In Section 2 we begin with some needed notation and terminology connected with a simple Euclidean Jordan algebra, say V. In our studies we employ a special group $K \subset Aut(V)$ and closed convex cone $\mathbf{a}^+ \subset V$, as well as (differentiable) K-increasing functions and their gradients. The useful property of such (continuous) gradients is that they send the cone $\mathbf{a}^+$ into itself. This can be illustrated and confirmed by a similar result of the above-mentioned Schur-Ostrowski Theorem. Based on this observation, we prove Theorem 1, which shows the property of anti-isotonicity of a certain operator generated by a Fan-like inequality (9) for simple Euclidean Jordan algebras. In Theorem 2 we show the isotonicity of the triangle operator induced by (9). In doing so, we utilize the preorder induced by the cone of all K-increasing functions.

In Theorem 3 we present a refinement of the above majorization inequality (8) for a simple Euclidean Jordan algebra. A particular case, related to a Maligranda's inequality [7] for real norms, is considered in Corollary 1.

2. Results for simple Euclidean Jordan algebras

We deal with a simple Euclidean Jordan algebra V of rank r, equipped with an inner product $\langle x,y\rangle = \operatorname{tr} xy$ for $x,y \in V$, and norm $\|x\| = \langle x,x\rangle^{1/2}$ for $x \in V$ [4, 6, 15]. By K we mean the connected component of the identity in the group $Aut(V)$ of all Jordan automorphisms. For a fixed Jordan frame $\{c_1,\dots,c_r\}$, we denote

$$\mathbf{a} = \left\{ \sum_{i=1}^{r} d_i c_i : d_i \in \mathbf{R} \right\} \quad \text{and} \quad \mathbf{a}^+ = \left\{ \sum_{i=1}^{r} \lambda_i c_i : \lambda_1 \geqslant \ldots \geqslant \lambda_r \right\}.$$

We use the notation $\lambda(x) = (\lambda_1(x),\dots,\lambda_r(x))$, where $\lambda_1(x) \geqslant \ldots \geqslant \lambda_r(x)$ are the eigenvalues of $x \in V$ arranged in nonincreasing order. We also denote $\gamma(x) = \sum_{i=1}^{r} \lambda_i(x) c_i$ for $x \in V$. It holds that $\gamma : V \to \mathbf{a}^+$ is K-invariant and the range of γ is $\mathbf{a}^+$.

It is known that the following Fan-like inequality holds on V:

$$\lambda(x+y) \prec \lambda(x)+\lambda(y) \quad \text{for } x,y \in V \tag{9}$$

(see [9, 15]).

For $x,y \in V$, we write $y \prec_K x$ provided that $y \in \operatorname{conv} Kx$, where $\operatorname{conv} Kx$ is the convex hull of the K-orbit $Kx = \{kx : k \in K\}$ (cf. (2)). For $x,y \in V$,

$$y \prec_K x \quad \text{iff} \quad \lambda(y) \prec \lambda(x). \tag{10}$$

A function Φ defined on V is said to be K-*invariant* if

$$\Phi(kx) = \Phi(x) \text{ for all } x \in V \text{ and } k \in K.$$

A function $\Phi : V \to \mathbf{R}$ is said to be K-*increasing*, if for $x, y \in V$,

$$y \prec_K x \text{ implies } \Phi(y) \leqslant \Phi(x).$$

It should be noted that the convexity and K-invariance of Φ implies K-increase of Φ.

Let $\mathbf{R}^V$ be the real linear space of all real functions defined on V. By $\mathscr{C}$, we denote the convex cone in $\mathbf{R}^V$ consisting of all K-increasing real functions defined on V. For any two real functions $\Phi : V \to \mathbf{R}$ and $\Psi : V \to \mathbf{R}$, we write $\Psi \leqslant_{\mathscr{C}} \Phi$, if the difference function $\Phi - \Psi$ is K-increasing on V (cf. (1)). Thus $\leqslant_{\mathscr{C}}$ is the cone preorder on $\mathbf{R}^V$ induced by $\mathscr{C}$.

