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SOME NOTES ON JENSEN–MERCER'S TYPE INEQUALITIES; EXTENSIONS AND REFINEMENTS WITH APPLICATIONS

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Abstract. In this paper we study inequalities corresponding to Jensen-Mercer's inequality. Some new extensions of Niezgoda's inequality and the integral version of Jensen-Mercer's inequality are given. The obtained inequalities do not only generalize the former ones, but our proofs are natural and simple. They clearly show the structure of such inequalities: they consist of two parts, a discrete or integral Jensen's inequality and then a majorization type inequality. Another purpose of the paper is to provide a deeper understanding of the methods used to refine Jensen-Mercer's and the corresponding inequalities. Moreover, some new refinements of these inequalities are obtained. Finally, some applications related to Fejér's and Hermite-Hadamard inequalities are given.

1. Introduction

A function $f : C \to \mathbb{R}$ defined on an interval $C \subset \mathbb{R}$ is called convex, if for any x, $y \in C$ and every $t \in [0,1]$ one has

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y).$$

Let the set I denote either $\{1,\ldots,n\}$ for some $n \geqslant 1$ or $\mathbb{N}_+ := \{1,2,\ldots\}$. We say that the numbers $(p_i)_{i\in I}$ represent a discrete probability distribution if $p_i \geqslant 0$ $(i \in I)$ and $\sum\limits_{i\in I} p_i = 1$.

There are a lot of important inequalities for convex functions. Perhaps the most useful among them are the discrete and integral Jensen's inequalities.

THEOREM 1. (discrete Jensen's inequality, see [13]) *Let $C \subset \mathbb{R}$ be an interval, and let $f : C \to \mathbb{R}$ be a convex function.*

(a) If $p_1,\ldots,p_n$ represent a discrete probability distribution, and $x_1,\ldots,x_n \in C$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i). \tag{1}$$

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(b) If $p_1, p_2, \ldots$ represent a discrete probability distribution, and $x_1, x_2, \ldots \in C$ such that the series $\sum_{i=1}^{\infty} p_i x_i$ and $\sum_{i=1}^{\infty} p_i f(x_i)$ are absolutely convergent, then $\sum_{i=1}^{\infty} p_i x_i$ lies in C and

$$f\left(\sum_{i=1}^{\infty} p_i x_i\right) \leqslant \sum_{i=1}^{\infty} p_i f(x_i). \tag{2}$$

THEOREM 2. (integral Jensen's inequality, see [13]) *Let φ be an integrable function on a probability space $(X, \mathscr{A}, \mu)$ taking values in an interval $C \subset \mathbb{R}$. Then $\int_X \varphi d\mu$ lies in C. If f is a convex function on C such that $f \circ \varphi$ is μ-integrable, then*

$$f\left(\int_X \varphi d\mu\right) \leqslant \int_X f \circ \varphi d\mu. \tag{3}$$

An interesting variant of Theorem 1 (a) was discovered by Mercer [12], namely:

THEOREM 3. (Jensen-Mercer's inequality) *If C is an interval, $f : C \to \mathbb{R}$ is a convex function, $p_1, \ldots, p_n$ represent a discrete probability distribution, and $x_1, \ldots, x_n \in [a, b] \subset C$, then*

$$f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) \leqslant f(a) + f(b) - \sum_{i=1}^{n} p_i f(x_i). \tag{4}$$

Mercer's original proof consists of two parts: first, a direct application of the discrete Jensen's inequality (Theorem 1 (a)) gives that

$$f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(a + b - x_i), \tag{5}$$

and then it is shown that

$$\sum_{i=1}^{n} p_i f(a + b - x_i) \leqslant f(a) + f(b) - \sum_{i=1}^{n} p_i f(x_i). \tag{6}$$

Obviously, inequality (6) is sharper than inequality (4), and the essence of Theorem 3 is contained in (6). Mercer's proof of inequality (6) is based on the following observation: the pairs a, b and x_i, $a + b - x_i$ $(i = 1, \ldots, n)$ possess the same mid-point. Another proof of Theorem 3 was given by Witkowski in paper [19]. It differs from the proof of Mercer, although it is similar in principle to that. Niezgoda [14] extended Theorem 3, and the principal tool in his treatment is majorization. His result is the next:

THEOREM 4. (Niezgoda's inequality) *Let $f : C \to \mathbb{R}$ be a continuous convex function on interval $C \subset \mathbb{R}$. Suppose $\mathbf{a} = (a_1, \ldots, a_m)$ with $a_j \in C$, and $\mathbf{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in C$ for all $i,\ j$. If $\mathbf{a}$ majorizes each row of $\mathbf{X}$, that is*

$$\mathbf{x}_i = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \mathbf{a} \text{ for each } i = 1, \ldots, n,$$

then we have the inequality

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} p_i x_{ij}\right) \leqslant \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} p_i f(x_{ij}), \tag{7}$$

where $p_1, \ldots, p_n$ represent a discrete probability distribution.

