Mathematical
Inequalities
& Applications
Volume 24, Number 4 (2021), 1075–1092 doi:10.7153/mia-2021-24-75

ON L_p INTERSECTION MEAN ELLIPSOIDS AND AFFINE ISOPERIMETRIC INEQUALITIES

XIN CAI AND FANGWEI CHEN*

(*Communicated by I. Perić*)

Abstract. In the paper, we discussed the L_p ($p \geqslant 1$) harmonic combination of convex bodies in $\mathbb{R}^n$. A variational formula for jth affine mean intersection $\tilde{\Lambda}_j$ of convex bodies is established when $1 \leqslant j \leqslant n-1$. Using the new L_p intersection ellipsoids associated with convex bodies, some affine isoperimetric equalities are obtained.

1. Introduction

Let $K \in \mathbb{R}^n$ be a convex body, a compact convex set with a nonempty interior. The relationship between the geometric invariants of K is very important, these geometric quantities are mainly described by some geometric equalities or geometric inequalities. Maybe the isoperimetric inequality is one of the most powerful inequalities in convex geometry, the ellipsoid often appears in solving the isoperimetric type problems and other extreme value problems. In particular, the L_p John ellipsoid [15], mixed L_p John ellipsoid [7], Orlicz-John ellipsoid [21], Orlicz-Legendre ellipsoid [22] are all a powerful tool to solve the isoperimetric types problem. The research of convex geometry theory in L_p space and Orlicz space is one of the hotspots in convex geometry, which has attracted the attention and interest of many mathematicians. In 1980s and 1990s, Firey, Lutwak and others studied the L_p Brunn-Minkowski theory and the dual L_p Brunn-Minkowski theory, which developed the classical Brunn-Minkowski theory in $\mathbb{R}^n$ (see [11, 10, 2, 12, 13, 16, 8, 14, 5, 18, 17]). The research on the relationship between the affine inequality and the ellipsoid in Euclidean space and L_p space has caused the concern of many scholars. Recently, Hu, Xiong and Zou defined the intersection mean ellipsoid in Euclidean space and proved some affine isoperimetric inequalities in Euclidean space (see [6]). The projection mean ellipsoid and the connection with the affine isoperimetric inequalities in Euclidean space are established in [23]. Inspired by paper of Hu, Xiong and Zou [6], in this paper we study the L_p intersection mean ellipsoid.

Mathematics subject classification (2020): 52A20, 52A40, 52A38.
Keywords and phrases: L_p affine quermassintegral, Minkowski inequality, L_p intersection ellipsoid, L_p dual mixed volume.
The work is supported in part by CNSF (Grant No. 11561012, 11861024), Guizhou Foundation for Science and Technology (Grant No. [2019]1055), Science and technology top talent support program of Guizhou Eduction Department (Grant No. [2017]069).
* Corresponding author.

Let $K \in \mathbb{R}^n$ be a convex body, a compact convex set with a nonempty interior, denote by $V(K)$ the volume of K in $\mathbb{R}^n$, it can be represented as

$$V(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(z,u)^n d\mathscr{H}^{n-1}(u),$$

where z is an interior point of K, the radical function $\rho_k(z,u) : \mathbb{S}^{n-1} \to \mathbb{R}^n$ of K with respect to z is defined by $\rho_k(z,u) = \sup\{\lambda > 0, z+\lambda u \in K\}$ and $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. If z is the origin, we simply write $\rho_k(u) = \rho_k(z,u)$.

Let $G_{n.j}$ denote the Grassman manifold of $\mathbb{R}^n$, μ_j is the Haar probability measure on $G_{n,j}$, $V_j(K\cap\xi)$ denotes the j-dimensional volume of intersection of K with a subspace $\xi \in G_{n,j}$. The total average volume of the j-th intersection of a convex body on $G_{n,j}$ is defined by Lutwak [9], which is called the dual affine quermassintegrals Φ_{n-j},

$$\tilde{\Phi}_{n-j}(K) = \frac{\omega_n}{\omega_j}\left(\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)\right)^{\frac{1}{n}}, \qquad j = 1,\cdots,n-1. \tag{1.1}$$

Specially with $\tilde{\Phi}_0(K) = V_n(K)$, $\tilde{\Phi}_n(K) = \omega_n$. We rewrite

$$\tilde{\Lambda}_j(K) = \tilde{\Phi}_{n-j}(K), \qquad j = 1,\cdots,n-1,$$

for convenience. It was proved by Grinberg [4] that the jth affine mean intersections are invariant under the volume preserving linear transforms. Moreover, he proved the following inequality

$$\tilde{\Lambda}_j(K) \leqslant \omega_n^{n-j} V(K)^j, \tag{1.2}$$

for $2 \leqslant j \leqslant n-1$, equality holds if and only if K is an origin-symmetric ellipsoid. Specially, when $j = 1$ and K is symmetric, inequality (1.2) becomes an identity; when $j = n-1$, inequality is

$$V(IK) \leqslant \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K)^{n-1}, \tag{1.3}$$

with equality if and only if K is an origin-symmetric ellipsoid. Where IK is the intersection body of K defined by

$$\rho_{IK}(u) = V_{n-1}(K\cap u^{\perp}), \ \ u \in \mathbb{S}^{n-1}.$$

More details see [3, 4, 11, 10].

The L_p dual mixed volume $\tilde{V}_{n,-p}(K,L)$ of convex bodies K and L is defined by Lutwak [11], which is a variation of volume V with respect to the L_p harmonic combination $K\hat{+}_p\varepsilon\cdot L$, which is

$$\tilde{V}_{n,-p}(K,L) = -\frac{p}{n}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} V(K\hat{+}_p\varepsilon\cdot L),$$

where $K\hat{+}_p\varepsilon\cdot L$ is defineded by $\rho_{K\hat{+}_p\varepsilon\cdot L} = (\rho_k^{-p} + \varepsilon\rho_L^{-p})^{-\frac{1}{p}}$, $\varepsilon > 0$.

In this paper, we discuss the L_p harmonic combination of convex bodies K and L. In section 3, we defined the jth L_p mixed dual affine mean intersection $\bar{\Lambda}_{j,-p}(K,L)$ of convex bodies K and L by,

$$\bar{\Lambda}_{j,-p}(K,L) = -\frac{p}{j\Lambda_j(K)}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\tilde{\Lambda}_j(K\hat{+}_p\varepsilon\cdot L), \qquad 1\leqslant j\leqslant n-1.$$

We show the jth L_p mixed dual affine mean intersection is affine invariant and can be represent as the integral of the L_p mixed volume of K and L. In section 4, by normalizing the jth L_p mixed dual affine mean intersection $\bar{\Lambda}_{j,-p}(K,L)$, we show that there exists a unique origin-symmetric ellipsoid solving the constrained minimization problem, that is

$$\min V(E) \qquad \text{subject to} \qquad \bar{\Lambda}_{j,-p}(K,E)\leqslant 1.$$

The ellipsoid is called the j-th L_p intersection mean ellipsoid of the convex body K, and is denoted by $S_{j,p}K$. Observe that $S_{j,p}K$ is closely related to $V_j(K\cap\cdot)$ of the convex body K. Moreover, we prove the following sharp affine isoperimetric inequalities.

THEOREM 1.1. *Suppose that K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $1\leqslant j\leqslant n-1$, $p\geqslant 1$. Then,*

$$\tilde{\Lambda}_j(K)\leqslant \omega_n^{\frac{n-j}{n}}V(S_{j,p}K)^{\frac{j}{n}},$$

when $2\leqslant j\leqslant n-1$, equality holds if and only if K is an origin-symmetric ellipsoid.

THEOREM 1.2. *Suppose that K is an origin-symmetric convex body in $\mathbb{R}^n$. Then,*

$$V(S^*_{1,p}K)V(K)\leqslant\omega_n^2,$$

with equality if and only if K is an origin-symmetric ellipsoid.