Throughout we adopt the convention that the Gateaux differentiability of a function $\Phi : V \to \mathbf{R}$ amounts to the existence of the directional derivative

$$\nabla_h \Phi(y) = \lim_{t \to 0} \frac{\Phi(y + th) - \Phi(y)}{t} \quad \text{for all } y, h \in V, \tag{11}$$

such that the map $V \ni h \to \nabla_h \Phi(y) \in \mathbf{R}$ is continuous and linear as a function of h.

For this reason, at each point $y \in V$, there exists the gradient $\nabla \Phi(y) \in V$ satisfying the condition

$$\nabla_h \Phi(y) = \langle \nabla \Phi(y), h \rangle \quad \text{for all } h \in V. \tag{12}$$

Since V is a simple Euclidean Jordan algebra, it follows from [6, Corollary 4] that the structure (V, K, γ) is so-called normal decomposition system [5]. Therefore $(V, K, \mathbf{a}^+)$ is so-called an Eaton triple with normal map γ (see [5, p. 817]). From this, by [10, Theorem 2.1], a Gateaux differentiable K-increasing function $\Phi : V \to \mathbf{R}$ with continuous gradient $\nabla \Phi(\cdot)$ satisfies the condition

$$\nabla \Phi(k\gamma(x)) \in k\mathbf{a}^+ \quad \text{for all } k \in K \text{ and } x \in V. \tag{13}$$

Two elements $x, y \in V$ are said to be K-*simultaneously diagonalizable*, if there exists a $k \in K$ such that $x \in k\mathbf{a}^+$ and $y \in k\mathbf{a}^+$ (see (3)).

The next fact is of importance for us:

$$x, y \in V \text{ are } K \text{ - simultaneously diagonalizable} \quad \Rightarrow \quad \lambda(x + y) = \lambda(x) + \lambda(y). \tag{14}$$

The following result, applied to $\Theta = \frac{1}{2} \| \cdot \|^2$, is a development of [12, Theorem 1].

THEOREM 1. *Let V be a simple Euclidean Jordan algebra. Let Φ, Ψ and Θ be Gateaux differentiable real functions on V with continuous gradients $\nabla \Phi(\cdot)$, $\nabla \Psi(\cdot)$ and $\nabla \Theta(\cdot)$, respectively. Assume that the functions Ψ, $\Phi - \Psi$ and $\Theta - \Phi$ are K-increasing on V, i.e., $0 \leqslant_{\mathscr{C}} \Psi \leqslant_{\mathscr{C}} \Phi \leqslant_{\mathscr{C}} \Theta$.*

If $x, y \in V$ then

$$\lambda(x + \nabla \Phi(y)) + \lambda(\nabla \Theta(y) - \nabla \Phi(y)) \prec \lambda(x + \nabla \Psi(y)) + \lambda(\nabla \Theta(y) - \nabla \Psi(y)). \tag{15}$$

Proof. By applying Fan-type inequality (9) for elements $x+\nabla\Psi(y)\in V$ and $\nabla\Phi(y) - \nabla\Psi(y)\in V$, we get

$$\lambda(x+\nabla\Phi(y)) = \lambda(x+\nabla\Psi(y)+\nabla\Phi(y)-\nabla\Psi(y)) \prec \lambda(x+\nabla\Psi(y))+\lambda(\nabla\Phi(y)-\nabla\Psi(y)). \tag{16}$$

Because all the n-vectors

$$\lambda(x+\nabla\Phi(y)),\ \lambda(x+\nabla\Psi(y)),\ \lambda(\nabla\Phi(y)-\nabla\Psi(y)),\ \lambda(\nabla\Theta(y)-\nabla\Phi(y))$$

belong to the convex cone $D\subset\mathbf{R}^n$ (see (3)), and the majorization preorder $\prec$ restricted to D is the cone preorder induced by $\operatorname{dual}D$ (see (4)), so inequality (16) implies

$$\begin{aligned}\lambda(x+\nabla\Phi(y)) &+ \lambda(\nabla\Theta(y)-\nabla\Phi(y))\\ &\prec \lambda(x+\nabla\Psi(y))+\lambda(\nabla\Phi(y)-\nabla\Psi(y))+\lambda(\nabla\Theta(y)-\nabla\Phi(y)).\end{aligned} \tag{17}$$