Inequality (7) (similarly to (4)) can also be divided into two parts, namely

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} p_i x_{ij}\right) \leqslant \sum_{i=1}^{n} p_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) \tag{8}$$

and

$$\sum_{i=1}^{n} p_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) \leqslant \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} p_i f(x_{ij}). \tag{9}$$

Inequality (8) follows easily from the discrete Jensen's inequality (Theorem 1 (a)) while inequality (9) can be verified by using majorization. It can be seen that inequality (9) is the essential part of Theorem 4.

The integral version of Jensen-Mercer's inequality is given in [2], and can be stated as follows.

THEOREM 5. *Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $\varphi : X \to [a,b]$ be a measurable function. Then for any continuous convex function $f : [a,b] \to \mathbb{R}$,*

$$f\left(a+b-\int_X \varphi\, d\mu\right) \leqslant \int_X f\circ(a+b-\varphi)\, d\mu \tag{10}$$

$$\leqslant \frac{b-\int_X \varphi\, d\mu}{b-a} f(b) + \frac{\int_X \varphi\, d\mu - a}{b-a} f(a) \leqslant f(a)+f(b) - \int_X f\circ\varphi\, d\mu. \tag{11}$$

In this result, inequalities in different types are clearly recognizable: inequality (10) is an easy consequence of the integral Jensen's inequality (Theorem 2), and the middle and the third inequalities come from the integral form of the Lah-Ribarič inequality (see [11]). It is worth emphasizing that the first term in (11) gives a refinement of the inequality

$$\int_X f\circ(a+b-\varphi)\, d\mu \leqslant f(a)+f(b) - \int_X f\circ\varphi\, d\mu, \tag{12}$$

which is the integral analogue of (6).

An integral version of Theorem 4 was given in [1].

In order to establish our results, we need some known results which are given in Section 2. In Section 3 we extend Niezgoda's inequality and the integral version of Jensen-Mercer's inequality. The obtained inequalities do not only generalize the studied ones, but our proofs are natural and simple. They clearly show the structure discussed above: first a discrete or integral Jensen's inequality is applied and then a majorization type inequality is used. The main purpose of Section 4 is to provide a deeper understanding of the methods used to refine Jensen-Mercer's and the corresponding inequalities. Moreover, some new refinements of these inequalities are obtained. Finally, in Section 5 some applications related to Fejér and Hermite-Hadamard inequalities are given.

2. Preliminary results

We introduce a majorization relation for finite sequences of real numbers.

DEFINITION 1. Let $C \subset \mathbb{R}$ be an interval. We say that $\mathbf{x} := (x_1, \ldots, x_n) \in C^n$ majorizes $\mathbf{y} := (y_1, \ldots, y_n) \in C^n$, written $\mathbf{x} \succ \mathbf{y}$, if

$$\sum_{i=1}^{k} x_{[i]} \geqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, \ldots, n-1 \text{ and } \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[1]} \geqslant x_{[2]} \geqslant \ldots \geqslant x_{[n]}$ and $y_{[1]} \geqslant y_{[2]} \geqslant \ldots \geqslant y_{[n]}$ are the entries of $\mathbf{x}$ and $\mathbf{y}$, respectively, in decreasing order.

The following classical result is often called majorization inequality (see [7]).

THEOREM 6. *Let $C \subset \mathbb{R}$ be an interval, and let $f : C \to \mathbb{R}$ be a convex function. If $\mathbf{x} := (x_1, \ldots, x_n) \in C^n$ and $\mathbf{y} := (y_1, \ldots, y_n) \in C^n$ such that $\mathbf{x}$ majorizes $\mathbf{y}$, then*

$$\sum_{i=1}^{n} f(x_i) \geqslant \sum_{i=1}^{n} f(y_i).$$

Inequalities (6), (9) and (12) can be proved by applying majorization inequality. This and our establishments about the structure of inequalities (4), (7) and (8–9) suggest that by applying generalizations of majorization inequality, Theorem 4 is expected to be extended. For this purpose we give two generalizations of majorization inequality.