THEOREM 1.3. *Suppose that K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. Then,*

$$V(IK)\leqslant\frac{\omega_{n-1}^n}{\omega_n^{n-2}}V(S_{n-1,p}K)^{n-1},$$

with equality if and only if K is an origin-symmetric ellipsoid.

2. Preliminaries

In this paper, we work in n-dimensional Euclidean space $\mathbb{R}^n$, endowed with the standard inner product $x\cdot y$ and Euclidean norm $\|x\|$. B^n and $\mathbb{S}^{n-1}$ denote the unit ball and unit sphere, respectively, the volume of B^n is denoted by $\omega_n=\frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$. For $1\leqslant j\leqslant n-1$, let $G_{n,j}$ be the Grassmann manifold of j dimensional linear space in $\mathbb{R}^n$, write V_j for the j-dimensional volume of a convex body in $\mathbb{R}^n$. The set of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric is denoted by $\mathscr{K}^n$, and the set of convex bodies containing the origin in their interiors is denoted by $\mathscr{K}_0^n$. Let $K \in \mathscr{K}^n$, its support function $h_k : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n.$$

It is easily seen that h_K is 1-homogeneous and subadditive. For $K \in \mathscr{K}_0^n$, the radial function of K is defined by

$$\rho_K(x) = \sup\{\lambda > 0 : \lambda x \in K\},\ x \in \mathbb{R}^n \setminus \{o\}.$$

We know that ρ_K is positive and 1-homogeneous. Moreover, for $T \in GL(n)$, we have $\rho_{TK}(x) = \rho_K(T^{-1}x)$. The polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1,\, y \in K\},$$

It is easy to check that $h_{K^*} = \rho_K^{-1}$, $(TK)^* = T^{-t}K^*$ for $T \in GL(n)$.

Let $K, K_i \subseteq \mathscr{K}_0^n$, $i \in \mathbb{N}$, then, $K_i \to K$ if and only if $\rho_{K_i} \xrightarrow{\delta_H} \rho_K$ uniformly on $\mathbb{S}^{n-1}$, where $\delta_H(K,L) = \max\limits_{u \in \mathbb{S}^{n-1}} |h_K(u) - h_L(u)|$, is the Hausdorff metric.

Let $K, L \in \mathscr{K}_0^n$, the L_p *harmonic combination* $\lambda \cdot K \hat{+}_p \mu \cdot L \in \mathscr{K}_0^n$ is defined by

$$\rho^{-p}_{\lambda \cdot K \hat{+}_p \mu \cdot L}(x) = \lambda \rho_K^{-p}(x) + \mu \rho_L^{-p}(x),\ x \in \mathbb{R}^n \setminus \{o\},$$

where $\lambda, \mu > 0$. Specially, $\lambda \cdot K = \lambda^{-\frac{1}{p}} K$.

Let $\xi \in G_{n,j}$ be j-dimensional subspace $(1 \leqslant j \leqslant n-1)$, then $K \cap \xi$ is an j-dimensional convex body in ξ, and $\rho_{K \cap \xi}(u) = \rho_K(u)$, for $u \in \mathbb{S}^{n-1} \cap \xi$. Moreover, it is easy to show that $(\lambda \cdot K \hat{+}_p \mu \cdot L) \cap \xi = \lambda \cdot (K \cap \xi) \hat{+}_p \mu \cdot (L \cap \xi)$. The volume of $K \cap \xi$ is

$$V_j(K \cap \xi) = \frac{1}{j} \int_{\mathbb{S}^{n-1} \cap \xi} \rho_K^j(u) d\mathscr{H}^{j-1}(u). \tag{2.1}$$

The L_p $(p \geqslant 1)$ dual mixed volume $\tilde{V}_{n,-p}(K,L)$ of $K, L \in \mathscr{K}_0^n$ is defined by (see [11])

$$\begin{aligned}\tilde{V}_{n,-p}(K,L) &= -\frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \hat{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) d\mathscr{H}^{n-1}(u).\end{aligned}$$

The dual Minkowski inequality says

$$\tilde{V}_{j,-p}(K \cap \xi, L \cap \xi)^j \geqslant V_j(K \cap \xi)^{j+p} V_j(L \cap \xi)^{-p}, \tag{2.2}$$

with equality if and only if $K \cap \xi$ and $L \cap \xi$ are dilations. For $\xi \in G_{n,j}$, we have

$$\tilde{V}_{j,-p}(K \cap \xi, L \cap \xi) = \frac{1}{j} \int_{\mathbb{S}^{n-1} \cap \xi} \rho_K^{j+p}(u) \rho_L^{-p}(u) d\mathscr{H}^{j-1}(u). \tag{2.3}$$

Let ε^n denote the class of n-dimensional origin-symmetric ellipsoids in $\mathbb{R}^n$, if $E \in \varepsilon^n$, denote by d_E its maximal principle radius and $u_E \in \mathbb{S}^{n-1}$ be its corresponding principal direction. Then, $h_E(u) \geqslant d_E |u \cdot u_E|$, for $u \in \mathbb{S}^{n-1}$.

The following Lemmas will be useful in the next section.

LEMMA 2.1. ([6]) *Suppose that* $\{T_i\}_{i\in\mathbb{N}} \subseteq SL(n)$, *then*

$$\|T_i\| \to \infty \Leftrightarrow \|T_i^{-1}\| \to \infty.$$

Therefore, $\{T_i\}_{i\in\mathbb{N}}$ *is bounded if and only if* $\{T_i^{-1}\}_{i\in\mathbb{N}}$ *is bounded.*

LEMMA 2.2. ([6]) *Suppose that* $\{E_i\}_{i\in\mathbb{N}} \subseteq \varepsilon^n$ *and* $V_n(E_i) = a > 0$, *for all* $i \in \mathbb{N}$. *Then* $\{E_i\}_{i\in\mathbb{N}}$ *is bounded if and only if* $\{E_i^*\}_{i\in\mathbb{N}}$ *is bounded.*

The following Lemma gathers some properties of the dual affine quermassintegral given by (1.1).

LEMMA 2.3. ([6]) *Suppose* K, $\{K_i\}_{i\subseteq\mathbb{N}} \subseteq \mathscr{K}_0^n$, *and* $1 \leqslant j \leqslant n-1$. *Then*
(1) $\tilde{\Lambda}_j(\lambda K) = \lambda^j \tilde{\Lambda}_j(K)$, *for* $\lambda > 0$;
(2) $\tilde{\Lambda}_j(TK) = |det(T)|^{\frac{j}{n}} \tilde{\Lambda}_j(K)$, *for* $T \in GL(n)$;
(3) $\tilde{\Lambda}_j(K_i) \to \tilde{\Lambda}_j(K)$, *if* $K_i \to K$.

3. L_p mixed dual affine quermassintegrals

In this section, we will establish a variational formula of $\tilde{\Lambda}_j(K)$, which also is called the jth affine mean intersection.

THEOREM 3.1. *Suppose* $K,L \in \mathscr{K}_0^n$, $1 \leqslant j \leqslant n-1$ *and* $p \geqslant 1$, $\tilde{\Lambda}_j(K)$ *be the* j*th affine mean intersection of* K, *then*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \bar{\Lambda}_j(K\hat{+}_p\varepsilon\cdot L) = -\frac{j}{p}\tilde{\Lambda}_j(K)\frac{\int_{G_{n,j}} \tilde{V}_{j,-p}(K\cap\xi, L\cap\xi) V_j(K\cap\xi)^{n-1} d\mu_j(\xi)}{\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)}.$$

Proof. For $K,L \in \mathscr{K}_0^n$, there exist r and R, $0 < r < R < \infty$, such that

$$rB^n \subseteq K \subseteq RB^n \text{ and } rB^n \subseteq L \subseteq RB^n.$$

According to the definition of L_p harmonic combination, we have

$$K\hat{+}_p\varepsilon\cdot L \subseteq K,\ \varepsilon > 0 \text{ and } K\hat{+}_p\varepsilon\cdot L \to K,\ \varepsilon \to 0^+.$$

For $\xi \in G_{n,j}$, we have

$$(K\hat{+}_p\varepsilon\cdot L)\cap\xi \nearrow K\cap\xi,\ \varepsilon \to 0^+.$$