Since $y\in V$, there exists a $k\in K$ such that $y=k\gamma(y)\in k\mathbf{a}^+$. In addition, Ψ and $\Phi-\Psi$ are K-increasing, so (13) implies that

$$\nabla\Psi(y)\in k\mathbf{a}^+,$$

$$\nabla\Phi(y)-\nabla\Psi(y)=\nabla(\Phi-\Psi)(y)\in k\mathbf{a}^+.$$

Thus $\nabla\Psi(y)$ and $\nabla\Phi(y)-\nabla\Psi(y)$ are K-simultaneously diagonalizable vectors. Now, it is a consequence of (14) applied to these two vectors that

$$\lambda(\nabla\Psi(y))+\lambda(\nabla\Phi(y)-\nabla\Psi(y))=\lambda(\nabla\Psi(y)+\nabla\Phi(y)-\nabla\Psi(y))=\lambda(\nabla\Phi(y)).$$

Therefore,

$$\lambda(\nabla\Phi(y)-\nabla\Psi(y))=\lambda(\nabla\Phi(y))-\lambda(\nabla\Psi(y)). \tag{18}$$

Also, $\Phi=\Psi+(\Phi-\Psi)$ and $\Theta-\Phi$ are K-increasing functions. So, we deduce from (13) that

$$\nabla\Phi(y)\in k\mathbf{a}^+,$$

$$\nabla\Theta(y)-\nabla\Phi(y)=\nabla(\Theta-\Phi)(y)\in k\mathbf{a}^+.$$

That is, the vectors $\nabla\Phi(y)$ and $\nabla\Theta(y)-\nabla\Phi(y)$ are K-simultaneously diagonalizable. Now, by applying (14) we infer that

$$\lambda(\nabla\Phi(y))+\lambda(\nabla\Theta(y)-\nabla\Phi(y))=\lambda(\nabla\Phi(y)+\nabla\Theta(y)-\nabla\Phi(y))=\lambda(\nabla\Theta(y)),$$

and, hence,

$$\lambda(\nabla\Theta(y)-\nabla\Phi(y))=\lambda(\nabla\Theta(y))-\lambda(\nabla\Phi(y)). \tag{19}$$

Next, we remind that the functions Ψ, $\Phi-\Psi$ and $\Theta-\Phi$ are K-increasing. As a result, the function

$$\Theta-\Psi=(\Theta-\Phi)+(\Phi-\Psi)$$

is K-increasing. By making use of (13) we conclude that

$$\nabla\Psi(y)\in k\mathbf{a}^+,$$

$$\nabla\Theta(y) - \nabla\Psi(y) = \nabla(\Theta - \Psi)(y) \in k\mathbf{a}^+,$$

which means that the vectors $\nabla\Psi(y)$ and $\nabla\Theta(y) - \nabla\Psi(y)$ are K-simultaneously diagonalizable. So, by using (14) we find that

$$\lambda(\nabla\Psi(y)) + \lambda(\nabla\Theta(y) - \nabla\Psi(y)) = \lambda(\nabla\Psi(y) + \nabla\Theta(y) - \nabla\Psi(y)) = \lambda(\nabla\Theta(y)),$$

and further

$$\lambda(\nabla\Theta(y) - \nabla\Psi(y)) = \lambda(\nabla\Theta(y)) - \lambda(\nabla\Psi(y)). \tag{20}$$

Finally, the usage of (17) with (18), (19) and (20) leads to the inequality

$$\begin{aligned}
&\lambda(x + \nabla\Phi(y)) + \lambda(\nabla\Theta(y) - \nabla\Phi(y)) \\
&\prec \lambda(x + \nabla\Psi(y)) + \lambda(\nabla\Phi(y)) - \lambda(\nabla\Psi(y)) + \lambda(\nabla\Theta(y)) - \lambda(\nabla\Phi(y)) \\
&= \lambda(x + \nabla\Psi(y)) - \lambda(\nabla\Psi(y)) + \lambda(\nabla\Theta(y)) = \lambda(x + \nabla\Psi(y)) + \lambda(\nabla\Theta(y) - \nabla\Psi(y)),
\end{aligned}$$

which completes the proof of the theorem. □

REMARK 1. Concerning Theorem 1, for $x, y \in V$ we define the operator

$$\Phi \to F_{x,y}(\Phi) = \lambda(x + \nabla\Phi(y)) + \lambda(\nabla\Theta(y) - \nabla\Phi(y))$$

with values in $\mathbf{R}^n$, where Φ ranges over the set of all Gateaux differentiable real functions defined on V.