The first one is called Fuchs' inequality (see [4]).

THEOREM 7. *Let $C \subset \mathbb{R}$ be an interval, and let $f : C \to \mathbb{R}$ be a convex function. If $(x_1, \ldots, x_n) \in C^n$, $(y_1, \ldots, y_n) \in C^n$ and $q_1, \ldots, q_n$ are real numbers such that*

(a) $x_1 \geqslant \ldots \geqslant x_n$ *and* $y_1 \geqslant \ldots \geqslant y_n$,

(b) $\sum_{k=1}^{r} q_k x_k \leqslant \sum_{k=1}^{r} q_k y_k \quad (r = 1, \ldots, n-1)$,

(c) $\sum_{k=1}^{n} q_k x_k = \sum_{k=1}^{n} q_k y_k$, *then*

$$\sum_{i=1}^{n} q_i f(x_i) \leqslant \sum_{i=1}^{n} q_i f(y_i).$$

The second result is called weighted Hardy-Littlewod-Pólya inequality (see [13]).

THEOREM 8. *Let $C \subset \mathbb{R}$ be an interval, and let $f : C \to \mathbb{R}$ be a continuous convex function. If $(x_1,\ldots,x_n) \in C^n$, $(y_1,\ldots,y_n) \in C^n$ and $q_1,\ldots,q_n$ are nonnegative numbers such that*

(a) $x_1 \geqslant \ldots \geqslant x_n$,

(b) $\sum_{k=1}^{r} q_k x_k \leqslant \sum_{k=1}^{r} q_k y_k \quad (r = 1,\ldots,n-1)$,

(c) $\sum_{k=1}^{n} q_k x_k = \sum_{k=1}^{n} q_k y_k$, *then*

$$\sum_{i=1}^{n} q_i f(x_i) \leqslant \sum_{i=1}^{n} q_i f(y_i).$$

We stress that the continuity of the function f and the nonnegativity of the numbers $q_1,\ldots,q_n$ cannot be omitted in the previous result.

Majorization inequality is not a completely special case of Theorem 8, since the continuity of f is not assumed in Theorem 6.

A recent refinement of the discrete Jensen's inequality will be used too. We need the following hypotheses:

(H_1) Let the index set I denote either $\{1,\ldots,n\}$ for some $n \geqslant 1$ or $\mathbb{N}_+$. Let the index set J denote either $\{1,\ldots,k\}$ for some $k \geqslant 1$ or $\mathbb{N}_+$.

(H_2) Let $(\lambda_j)_{j\in J}$ represent a positive probability distribution. For each $j \in J$ let π_j be a permutation of the set I.

The following result comes from [5] (for the integral version, see [6]).

THEOREM 9. *Assume $(H_1 - H_2)$. Let $C \subset \mathbb{R}$ be an interval, and $f : C \to \mathbb{R}$ be a convex function. If $(x_i)_{i\in I}$ is a sequence from C and $(p_i)_{i\in I}$ represents a positive probability distribution such that the series $\sum_{i\in I} p_i x_i$ and $\sum_{i\in I} p_i f(x_i)$ are absolutely convergent, then*

$$f\left(\sum_{i\in I} p_i x_i\right) \leqslant C_{per} = C_{per}(f, \mathbf{x}, \mathbf{p}, \lambda, \pi) \tag{13}$$

$$:= \sum_{i\in I}\left(\sum_{j\in J} \lambda_j p_{\pi_j(i)}\right) f\left(\frac{\sum_{j\in J} \lambda_j p_{\pi_j(i)} x_{\pi_j(i)}}{\sum_{j\in J} \lambda_j p_{\pi_j(i)}}\right) \leqslant \sum_{i\in I} p_i f(x_i). \tag{14}$$

3. New Jensen-Mercer type inequalities

First, we prove an extension of Niezgoda's inequality.