Since V_j is continuous and positive, so

$$V_j\big((K\hat{+}_p\varepsilon\cdot L)\cap\xi\big) \nearrow V_j(K\cap\xi),\ \varepsilon \to 0^+.$$

By the monotone convergence theorem, we obtained that

$$\lim_{\varepsilon\to 0^+} \int_{G_{n,j}} V_j\big((K\hat{+}_p\varepsilon\cdot L)\cap\xi\big)^n d\mu_j(\xi) = \int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi). \tag{3.1}$$

In order to compute the derivation of $V_j(K\hat{+}_p\varepsilon\cdot L)$, by (2.1) and (2.3), we have

$$\lim_{\varepsilon\to 0^+}\frac{(\rho_K^{-p}+\varepsilon\rho_L^{-p})^{-\frac{j}{p}}-\rho_K^j}{\varepsilon}=-\frac{j}{p}\rho_K^{j+p}\rho_L^{-p},\ on\ \mathbb{S}^{n-1}\cap\xi.$$

Moreover, note that

$$\begin{aligned}\left|\frac{(\rho_K^{-p}+\varepsilon\rho_L^{-p})^{-\frac{j}{p}}-\rho_K^j}{\varepsilon}\right|&=\left|\frac{(\rho_K^{-p}+\varepsilon\rho_L^{-p})^{-\frac{j}{p}}-\rho_K^j}{(\rho_K^{-p}+\varepsilon\rho_L^{-p})-\rho_K^{-p}}\right|\left|\frac{(\rho_K^{-p}+\varepsilon\rho_L^{-p})-\rho_K^{-p}}{\varepsilon}\right|\\&=\left|\frac{[(\rho_K^{-p}+\varepsilon\rho_L^{-p})^{-1}]^{\frac{j}{p}}-[(\rho_K^{-p})^{-1}]^{\frac{j}{p}}}{(\rho_K^{-p}+\varepsilon\rho_L^{-p})-\rho_K^{-p}}\right|\rho_L^{-p}\\&\leqslant\frac{j}{p}\rho_K^{j+p}\rho_L^{-p}\leqslant\frac{j}{p}R^{j+p}r^{-p},\end{aligned}\tag{3.2}$$

uniformly on $\mathbb{S}^{n-1}\cap\xi$. By the Lebesgue dominated conbergence theorem and (2.3), we have

$$\begin{aligned}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)&=\frac{1}{j}\lim_{\varepsilon\to 0^+}\int_{\mathbb{S}^{n-1}\cap\xi}\frac{(\rho_K^{-p}+\varepsilon\rho_L^{-p})^{-\frac{j}{p}}-\rho_K^j}{\varepsilon}d\mathscr{H}^{j-1}\\&=-\frac{j}{p}\tilde{V}_{j,-p}(K\cap\xi,L\cap\xi).\end{aligned}\tag{3.3}$$

Now, we prove $\{\varepsilon^{-1}[V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)^n-V_j(K\cap\xi)^n]:\varepsilon>0,\xi\in G_{n,j}\}$ is uniformly bounded. By (3.2), we obtain

$$\begin{aligned}\left|\frac{V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)-V_j(K\cap\xi)}{\varepsilon}\right|&\leqslant\frac{1}{j}\int_{\mathbb{S}^{n-1}}\left|\frac{(\rho_K^{-p}+\varepsilon\rho_L^{-p})^{-\frac{j}{p}}-\rho_K^j}{\varepsilon}\right|d\mathscr{H}^{j-1}\\&\leqslant\frac{j}{p}\omega_j r^{-p}R^{j+p}.\end{aligned}$$

Then,

$$\begin{aligned}&\varepsilon^{-1}|V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)^n-V_j(K\cap\xi)^n|\\&\quad=\left|\frac{V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)^n-V_j(K\cap\xi)^n}{V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)-V_j(K\cap\xi)}\right|\left|\frac{V_j((K\hat{+}_p\varepsilon\cdot L)\cap\xi)-V_j(K\cap\xi)}{\varepsilon}\right|\\&\quad\leqslant\frac{j}{p}\omega_jR^{p+j}r^{-p}nV_j(K\cap\xi)^{n-1}\\&\quad\leqslant\frac{jn}{p}\omega_j^n r^{-p}R^{jn+p}.\end{aligned}$$

Therefore, according to the definition of $\tilde{\Lambda}_j$, formula (3.1), the above estimate and the Lebesgue dominated convergence theorem, and (3.3), we obtain that

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \tilde{\Lambda}_j(K \hat{+}_p \varepsilon \cdot L) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \frac{\omega_n}{\omega_j}\left(\int_{G_{n,j}} V_j((K \hat{+}_p \varepsilon \cdot L) \cap \xi)^n d\mu_j(\xi)\right)^{\frac{1}{n}} \\
&= \frac{\omega_n}{n\omega_j}\left(\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)\right)^{\frac{1}{n}-1} \\
&\quad \times \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\left(\int_{G_{n,j}} V_j((K \hat{+}_p \varepsilon \cdot L)\cap\xi)^n d\mu_j(\xi)\right) \\
&= \tilde{\Lambda}_j(K)\left(\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)\right)^{-1} \\
&\quad \times \int_{G_{n,j}} V_j(K\cap\xi)^{n-1}\cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} V_j((K \hat{+}_p \varepsilon \cdot L)\cap\xi) d\mu_j(\xi) \\
&= -\frac{j}{p}\tilde{\Lambda}_j(K)\frac{\int_{G_{n,j}} \tilde{V}_{j,-p}(K\cap\xi, L\cap\xi) V_j(K\cap\xi)^{n-1} d\mu_j(\xi)}{\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)}.
\end{aligned}$$

We obtained the desired formula. □

Specially, if we take $p=1$, it becomes the Theorem 3.1 obtain in [6]. We introduce the j-th affine intersection measure of K.

DEFINITION 3.1. ([6]) Suppose that $K \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1$. The geometric measure

$$\tilde{\mu}_j(K,\omega) = \left(\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)\right)^{-1} \int_\omega V_j(K\cap\xi)^n d\mu_j(\xi),$$

for a Borel set $\omega \subseteq G_{n,j}$ is called the j-th affine intersection measure of K. The affine intersection measure $\tilde{\mu}_j(K,\cdot)$ is a probability measure on $G_{n,j}$, and it is absolutely continuous with respect to μ_j. Observe that $\tilde{\mu}_j(\lambda K,\cdot) = \tilde{\mu}_j(K,\cdot)$ for $\lambda > 0$. Specially, $\tilde{\mu}_0(K,\cdot) = \mu_0$, $\tilde{\mu}_n(K,\cdot) = \mu_n$ and $\tilde{\mu}_j(B^n,\cdot) = \mu_j$.

Now, we give the definition of the jth L_p mixed dual affine mean intersection of K and L.

DEFINITION 3.2. Suppose that $K, L \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1$, $p \geqslant 1$. The geometric inequality

$$\bar{\Lambda}_{j,-p}(K,L) = \int_{G_{n,j}} \frac{\tilde{V}_{j,-p}(K\cap\xi, L\cap\xi)}{V_j(K\cap\xi)} d\tilde{\mu}_j(K,\xi),$$

is called the jth L_p mixed dual affine mean intersection of K and L.

Theorem 3.1 grants that

$$\bar{\Lambda}_{j,-p}(K,L) = -\frac{p}{j\tilde{\Lambda}_j(K)} \lim_{\varepsilon\to 0^+} \frac{\tilde{\Lambda}_j(K\hat{+}_p\varepsilon\cdot L) - \tilde{\Lambda}_j(K)}{\varepsilon}, \tag{3.4}$$

Specially, $\bar{\Lambda}_{j,-p}(K,K) = 1$, $\bar{\Lambda}_{n,-p}(K,L) = \frac{\tilde{V}_{n,-p}(K,L)}{V_n(K)}$. Therefore, the jth L_p mixed dual affine mean intersection $\bar{\Lambda}_{j,-p}(K,L)$, $1 \leqslant j \leqslant n-1$, $p \geqslant 1$ is an extension of the normalized L_p dual mixed volume $\frac{\tilde{V}_{n,-p}(K,L)}{V_n(K)}$.