It is worth emphasizing that Theorem 1 states that $F_{x,y}(\cdot)$ is *anti-isotone* with respect to the preorder pair $(\leqslant_{\mathscr{C}}, \prec)$ on the "function interval" $[0, \Theta]$ generated by $\leqslant_{\mathscr{C}}$.

Having in mind triangle-like inequality (9), we introduce the operator $\Delta : V \times V \to \mathbf{R}^n$ by

$$\Delta(x, y) = \lambda(x) + \lambda(y) - \lambda(x + y) \quad \text{for } x, y \in V. \tag{21}$$

THEOREM 2. *Let V be a simple Euclidean Jordan algebra. Let Φ and Ψ be Gateaux differentiable real functions on V with continuous gradients $\nabla\Phi(\cdot)$ and $\nabla\Psi(\cdot)$, respectively. Assume that the functions Ψ and $\Phi - \Psi$ are K-increasing on V, i.e., $0 \leqslant_{\mathscr{C}} \Psi \leqslant_{\mathscr{C}} \Phi$.*

If $x, y \in V$ then

$$\Delta(x, \nabla\Psi(y)) \leqslant_{\operatorname{dual} D} \Delta(x, \nabla\Phi(y)). \tag{22}$$

If, in addition, $\Delta(x, \nabla\Psi(y)) \in D$, then

$$\Delta(x, \nabla\Psi(y)) \prec \Delta(x, \nabla\Phi(y)). \tag{23}$$

Proof. We proceed as in the proof of Theorem 1 with $\Theta = \Phi$. In particular, equality (18) still holds valid. By using Ky Fan's inequality (9), we get

$$\lambda(x + \nabla\Phi(y)) \leqslant_{\operatorname{dual} D} \lambda(x + \nabla\Psi(y)) + \lambda(\nabla\Phi(y)) - \lambda(\nabla\Psi(y)), \tag{24}$$

because the n-vectors $\lambda(x+\nabla\Phi(y))$, $\lambda(x+\nabla\Psi(y))$ and $\lambda(\nabla\Phi(y))-\lambda(\nabla\Psi(y)) = \lambda(\nabla\Phi(y)-\nabla\Psi(y))$ belong to the convex cone D.

By subtracting the term $\lambda(x)+\lambda(\nabla\Phi(y))$ from the both sides of inequality (24), we obtain

$$\lambda(x+\nabla\Phi(y))-\lambda(x)-\lambda(\nabla\Phi(y)) \leqslant_{\operatorname{dual} D} \lambda(x+\nabla\Psi(y))-\lambda(x)-\lambda(\nabla\Psi(y)),$$

which, by (21), can be rewritten as

$$-\Delta(x,\nabla\Phi(y)) \leqslant_{\operatorname{dual} D} -\Delta(x,\nabla\Psi(y)). \tag{25}$$

This easily gives (22), as claimed.

To see the second part of the assertion, assume that $\Delta(x,\nabla\Psi(y)) \in D$. Hence, for any $d \in D$, we have

$$\langle d,(\Delta(x,\nabla\Psi(y)))_{\downarrow}\rangle = \langle d,\Delta(x,\nabla\Psi(y))\rangle \leqslant \langle d,\Delta(x,\nabla\Phi(y))\rangle \leqslant \langle d,(\Delta(x,\nabla\Phi(y)))_{\downarrow}\rangle. \tag{26}$$

The former inequality holds by (22), while the latter follows from (6).

Now, in light of (26) and (5) we see that (23) holds true. □

REMARK 2. In Theorem 2, for any $x,y \in V$ the assertions (22) and (23) can be viewed as the *isotonicity* with respect to the preorder pairs $(\leqslant_{\mathscr{C}},\leqslant_{\operatorname{dual} D})$ and $(\leqslant_{\mathscr{C}},\prec)$, respectively, of the operator

$$\Phi \to \Delta_{x,y}(\Phi) = \Delta(x,\nabla\Phi(y)) = \lambda(x)+\lambda(\nabla\Phi(y))-\lambda(x+\nabla\Phi(y)),$$

where Φ ranges over the "function interval" $[0,\Theta]$ included in the set of all Gateaux differentiable real functions defined on V.