THEOREM 10. *Let the set I denote either $\{1,\ldots,n\}$ for some $n \geqslant 1$ or $\mathbb{N}_+$. Let $C \subset \mathbb{R}$ be an interval, let $f : C \to \mathbb{R}$ be a convex function, and let $(p_i)_{i\in I}$ represent a discrete probability distribution. Let $m_i \geqslant 2$ be an integer $(i \in I)$, and for each $i \in I$ let α_{ik}, $\beta_{ik} \in \mathbb{N}_+$ $(k = 1,\ldots,l_i)$ be such that*

$$1 =: \beta_{i0} \leqslant \alpha_{i1} < \beta_{i1} < \ldots < \alpha_{il_i} < \beta_{il_i} \leqslant \alpha_{il_i+1} := m_i + 1,$$

and that $L_i := \sum_{k=1}^{l_i+1} (\alpha_{ik} - \beta_{ik-1}) > 0$. Let

$$q_{ij} \geqslant 0, \quad i \in I, \quad j \in \{1,\ldots,m_i\}$$

such that

$$\sum_{k=1}^{l_i+1} \sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij} = L_i, \quad i \in I, \tag{15}$$

and let

$$a_{ij}, x_{ij} \in C, \quad i \in I, \quad j \in \{1,\ldots,m_i\}.$$

Assume further that one of the following groups of conditions is satisfied: either

(a_1) $x_{i1} \geqslant x_{i2} \geqslant \ldots \geqslant x_{im_i}$ $(i \in I)$,

(b_1) $\sum_{k=1}^{r} q_{ik}x_{ik} \leqslant \sum_{k=1}^{r} q_{ik}a_{ik}$ $(r = 1,\ldots,m_i - 1,\ i \in I)$,

(c_1) $\sum_{k=1}^{m_i} q_{ik}x_{ik} = \sum_{k=1}^{m_i} q_{ik}a_{ik}$ $(i \in I)$,

(d_1) *f is continuous,*

or

(a_2) $q_{ij} = 1$ $(i \in I,\ j \in \{1,\ldots,m_i\})$,

(b_2) $(x_{i1},\ldots,x_{im_i}) \prec (a_{i1},\ldots,a_{im_i})$ $(i \in I)$.

If either the index set I is finite or $I = \mathbb{N}_+$ and then the series

$$\sum_{i\in I} \frac{p_i}{L_i} \left| \sum_{k=1}^{l_i+1} \sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}x_{ij} \right|, \quad \sum_{i\in I} \frac{p_i}{L_i} \left| f\left(\sum_{k=1}^{l_i+1} \sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}x_{ij} \right) \right|, \tag{16}$$

and

$$\sum_{i\in I} \frac{p_i}{L_i} \left(\sum_{k=1}^{l_i+1} \sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij} f(x_{ij}) \right), \quad \sum_{i\in I} \frac{p_i}{L_i} \left(\sum_{j=1}^{m_i} q_{ij} f(a_{ij}) - \sum_{k=1}^{l_i} \sum_{j=\alpha_{ik}}^{\beta_{ik}-1} q_{ij} f(x_{ij}) \right) \tag{17}$$

are convergent, then

$$f\left(\sum_{i\in I}\frac{p_i}{L_i}\left(\sum_{j=1}^{m_i}q_{ij}a_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_{ik}}^{\beta_{ik}-1}q_{ij}x_{ij}\right)\right) \tag{18}$$

$$\leqslant\sum_{i\in I}p_i f\left(\frac{1}{L_i}\left(\sum_{j=1}^{m_i}q_{ij}a_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_{ik}}^{\beta_{ik}-1}q_{ij}x_{ij}\right)\right) \tag{19}$$

$$\leqslant\sum_{i\in I}\frac{p_i}{L_i}\left(\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1}q_{ij}f\left(x_{ij}\right)\right) \tag{20}$$

$$\leqslant\sum_{i\in I}\frac{p_i}{L_i}\left(\sum_{j=1}^{m_i}q_{ij}f\left(a_{ij}\right)-\sum_{k=1}^{l_i}\sum_{j=\alpha_{ik}}^{\beta_{ik}-1}q_{ij}f\left(x_{ij}\right)\right). \tag{21}$$

Proof. The first case: Conditions (a_1), (b_1) (c_1) and (d_1) are satisfied. It follows from (c_1) that

$$\sum_{j=1}^{m_i}q_{ij}a_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_{ik}}^{\beta_{ik}-1}q_{ij}x_{ij}=\sum_{j=1}^{m_i}q_{ij}x_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_{ik}}^{\beta_{ik}-1}q_{ij}x_{ij}$$

$$=\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1}q_{ij}x_{ij},\quad i\in I, \tag{22}$$

where the summation index k varies from 2 if $\beta_{i0}=\alpha_{i1}$, and it varies to l_i if $\beta_{il_i}=\alpha_{il_i+1}$.