PROPOSITION 3.2. *Suppose that* $K,L \in \mathscr{K}_0^n$, $\{K_i\}_{i\in\mathbb{N}}$ *and* $\{L_i\}_{i\in\mathbb{N}} \subset \mathscr{K}_0^n$, *and* $1 \leqslant j \leqslant n-1$, $p \geqslant 1$. *Then,*

(1) $\bar{\Lambda}_{j,-p}(\lambda K,\mu L) = \lambda^p\mu^{-p}\bar{\Lambda}_{j,-p}(K,L)$ *for* $\lambda > 0$, $\mu > 0$;

(2) $\bar{\Lambda}_{j,-p}(TK,TL) = \bar{\Lambda}_{j,-p}(K,L)$ *for* $T \in GL(n)$;

(3) $\bar{\Lambda}_{j,-p}(K_i,L_i) \to \bar{\Lambda}_{j,-p}(K,L)$ *if* $K_i \to K, L_i \to L$.

Proof. By Definition 3.2 and formula (2.3), $\tilde{\mu}_j(\lambda K,\cdot) = \tilde{\mu}_j(K,\cdot)$ and the homogeneity of V_j, the first assertion follows.

The second assertion is obtained by Lemma 2.3, formula (3.4) and the fact

$$TK\hat{+}_p\varepsilon\cdot TL = T(K\hat{+}_p\varepsilon\cdot L),$$

for $T \in GL(n)$.

According to Definition 3.1, and Definition 3.2, $\bar{\Lambda}_{j,-p}(K,L)$ can be represent by the following formula

$$\bar{\Lambda}_{j,-p}(K,L) = \left(\frac{\omega_n}{\omega_j}\right)^n \tilde{\Lambda}_j(K)^{-n} \int_{G_{n,j}} \tilde{V}_{j,-p}(K\cap\xi, L\cap\xi)V_j(K\cap\xi)^{n-1} d\mu_j(\xi).$$

If $K_i \to K$ and $L_i \to L$, there exists $0 < r < R < \infty$, such that $rB^n \subseteq K,K_i,L,L_i \subseteq RB^n$. So $\tilde{V}_{j,-p}(K\cap\xi, L\cap\xi)V_j(K\cap\xi)^{n-1} \leqslant \omega_j^n R^{jn+p}r^{-p}$, which shows $\{\tilde{V}_{j,-p}(K\cap\xi, L\cap\xi)V_j(K\cap\xi)^{n-1} : i \in \mathbb{N}\}$ is uniformly bounded on $G_{n,j}$. Combining with the Lebesgue dominated convergence theorem and Lemma 2.3, assertion (3) follows. □

The following Propositions about the measure of $\mu_j(K,\omega)$ are obtained in [6].

PROPOSITION 3.3. ([6]) *Suppose that* $K \in \mathscr{K}_0^n$, $T \in SL(n)$ *and* $1 \leqslant j \leqslant n-1$. *Then for a Borel set* $\omega \subseteq G_{n,j}$, $\tilde{\mu}_j(TK,\omega) = \tilde{\mu}_j(K,T^{-1}\omega)$.

PROPOSITION 3.4. ([6]) *Suppose that* $K \in \mathscr{K}_0^n$, $\{K_i\}_{i\in\mathbb{N}} \subseteq \mathscr{K}_0^n$ *and* $1 \leqslant j \leqslant n-1$. *If* $K_i \to K$, *then* $\tilde{\mu}_j(K_i,\cdot) \to \tilde{\mu}_j(K,\cdot)$ *weakly.*

4. L_p intersection mean ellipsoids

In this section, we define a family of new ellipsoids associated with convex bodies according to solve the following optimization problems.

PROBLEM $P_{j,p}$. Suppose that K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, $1 \leqslant j \leqslant n-1$, and $p \geqslant 1$. Among all origin-symmetric ellipsoids E, find one to solve the constrained minimization problem

$$\min_E V(E) \quad \text{subject to} \quad \bar{\Lambda}_{j,-p}(K,E) \leqslant 1.$$

PROBLEM $\bar{P}_{j,p}$. Suppose that K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, $1 \leqslant j \leqslant n-1$, and $p \geqslant 1$. Among all origin-symmetric ellipsoids E, find one to solve the constrained minimization problem

$$\min_E \bar{\Lambda}_{j,-p}(K,E) \quad \text{subject to} \quad V(E) \leqslant \omega_n.$$

Firstly, we will show the solution of Problem $P_{j,p}$ and Problem $\bar{P}_{j,p}$ only differ by a scale factor in the following Lemma.

LEMMA 4.1. *Suppose that $K \in \mathscr{K}_0^n$, and $1 \leqslant j \leqslant n-1$.*
(1) *If E_0 is a solution to Problem $P_{j,p}$, then*

$$\left(\frac{\omega_n}{V(E_0)}\right)^{\frac{1}{n}} E_0$$

is a solution to $\bar{P}_{j,p}$.
(2) *If E_1 is a solution to Problem $\bar{P}_{j,p}$, then*

$$\bar{\Lambda}_{j,-p}(K,E_1)^{\frac{1}{p}} E_1$$

is a solution to Problem $P_{j,p}$.

Proof. (1) Assume that $E \in \{E \in \varepsilon^n : V(E) \leqslant \omega_n\}$. By Proposition 3.2, we have

$$\bar{\Lambda}_{j,-p}(K, \bar{\Lambda}_{j,-p}(K,E)^{\frac{1}{p}} E) = 1.$$

Then

$$V(E_0) \leqslant V(\bar{\Lambda}_{j,-p}(K,E)^{\frac{1}{p}} E) = \bar{\Lambda}_{j,-p}(K,E)^{\frac{n}{p}} V(E).$$

Therefore, from $\bar{\Lambda}_{j,-p}(K,E_0) \leqslant 1$ and Proposition 3.2, we have

$$\bar{\Lambda}_{j,-p}(K,E) \geqslant \left(\frac{V(E_0)}{V(E)}\right)^{\frac{p}{n}} \geqslant \left(\frac{V(E_0)}{\omega_n}\right)^{\frac{p}{n}} \bar{\Lambda}_{j,-p}(K,E_0) = \bar{\Lambda}_{j,-p}\left(K, \left(\frac{\omega_n}{V(E_0)}\right)^{\frac{1}{n}} E_0\right).$$

On the other hand, note that $V\left(\left(\frac{\omega_n}{V(E_0)}\right)^{\frac{1}{n}}E_0\right)=\omega_n$, so we obtain that $\left(\frac{\omega_n}{V(E_0)}\right)^{\frac{1}{n}}E_0$ is a solution to Problem $\bar{P}_{j,p}$.

(2) Assume that $E\in\{E\in\varepsilon^n:\bar{\Lambda}_{j,-p}(K,E)\leqslant 1\}$. Since $V\left(\left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{n}}E\right)=\omega_n$, it follows that

$$\bar{\Lambda}_{j,-p}(K,E_1)\leqslant\bar{\Lambda}_{j,-p}\left(K,\left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{n}}E\right)=\left(\frac{V(E)}{\omega_n}\right)^{\frac{p}{n}}\bar{\Lambda}_{j,-p}(K,E).$$

The above inequality can also be rewritten as

$$V(\bar{\Lambda}_{j,-p}(K,E_1)^{\frac{1}{p}}E_1)=\bar{\Lambda}_{j,-p}(K,E_1)^{\frac{n}{p}}V(E_1)\leqslant\frac{V(E)}{\omega_n}\bar{\Lambda}_{j,-p}(K,E)^{\frac{n}{p}}V(E_1)\leqslant V(E).$$

Because $\bar{\Lambda}_{j,-p}(K,\bar{\Lambda}_{j,-p}(K,E_1)^{\frac{1}{p}}E_1)=1$, so we obtain that $\bar{\Lambda}_{j,-p}(K,E_1)^{\frac{1}{p}}E_1$ is a solution to Problem $P_{j,p}$. □

LEMMA 4.2. *Suppose that $K\in\mathscr{K}_0^n$ and $1\leqslant j\leqslant n-1$. Then*

(1) $\min\{V(E):E\in\varepsilon^n,\bar{\Lambda}_{j,-p}(K,E)\leqslant 1\}=\min\{V(E):E\in\varepsilon^n,\bar{\Lambda}_{j,-p}(K,E)=1\}$;

(2) $\min\{\bar{\Lambda}_{j,-p}(K,E):E\in\varepsilon^n,V(E)\leqslant\omega_n\}=\min\{\bar{\Lambda}_{j,-p}(K,E):E\in\varepsilon^n,V(E)=\omega_n\}$.