THEOREM 3. *Let V be a simple Euclidean Jordan algebra. Let Φ and Θ be Gateaux differentiable real functions on V with continuous gradients $\nabla\Phi(\cdot)$ and $\nabla\Theta(\cdot)$, respectively. Assume that the functions Φ and $\Theta-\Phi$ are K-increasing on V, i.e., $0 \leqslant_{\mathscr{C}} \Phi \leqslant_{\mathscr{C}} \Theta$.*

If $x,y \in V$ then

$$\lambda(x+\nabla\Theta(y)) \prec \lambda(x+\nabla\Phi(y))+\lambda(\nabla\Theta(y)-\nabla\Phi(y)) \prec \lambda(x)+\lambda(\nabla\Theta(y)). \tag{27}$$

Proof. It follows by the substitution $\Psi = 0$ that

$$0 \leqslant_{\mathscr{C}} \Psi \leqslant_{\mathscr{C}} \Phi \leqslant_{\mathscr{C}} \Theta. \tag{28}$$

In other words, the functions Ψ, $\Phi-\Psi$ and $\Theta-\Phi$ are K-increasing on V. Thus all needed assumptions in Theorem 1 are fulfilled for these functions. So, inequality (15) is met and takes the form

$$\begin{aligned}\lambda(x+\nabla\Phi(y))+\lambda(\nabla\Theta(y)-\nabla\Phi(y)) &\prec \lambda(x+\nabla\Psi(y))+\lambda(\nabla\Theta(y)-\nabla\Psi(y))\\ &= \lambda(x)+\lambda(\nabla\Theta(y)),\end{aligned} \tag{29}$$

since $\Psi = 0$ and $\nabla\Psi(y) = 0$. Thus the right-hand side of (27) is proven.

In order to show that the left-hand side of (27) is also satisfied, we put $\Psi = \Theta$. Then we see that

$$0 \leqslant_{\mathscr{C}} \Phi \leqslant_{\mathscr{C}} \Psi \leqslant_{\mathscr{C}} \Theta. \tag{30}$$

Due to Theorem 1 applied to assumption (30), we derive

$$\begin{aligned}\lambda(x+\nabla\Theta(y)) &= \lambda(x+\nabla\Psi(y)) + \lambda(\nabla\Theta(y) - \nabla\Psi(y)) \\ &\prec \lambda(x+\nabla\Phi(y)) + \lambda(\nabla\Theta(y) - \nabla\Phi(y)),\end{aligned} \tag{31}$$

because $\nabla\Psi(y) = \nabla\Theta(y)$ for $\Psi = \Theta$. This completes the proof of Theorem 3. □

COROLLARY 1. ([11]) *Let V be a simple Euclidean Jordan algebra. Let $0 \leqslant t \leqslant 1$.*

If $x, y \in V$ then

$$\lambda(x+y) \prec \lambda(x+ty) + (1-t)\lambda(y) \prec \lambda(x) + \lambda(y). \tag{32}$$

Proof. The function $\|\cdot\|$ is convex and K-invariant. It now follows that $\|\cdot\|$ is K-increasing on V. Therefore the function $\frac{1}{2}\|\cdot\|^2$ is K-increasing on V, too.

By putting $\Phi = \Phi_t = t\frac{1}{2}\|\cdot\|^2$ and $\Theta = \frac{1}{2}\|\cdot\|^2$, we obtain

$$0 \leqslant_{\mathscr{C}} \Phi \leqslant_{\mathscr{C}} \frac{1}{2}\|\cdot\|^2 = \Theta.$$

In fact, the functions Φ and $\frac{1}{2}\|\cdot\|^2 - \Phi = (1-t)\frac{1}{2}\|\cdot\|^2$ are K-increasing on V with $0 \leqslant t \leqslant 1$. Moreover, $\nabla\Phi(y) = ty$ and $\nabla\Theta(y) = y$.

Now, thanks to Theorem 3 we see that (27) reduces to (32), as wanted. □

REMARK 3. The Ky Fan inequality is known to hold in general Euclidean Jordan algebras. It might be interesting to see whether all the results stated in the present article continue to hold in general Euclidean Jordan algebras.

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