According to this and (15)

$$\frac{1}{L_i}\left(\sum_{j=1}^{m_i}q_{ij}a_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_k}^{\beta_k-1}q_{ij}x_{ij}\right)=\frac{1}{L_i}\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1}q_{ij}x_{ij}\in C,\quad i\in I. \tag{23}$$

Since $(p_i)_{i\in I}$ represent a discrete probability distribution, the series in (16) are convergent, and (23) holds, the discrete Jensen's inequality can be applied, and we obtain that

$$f\left(\sum_{i\in I}p_i\left(\frac{1}{L_i}\left(\sum_{j=1}^{m_i}q_{ij}a_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_k}^{\beta_k-1}q_{ij}x_{ij}\right)\right)\right)$$

$$\leqslant\sum_{i\in I}p_i f\left(\frac{1}{L_i}\left(\sum_{j=1}^{m_i}q_{ij}a_{ij}-\sum_{k=1}^{l_i}\sum_{j=\alpha_k}^{\beta_k-1}q_{ij}x_{ij}\right)\right)$$

$$=\sum_{i\in I}p_i f\left(\frac{1}{L_i}\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1}q_{ij}x_{ij}\right).$$

which is exactly the inequality (19).

Another application of the discrete Jensen's inequality gives that

$$f\left(\frac{1}{L_i}\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}x_{ij}\right) \leqslant \frac{1}{L_i}\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}f(x_{ij}), \quad i\in I. \tag{24}$$

By using the convergence of the first series in (17), it follows from (24) that

$$\sum_{i\in I} p_i f\left(\frac{1}{L_i}\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}x_{ij}\right) \leqslant \sum_{i\in I}\frac{p_i}{L_i}\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}f(x_{ij}), \tag{25}$$

and thus inequality (20) is proved.

Under conditions (a_1), (b_1), (c_1) and (d_1), Theorem 8 implies that

$$\sum_{j=1}^{m_i} q_{ij}f(x_{ij}) \leqslant \sum_{j=1}^{m_i} q_{ij}f(a_{ij}), \quad i\in I,$$

and hence we can find by (22), that

$$\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} q_{ij}f(x_{ij}) \leqslant \sum_{j=1}^{m_i} q_{ij}f(a_{ij}) - \sum_{k=1}^{l_i}\sum_{j=\alpha_{ik}}^{\beta_{ik}-1} q_{ij}f(x_{ij}), \quad i\in I.$$

Multiplying both sides of the previous inequality by $\frac{p_i}{L_i}$ $(i\in I)$ and then take the sum of the products, inequality (21) is obtained by using (25) and the convergence of the second series in (17).

The second case: Conditions (a_2) and (b_2) are satisfied.

We can copy the proof of the first case by using Theorem 6 instead of Theorem 8. It is easy to check that the number of terms in the sum

$$\sum_{k=1}^{l_i+1}\sum_{j=\beta_{ik-1}}^{\alpha_{ik}-1} x_{ij}$$

is just L_i, and hence condition (15) automatically holds.

The proof is complete. □

REMARK 1. (a) Under special choices of parameters in the previous result (the numbers m_i, α_{ik}, β_{ik}, l_i, a_{ij} do not depend on the index i, the index set $I=\{1,\dots,n\}$ is finite, and conditions (a_2) and (b_2) are satisfied) inequalities (18–21) contain

$$f\left(\frac{1}{L}\left(\sum_{j=1}^{m} a_j - \sum_{k=1}^{l}\sum_{j=\alpha_k}^{\beta_k-1}\sum_{i=1}^{n} p_i x_{ij}\right)\right) \leqslant \frac{1}{L}\left(\sum_{j=1}^{m} f(a_j) - \sum_{k=1}^{l}\sum_{j=\alpha_k}^{\beta_k-1}\sum_{i=1}^{n} p_i f(x_{ij})\right) \tag{26}$$

as a particular case. Inequality (26) is proved in Corollary 1 of [10] and in Corollary 2.6 of [1]. In each of these papers (26) is derived from integral inequalities, and it is difficult to discover the true structure of this inequality. Theorem 10 does not only generalize (26), but the argument is natural and simple. It clearly shows that (18–21) follow the structure of inequalities discussed in the introduction: the discrete Jensen's inequality is applied twice, and then a majorization type inequality is used.