Proof. (1) Set $A=\min\{V(E):E\in\varepsilon^n,\bar{\Lambda}_{j,-p}(K,E)\leqslant 1\}$, and $B=\min\{V(E):E\in\varepsilon^n,\bar{\Lambda}_{j,-p}(K,E)=1\}$. Given an ellipsoid $E_0\in A$ with $\bar{\Lambda}_{j,-p}(K,E_0)<1$. By Proposition 3.2, we have

$$\bar{\Lambda}_{j,-p}(K,\bar{\Lambda}_{j,-p}(K,E_0)^{\frac{1}{p}}E_0)=1.$$

That is the ellipsoid $\bar{\Lambda}_{j,-p}(K,E_0)^{\frac{1}{p}}E_0\in A$. Since

$$V(\bar{\Lambda}_{j,-p}(K,E_0)^{\frac{1}{p}}E_0)=\bar{\Lambda}_{j,-p}(K,E_0)^{\frac{n}{p}}V(E_0)<V(E_0).$$

That means E_0 cannot be a minimum of A, then we prove the equivalence.

The same method can be applied in the second assertion. So we complete the proof. □

In order to prove the existence of the solution of Problem $\bar{P}_{j,p}$, we need the following Lemma see ([19, 20]).

LEMMA 4.3. *Let f is a continuous function on $\mathbb{S}^{n-1}$, $\xi\subset G_{n,j}$ be a j-dimensional subspace of $G_{n,j}$, $1\leqslant j\leqslant n-1$. Then*

$$\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}f(u)d\mathscr{H}^{n-1}(u)=\int_{G_{n,j}}\frac{1}{j\omega_j}\int_{\mathbb{S}^{n-1}\cap\xi}f(v)d\mathscr{H}^{j-1}(v)d\mu_j(\xi),\tag{4.1}$$

Now we can give the existence of the solution of Problem $\bar{P}_{j,p}$.

THEOREM 4.4. *Their exists a solution to Problem $\bar{P}_{j,p}$.*

Proof. If $K \in \mathscr{K}_0^n$, there exists r and R $(0 < r < R < \infty)$, such that $rB^n \subseteq K \subseteq RB^n$. If E is an origin-symmetric ellipsoid, by Definition 3.2, formula (2.2), (2.3), (4.1) and the fact that $\int_{\mathbb{S}^{n-1}} |u\cdot v|^p d\mathscr{H}^{n-1}(v) < \infty$ for $u \in \mathbb{S}^{n-1}$, we have the following computation

$$\begin{aligned}
\bar{\Lambda}_{j,-p}(K,E) &= \frac{\int_{G_{n,j}} \tilde{V}_{j,-p}(K\cap\xi, E\cap\xi)V_j(K\cap\xi)^{n-1}d\mu_j(\xi)}{\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)} \\
&\geqslant \frac{\int_{G_{n,j}} \tilde{V}_{j,-p}(K\cap\xi, E\cap\xi)V_j(rB^n\cap\xi)^{n-1}d\mu_j(\xi)}{\int_{G_{n,j}} V_j(RB^n\cap\xi)^n d\mu_j(\xi)} \\
&= \left(\frac{r}{R}\right)^{jn} \frac{1}{r^j\omega_j} \int_{G_{n,j}} \frac{1}{j} \int_{\mathbb{S}^{n-1}\cap\xi} \rho_K^{j+p}\rho_E^{-p} d\mathscr{H}^{j-1}(u)d\mu_j(\xi) \\
&\geqslant \left(\frac{r}{R}\right)^{jn} r^p d_{E^*}^p \int_{G_{n,j}} \frac{1}{j\omega_j} \int_{\mathbb{S}^{n-1}\cap\xi} |u\cdot u_{E^*}|^p d\mathscr{H}^{j-1}(u)d\mu_j(\xi) \\
&= \left(\frac{r}{R}\right)^{jn} r^p d_{E^*}^p \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}\cap\xi} |u\cdot u_{E^*}|^p d\mathscr{H}^{n-1}(u). \qquad (4.2)
\end{aligned}$$

Thus, for many minimizing sequence of ellipsoids $\{E_i\}_{i\in\mathbb{N}} \subseteq \varepsilon^n$ for Problem $\bar{P}_{j,p}$, when i is sufficiently large, then

$$\bar{\Lambda}_{j,-p}(K,E_i) \leqslant \bar{\Lambda}_{j,-p}(K,B^n) < \infty. \qquad (4.3)$$

From above estimate, we can know the maximal principle radius sequence $\{d_{E_i^*}\}_{i\in\mathbb{N}}$ is bounded. And $V(E_i) = \omega_n$, $i \in \mathbb{N}$, and Lemma 2.2, we obtain that any minimizing sequence of ellipsoids $\{E_i\}_{i\in\mathbb{N}}$ for Problem $\bar{P}_{j,p}$ is bounded. By the Blaschke selection theorem, there exists a convergent subsequence $\{E_{i_k}\}_{i\in\mathbb{N}}$ converging to an origin-symmetric ellipsoid E_0. From the continuity of volume, we have $V(E_0) = \omega_n > 0$. This implies that E_0 is non degenerate, and E_0 is a solution to Problem $\bar{P}_{j,p}$. This completes the proof. $\square$

THEOREM 4.5. *There exists a unique solution to Problem $\bar{P}_{j,p}$.*

Proof. Assume that $E_1, E_2 \in \varepsilon^n, E_1 \neq E_2$ are solutions of Problem $\bar{P}_{j,p}$. We assume that $E_i = T_iB^n$, T_i is symmetric and positive definite with $det(T_i) = 1, i = 1,2$. And $T_1 \neq \lambda T_2$ for all $\lambda > 0$, according to the Minkowski inequality for symmetric and positive definite matrices, it follows that

$$det\left(\frac{T_1^{-p}+T_2^{-p}}{2}\right)^{\frac{1}{n}} > \frac{1}{2}det(T_1^{-p})^{\frac{1}{n}} + \frac{1}{2}det(T_2^{-p})^{\frac{1}{n}} = 1.$$

Let

$$T_3^{-p} = det\left(\frac{T_1^{-p}+T_2^{-p}}{2}\right)^{-\frac{1}{n}} \frac{T_1^{-p}+T_2^{-p}}{2}, and\ E_3 = T_3B^n.$$

So, $T_3 \in SL(n)$, and for all $u \in \mathbb{S}_{n-1}$, we have

$$\begin{aligned}
\rho_{E_3}^{-p}(u) &= |T_3^{-p}u| \\
&= det\left(\frac{T_1^{-p}+T_2^{-p}}{2}\right)^{-1}\left|\frac{T_1^{-p}u+T_2^{-p}u}{2}\right| \\
&< \left|\frac{T_1^{-p}u+T_2^{-p}u}{2}\right| \leqslant \frac{1}{2}|T_1^{-p}u| + \frac{1}{2}|T_2^{-p}u| \\
&= \frac{1}{2}\rho_{E_1}^{-p}(u) + \frac{1}{2}\rho_{E_2}^{-p}(u).
\end{aligned}$$

Therefore, from (2.3) and Definition 3.2, we have

$$\begin{aligned}
\bar{\Lambda}_{j,-p}(K,E_3) &< \frac{1}{2}\bar{\Lambda}_{j,-p}(K,E_1) + \frac{1}{2}\bar{\Lambda}_{j,-p}(K,E_2) \\
&= \bar{\Lambda}_{j,-p}(K,E_1) = \bar{\Lambda}_{j,-p}(K,E_2).
\end{aligned}$$

But the fact that $T_3 \in SL(n)$ and the assumption on E_1 and E_2, we obtain

$$\bar{\Lambda}_{j,-p}(K,E_3) \geqslant \bar{\Lambda}_{j,-p}(K,E_1) = \bar{\Lambda}_{j,-p}(K,E_2).$$

which contradicts the above assumption. This completes the proof. □

According to Theorem 4.4 and Theorem 4.5, we introduce the following ellipsoids.

DEFINITION 4.1. Let $K \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1$, $p \geqslant 1$. Among all origin-symmetric ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min_E V(E)\ \ subject\ to\ \ \bar{\Lambda}_{j,-p}(K,E) \leqslant 1,$$

is called the L_p *intersection mean ellipsoid* of order j of K, and denoted by $S_{j,p}(K)$.

Among all origin-symmetric ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min_E \bar{\Lambda}_{j,-p}(K,E)\ \ subject\ to\ \ V(E) = \omega_n,$$

is called the *normalized* L_p *intersection mean ellipsoid* of order j of K, and denoted by $\bar{S}_{j,p}(K)$.

COROLLARY 4.6. *Suppose that $K \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1$, $p \geqslant 1$. Then for $T \in GL(n)$,*

$$S_{j,p}(TK) = T(S_{j,p}K).$$

Proof. According to Definition 4.1 and Proposition 3.2, if $T \in GL(n)$, we have

$$\bar{\Lambda}_{j,-p}(K, T^{-1}(S_{j,p}(TK))) = \bar{\Lambda}_{j,-p}(TK, S_{j,p}(TK)) \leqslant 1.$$

So $V(S_{j,p}K) \leqslant V(T^{-1}(S_{j,p}(TK)))$. Then we have $V(T(S_{j,p}K)) \leqslant V(S_{j,p}(TK))$. Since $\bar{\Lambda}_{j,-p}(TK, T(S_{j,p}K)) = \bar{\Lambda}_{j,-p}(K, S_{j,p}K) \leqslant 1$, then $T(S_{j,p}K) \in \{E \in \varepsilon^n : \bar{\Lambda}_{j,-p}(TK, E) \leqslant 1\}$. From Theorem 4.5, it follows that $S_{j,p}(TK) = T(S_{j,p}K)$, we complete the proof. □

COROLLARY 4.7. *Suppose that $K \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1, p \geqslant 1$. Then for $E \in \varepsilon^n$, we have*

$$S_{j,p}E = E$$

Proof. By Corollary 4.6, it suffices to prove that $S_{j,p}B^n = B^n$. Let $S_{j,p}B^n = TB^n$, $T \in GL(n)$. By Lemma 2.3, we have

$$\tilde{\Lambda}_j(S_{j,p}B^n) = |detT|^{\frac{j}{n}} \tilde{\Lambda}_j(B^n).$$

By Lemma 4.2 and Lemma 5.2, we have

$$1 = \bar{\Lambda}_{j,-p}(B^n, S_{j,p}B^n) \geqslant \left(\frac{\tilde{\Lambda}_j(B^n)}{\tilde{\Lambda}_j(S_{j,p}B^n)}\right)^{\frac{p}{j}} = \left(\frac{1}{|detT|}\right)^{\frac{p}{n}} = \left(\frac{V(B^n)}{V(S_{j,p}B^n)}\right)^{\frac{p}{n}}.$$

Hence, $V(B^n) \leqslant V(S_{j,p}B^n)$. On the other hand $\bar{\Lambda}_{j,-p}(B^n, B^n) = 1$, we have $B^n \in \{E \in \varepsilon^n : \bar{\Lambda}_{j,-p}(B^n, E) \leqslant 1\}$. From Theorem 4.5, we have $S_{j,p}B^n = B^n$. □

LEMMA 4.8. *Suppose that $K, \{K_i\}_{i\in\mathbb{N}} \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1$, $p \geqslant 1$. If $K_i \to K$, then $\{\bar{S}_{j,p}K, \bar{S}_{j,p}K_i, i \in \mathbb{N}\}$ is bounded from above.*

Proof. Since $K_i \in \mathscr{K}_0^n, K_i \to K \in \mathscr{K}_0^n$, there exists r and R, $0 < r < R < \infty$, such that

$$rB^n \subseteq K \subseteq RB^n, \ \ and \ rB^n \subseteq K_i \subseteq RB^n,$$

for all $i \in \mathbb{N}$.

Since $\bar{S}_{j,p}K \in \varepsilon^n$,by (4.2), we have

$$\bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K) \geqslant \left(\frac{r}{R}\right)^{jn} r^p d^p_{\bar{S}^*_{j,p}K} \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} |u \cdot u_E^*|^p d\mathscr{H}^{n-1}(u)$$

Here, $\bar{S}^*_{j,p}K$ is the polar body of $\bar{S}_{j,p}K$. from Definition 4.1, we have

$$\bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K) \leqslant \bar{\Lambda}_{j,-p}(K, B^n) \leqslant \left(\frac{R}{r}\right)^{jn+p}$$

Therefore, it follows that

$$d_{\bar{S}^*_{j,p}K} \leqslant \left(\frac{n\omega_n}{\int_{\mathbb{S}^{n-1}} |u \cdot u_E^*|^p d\mathscr{H}^{n-1}(u)}\right)^{\frac{1}{p}} \left(\frac{R^{2jn+p}}{r^{2jn+2p}}\right)^{\frac{1}{p}}$$

For $\bar{S}^*_{j,p}K_i, i \in \mathbb{N}$, the proof is similar. Thus, we have $\{\bar{S}_{j,p}K, \bar{S}_{j,p}K_i, i \in \mathbb{N}\}$ is bounede from above. It completes the proof. □

THEOREM 4.9. *Suppose that $K, K_i \in \mathscr{K}_0^n, i \in \mathbb{N}$ and $1 \leqslant j \leqslant n-1, p \geqslant 1$. If $K_i \to K$, then*

$$\lim_{i\to\infty} S_{j,p}K_i = S_{j,p}K.$$

Proof. From Lemma 4.8, there exists a constant $0 < R < \infty$, such that all the ellipsoids $\bar{S}_{j,p}K, \bar{S}_{j,p}K_i, i \in \mathbb{N}$ are in the set

$$\varepsilon = \{E \in \varepsilon^n : V(E) = \omega_n \; and \; E \subseteq RB^n\}$$

By the compactness of ε^n, the boundedness of $\{K, K_i, \ \ i \in \mathbb{N}\}$ and Proposition 3.2(3), it follows that

$$\lim_{i\to\infty} \bar{\Lambda}_{j,-p}(K_i, E) = \bar{\Lambda}_{j,-p}(K, E), \; uniformly \; in \; E \in \varepsilon^n.$$

By Definition 4.1, we have

$$\begin{aligned}
\lim_{i\to\infty} \bar{\Lambda}_{j,-p}(K_i, \bar{S}_{j,p}K_i) &= \lim_{i\to\infty} \min_{E\in\varepsilon^n} \bar{\Lambda}_{j,-p}(K_i, E) \\
&= \min_{E\in\varepsilon^n} \lim_{i\to\infty} \bar{\Lambda}_{j,-p}(K_i, E) \\
&= \min_{E\in\varepsilon^n} \bar{\Lambda}_{j,-p}(K, E) \\
&= \bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K). \qquad (4.4)
\end{aligned}$$

If we assume $E_0 \neq \bar{S}_{j,p}K$. Since $\{\bar{S}_{j,p}K_i\}_{i\in\mathbb{N}} \subseteq \varepsilon^n$, from the compactness of ε^n, the Blaschke selection theorem and $E_0 \neq \bar{S}_{j,p}K$, we can know there exists a convergent subsequence $\{\bar{S}_{j,p}K_{i_k}\}_{k\in\mathbb{N}}$ such that $\bar{S}_{j,p}K_{i_k} \to E_0 \in \varepsilon^n$, but $E_0 \neq \bar{S}_{j,p}K$. So

$$\begin{aligned}
\bar{\Lambda}_{j,-p}(K, E_0) &= \bar{\Lambda}_{j,-p}(K, \lim_{k\to\infty} \bar{S}_{j,p}K_{i_k}) \\
&= \lim_{k\to\infty} \bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K_{i_k}) \\
&= \lim_{k\to\infty} \bar{\Lambda}_{j,-p}(K_{i_k}, \bar{S}_{j,p}K_{i_k}) \\
&= \bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K).
\end{aligned}$$

Therefore, by Definition 4.1, we have $\bar{S}_{j,p}K = E_0$, which is a contradiction. That is, $\lim_{i\to\infty} \bar{S}_{j,p}K_i = \bar{S}_{j,p}K$. From this limit, (4.4) and $S_{j,p}K = \bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K)^{\frac{1}{p}} \bar{S}_{j,p}K$, we have

$$\lim_{i\to\infty} S_{j,p}K_i = \lim_{i\to\infty} \bar{\Lambda}_{j,-p}(K_i, \bar{S}_{j,p}K_i)^{\frac{1}{p}} \bar{S}_{j,p}K_i = \bar{\Lambda}_{j,-p}(K, \bar{S}_{j,p}K)^{\frac{1}{p}} \bar{S}_{j,p}K = S_{j,p}K,$$

the desired statement is obtained. This completes the proof. □

5. New sharp affine isoperimetric inequalities

In the section, we will give some new sharp affine isoperimetric inequalities for jth L_p mixed dual affine mean intersection and L_p intersection mean ellipsoids. The following Lemma will be useful.

LEMMA 5.1. ([1]) *Suppose that* $K,L \in \mathscr{K}_0^n$ *and* $2 \leqslant j \leqslant n-1$*. If* $K \cap \xi$ *is a dilate of* $L \cap \xi$ *for each* $\xi \in G_{n,j}$*, then* K *is a dilate of* L*.*

Now we give the inequality of the jth L_p mixed dual affine mean intersection of K and L.

LEMMA 5.2. *Suppose that* $K\,L \in \mathscr{K}_0^n$ *and* $1 \leqslant j \leqslant n-1, p \geqslant 1$*. We have*

$$\bar{\Lambda}_{j,-p}(K,L) \geqslant \left(\frac{\tilde{\Lambda}_j(K)}{\tilde{\Lambda}_j(L)}\right)^{\frac{p}{j}}.$$

When $2 \leqslant j \leqslant n-1$*, equality holds if and only if* K *is a dilate of* L*. Moreover, if* K,L *are origin-symmetric, we have*

$$\bar{\Lambda}_{1,-p}(K,L) \geqslant \left(\frac{V(K)}{V(L)}\right)^{\frac{p}{n}},$$

equality holds if and only if K *is a dilate of* L*.*

Proof. By Definition 3.1, formula (2.2), the dual Minkowski inequality, the Jensen inequality, Definition 3.2 and the definition of $\tilde{\Lambda}_j$, we have

$$\begin{aligned}
\bar{\Lambda}_{j,-p}(K,L) &= \int_{G_{n,j}} \frac{\tilde{V}_{j,-p}(K\cap\xi, L\cap\xi)}{V_j(K\cap\xi)} d\tilde{\mu}_j(K,\xi) \\
&\geqslant \int_{G_{n,j}} \frac{V_j(K\cap\xi)^{\frac{j+p}{j}} V_j(L\cap\xi)^{-\frac{p}{j}}}{V_j(K\cap\xi)} d\tilde{\mu}_j(K,\xi) \\
&= \int_{G_{n,j}} \left(\frac{V_j(L\cap\xi)^n}{V_j(K\cap\xi)^n}\right)^{-\frac{p}{jn}} d\tilde{\mu}_j(K,\xi) \\
&\geqslant \left(\int_{G_{n,j}} \frac{V_j(L\cap\xi)^n}{V_j(K\cap\xi)^n} d\tilde{\mu}_j(K,\xi)\right)^{-\frac{p}{jn}} \\
&= \left(\int_{G_{n,j}} \frac{V_j(L\cap\xi)^n}{V_j(K\cap\xi)^n} \left(\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)\right)^{-1} V_j(K\cap\xi)^n d\mu_j(\xi)\right)^{-\frac{p}{jn}} \\
&= \left(\frac{\int_{G_{n,j}} V_j(L\cap\xi)^n d\mu_j(\xi)}{\int_{G_{n,j}} V_j(K\cap\xi)^n d\mu_j(\xi)}\right)^{-\frac{p}{jn}} \\
&= \left(\frac{\Lambda_j(K)}{\Lambda_j(L)}\right)^{\frac{p}{j}}.
\end{aligned}$$

With equalities conditions for each $\xi \in G_{n,j}$, there is some $\lambda_\xi > 0$, such that $L\cap\xi = \lambda_\xi(K\cap\xi)$, and $\frac{V_j(L\cap\xi)}{V_j(K\cap\xi)}$ is a constant on $G_{n,j}$. So we know λ_ξ is a constant on $G_{n,j}$, and $K\cap\xi$ is a dilate of $L\cap\xi$, for each $\xi \in G_{n,j}$. By Lemma 5.1, when $2 \leqslant j \leqslant n-1$, we have K is a dilate of L. Secondly, suppose that $j=1$ and K,L are origin-symmetric. For $\xi \in G_{n,1}$, $\xi = Ru$ for some $u \in \mathbb{S}^{n-1}$, then $\mathbb{S}^{n-1}\cap\xi = \{-u,u\}$, $V_1(K\cap\xi) = 2\rho_K(u)$. By the dual Minkowski inequality, we have

$$\begin{aligned}\bar{\Lambda}_{1,-p}(K,L) &= \frac{\int_{G_{n,1}} \tilde{V}_{1,-p}(K\cap\xi,L\cap\xi)V_1(K\cap\xi)^{n-1}d\mu_1(\xi)}{\int_{G_{n,1}} V_1(K\cap\xi)^n d\mu_1(\xi)}\\ &= \frac{\int_{\mathbb{S}^{n-1}} \rho_k^{n+p}(u)\rho_L^{-p}(u)d\mathscr{H}^{n-1}(u)}{\int_{\mathbb{S}^{n-1}} \rho_k^{n}(u)d\mathscr{H}^{n-1}(u)}\\ &= \frac{\tilde{V}_{n,-p}(K,L)}{V(K)} \geqslant \left(\frac{V(K)}{V(L)}\right)^{\frac{p}{n}}.\end{aligned}$$

With equality if and only if K is a dilate of L. It completes the proof. □

THEOREM 5.3. *Suppose that $K \in \mathscr{K}_0^n$ and $1 \leqslant j \leqslant n-1, p \geqslant 1$. We have*

$$\tilde{\Lambda}_j(K) \leqslant \omega_n^{\frac{n-j}{n}} V(S_{j,p}K)^{\frac{j}{n}}.$$

When $2 \leqslant j \leqslant n-1$, equality holds if and only if K is an origin-symmetric ellipsoid.

Proof. According to Lemma 4.2 and Lemma 5.2, we have

$$1 = \bar{\Lambda}_{j,-p}(K,S_{j,p}K) \geqslant \left(\frac{\tilde{\Lambda}_j(K)}{\tilde{\Lambda}_j(S_{j,p}K)}\right)^{\frac{p}{j}}.$$

Thus, $\tilde{\Lambda}_j(K) \leqslant \tilde{\Lambda}_j(S_{j,p}K)$. When $2 \leqslant j \leqslant n-1$, equality holds if and only if K is an origin-symmetric ellipsoid. For $S_{j,p}K \in \varepsilon^n$, from Theorem 4.5, we have

$$V\left(\left(\frac{V(S_{j,p}K)}{\omega_n}\right)^{\frac{1}{n}} B^n\right) = V(S_{j,p}K)), \text{ then } \left(\frac{V(S_{j,p}K)}{\omega_n}\right)^{\frac{1}{n}} B^n = S_{j,p}K,$$

by Lemma 2.3 and the fact $\tilde{\Lambda}_j(B^n) = \omega_n$, we have

$$\tilde{\Lambda}_j(S_{j,p}K) = \tilde{\Lambda}_j\left(\left(\frac{V(S_{j,p}K)}{\omega_n}\right)^{\frac{1}{n}} B^n\right) = \left(\frac{V(S_{j,p}K)}{\omega_n}\right)^{\frac{j}{n}} \tilde{\Lambda}_j(B^n) = \omega_n^{\frac{n-j}{n}} V(S_{j,p}K)^{\frac{j}{n}}.$$

It completes the proof. □

THEOREM 5.4. *Suppose that K is an origin-symmetric convex body in $\mathbb{R}^n$. Then*

$$V(S_{1,p}^*K)V(K) \leqslant \omega_n^2,$$

with equality if and only if K is an origin-symmetric ellipsoid.

Proof. From Definition 4.1 and Lemma 5.2, it follows that

$$1=\bar{\Lambda}_{1,-p}(K,S_{1,p}K)\geqslant\left(\frac{V(K)}{V(S_{1,p}K)}\right)^{\frac{p}{n}}.$$

Thus, $V(K)\leqslant V(S_{1,p}K)$, with equality if and only if K is an origin-symmetric ellipsoid.

For $S_{1,p}K\in\varepsilon^n$, by the Blaschke-Santaló inequality, we have

$$V(S^*_{1,p}K)V(K)\leqslant V(S^*_{1,p}K)V(S_{1,p}K)=\omega_n^2.$$

It completes the proof. □

THEOREM 5.5. *Suppose that $K\in\mathscr{K}_0^n$. Then,*

$$V(IK)\leqslant\frac{\omega_{n-1}^n}{\omega_n^{n-2}}V(S_{n-1,p}K)^{n-1},$$

with equality if and only if K is an origin-symmetric ellipsoid.

Proof. For $K\in\mathscr{K}_0^n$, by the definition of $\tilde{\Lambda}_{n-1}(K)$, it follows that

$$\begin{aligned}\tilde{\Lambda}_{n-1}(K)&=\frac{\omega_n}{\omega_{n-1}}\left(\int_{G_{n,n-1}}V_{n-1}(K\cap\xi)^n d\mu_{n-1}(\xi)\right)^{\frac{1}{n}}\\&=\frac{\omega_n}{\omega_{n-1}}\left(\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}V_{n-1}(K\cap u^\perp)^n d\mathscr{H}^{n-1}(u)\right)^{\frac{1}{n}}\\&=\frac{\omega_n}{\omega_{n-1}}\left(\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\rho_{IK}^n(u)d\mathscr{H}^{n-1}(u)\right)^{\frac{1}{n}}\\&=\frac{\omega_n^{\frac{n-1}{n}}}{\omega_{n-1}}V(IK)^{\frac{1}{n}}.\end{aligned}$$

From Definition 4.1 and Lemma 5.2, we have

$$1=\bar{\Lambda}_{n-1,-p}(K,S_{n-1,p}K)\geqslant\left(\frac{\tilde{\Lambda}_{n-1}(K)}{\tilde{\Lambda}_{n-1}(S_{n-1,p}K)}\right)^{\frac{p}{n-1}}=\left(\frac{V(IK)}{V(I(S_{n-1,p}K))}\right)^{\frac{p}{n(n-1)}}.$$

Thus, $V(IK)\leqslant V(I(S_{n-1,p}K))$, with equality if and only if K is an origin-symmetric ellipsoid. Since $S_{n-1,p}K\in\varepsilon^n$, by the fact that $IE=\frac{\omega_{n-1}}{\omega_n}V(E)E^*$ (see, e. g. [4]), and the Blaschke-Santaló inequality, we have

$$V(I(S_{n-1,p}K))=\frac{\omega_{n-1}^n}{\omega_n^n}V(S_{n-1,p}K)^nV(S^*_{n-1,p}K)=\frac{\omega_{n-1}^n}{\omega_n^{n-2}}V(S_{n-1,p}K)^{n-1}.$$

It completes the proof. □

Acknowledgements. The authors would like to strongly thank the anonymous referee for the very valuable comments and helpful suggestions that directly lead to improve the original manuscript.

REFERENCES

[1] Y. BURAGO AND V. ZALGALLER, *Geometric Inequality*, Springer, New York, 1988.

[2] W. FIREY, *p-means of convex bodies*, Math. Scand **10** (1962), 17–24.

[3] R. GARDNER, *Geometric Tomography*, Cambridge University Press, Cambridge, 2006.

[4] E. GRINBERG, *Isoperimetric inequalities and identities for k-dimensional cross-sections of convex bodies*, Math. Ann. **291** (1991), 75–86.

[5] O. GULERYUZ, E. LUTWAK, D. YANG, AND G. ZHANG, *Information-theoretic inequalities for contoured probability distributions*, IEEE Trans. Inf. Theory **48** (2002), 2377–2383.

[6] J. HU, G. XIONG, AND D. ZOU, *Affine isoperimetric inequalities for intersection mean ellipsoids*, Calc. Var. **58, 191** (2019).

[7] J. HU, G. XIONG, AND D. ZOU, *On mixed L_p John ellipsoids*, Adv. Geom. **19** (2019), 297–312.

[8] D. HUG, E. LUTWAK, D. YANG, AND G. ZHANG, *On the L_p Minkowski problem for polytopes*, Discrete Comput. Geom. **33** (2005), 699–715.

[9] E. LUTWAK, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

[10] E. LUTWAK, *The Brunn-Minkowski-Firey Theory I: Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[11] E. LUTWAK, *The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.

[12] E. LUTWAK, D. YANG, AND G. ZHANG, *A new affine invariant for polytopes and schneider's projection problem*, Trans. Amer. Math. Soc. **353** (5) (2001), 1767–1779.

[13] E. LUTWAK, D. YANG, AND G. ZHANG, *Sharp affine L_p sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38.

[14] E. LUTWAK, D. YANG, AND G. ZHANG, *Moment-entropy inequalities*, Ann. Probab. **32** (2004), 757–774.

[15] E. LUTWAK, D. YANG, AND G. ZHANG, *Lp John ellipsoids*, Proc. Lond. Math. Soc. **90** (2005), 497–520.

[16] E. LUTWAK, D. YANG, AND G. ZHANG, *Optimal Sobolev norms and the L^p Minkowski problem*, Int. Math. Res. Not. **2006** (2006).

[17] E. LUTWAK, D. YANG, AND G. ZHANG, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), 365–387.

[18] E. LUTWAK, D. YANG, AND G. ZHANG, *Orlicz projection bodies*, Adv. Math. **223** (2010), 220–242.

[19] D. REN, *Topics in Integral Geometry*, World Scientific, Singapore, 1994.

[20] L. SANTALÓ, *Integral Geometry and Geometric Probability*, Cambridge University Press, Cambridge, 2004.

[21] D. ZOU AND G. XIONG, *Orlicz-John ellipsoids*, Adv. Math. **265** (2014), 132–168.

[22] D. ZOU AND G. XIONG, *Orlicz-Legendre ellipsoids*, J. Geom. Anal. **26** (2016), 2474–2502.

[23] D. ZOU AND G. XIONG, *New affine inequalities and projection mean ellipsoids*, Calc. Var. **58, 44** (2019).

(Received March 17, 2021)

Xin Cai
School of Mathematics and Statistics
Guizhou University of Finance and Economics
Guiyang, Guizhou 550025, People's Republic of China
e-mail: 760311694@qq.com

Fangwei Chen
School of Mathematics and Statistics
Guizhou University of Finance and Economics
Guiyang, Guizhou 550025, People's Republic of China
e-mail: cfw-yy@126.com