(b) We emphasize that the index set I in Theorem 10 can also be an infinite set. There are few results dealing with variants of Jensen-Mercer's inequality in which the index set is not finite. The next inequality is proved in Theorem 4.1 and in Corollary 4.1 of [15]: Let $\sum_{n=1}^{\infty} p_i x_i$ be a convex combination of points $x_i \in [a,b]$. Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(a+b-\sum_{n=1}^{\infty} p_i x_i\right) \leqslant \frac{\sum_{n=1}^{\infty} p_i x_i - a}{b-a} f(a) + \frac{b-\sum_{n=1}^{\infty} p_i x_i}{b-a} f(b) \tag{27}$$

$$\leqslant f(a)+f(b)-\sum_{n=1}^{\infty} p_i f(x_i). \tag{28}$$

It is easy to see that inequality between the first and the third term in (27–28) is a very simple case of (18–21).

The second result concerns integrals. In our approach the emphasis is on the applicability of either Theorem 6 or Theorem 7 or Theorem 8, in contrast to papers [10] and [1], where the extension of majorization to integrals is applied. We restrict the discussion to a simple case, but the result could be formulated in a more general form similar to Theorem 10.

THEOREM 11. *Let $C \subset \mathbb{R}$ be an interval, let $f : C \to \mathbb{R}$ be a convex function, and let $(X, \mathscr{A}, \mu)$ be a probability space. Assume further that $m \geqslant 2$ is an integer,*

$$q_j \in \mathbb{R}, \quad j \in \{1,\ldots,m\}, \quad q_m > 0,$$

and

$$\varphi_j, \psi_j : X \to C, \quad j \in \{1,\ldots,m\}$$

are measurable functions such that
either

(a_1) $\varphi_1 \geqslant \varphi_2 \geqslant \ldots \geqslant \varphi_m$ *and* $\psi_1 \geqslant \psi_2 \geqslant \ldots \geqslant \psi_m$,

(b_1) $\sum_{k=1}^{r} q_k \varphi_k \leqslant \sum_{k=1}^{r} q_k \psi_k$ $(r = 1,\ldots,m-1)$,

(c_1) $\sum_{k=1}^{m} q_k \varphi_k = \sum_{k=1}^{m} q_k \psi_k$,

or

(a_2) $\varphi_1 \geqslant \varphi_2 \geqslant \ldots \geqslant \varphi_m$,

$(b_2$–$c_2)$ *conditions* (b_1) *and* (c_1) *hold,*

(d_2) $q_j \geqslant 0$ $(j \in \{1,\ldots,m\})$,

(e_2) f *is continuous.*

or

(a_3) $q_j = 1$ $(j \in \{1,\ldots,m\})$,

(b_3) $(\varphi_1(x),\ldots,\varphi_m(x)) \prec (\psi_1(x),\ldots,\psi_m(x))$ $(x \in X)$.

If the functions

$$\varphi_m, \quad f\circ\varphi_m, \quad \sum_{j=1}^{m} q_j f\circ\psi_j, \quad \sum_{j=1}^{m-1} q_j f\circ\varphi_j \tag{29}$$

are μ*-integrable, then*

$$f\left(\int_X \frac{1}{q_m}\left(\sum_{j=1}^{m} q_j\psi_j - \sum_{j=1}^{m-1} q_j\varphi_j\right)d\mu\right)$$

$$\leqslant \int_X f\circ\left(\frac{1}{q_m}\left(\sum_{j=1}^{m} q_j\psi_j - \sum_{j=1}^{m-1} q_j\varphi_j\right)\right)d\mu \tag{30}$$

$$\leqslant \frac{1}{q_m}\int_X\left(\sum_{j=1}^{m} q_j f\circ\psi_j\right)d\mu - \frac{1}{q_m}\int_X\left(\sum_{j=1}^{m-1} q_j f\circ\varphi_j\right)d\mu. \tag{31}$$

Proof. The proof is similar to the proof of Theorem 10.

Since f is Borel-measurable, the composite functions are all measurable.

We consider only the first case: Conditions (a_1), (b_1), and (c_1) are satisfied.

By (c_1), and by $q_m > 0$,

$$\varphi_m = \frac{1}{q_m}\left(\sum_{j=1}^{m} q_j\psi_j - \sum_{j=1}^{m-1} q_j\varphi_j\right).$$

As a direct application of the integral Jensen's inequality (φ_m and $f\circ\varphi_m$ are μ-integrable), we have inequality (30).

Next, we can use Theorem 7 which implies that

$$\sum_{j=1}^{m-1} q_j f(\varphi_j(x)) + q_m f\left(\frac{1}{q_m}\left(\sum_{j=1}^{m} q_j\psi_j(x) - \sum_{j=1}^{m-1} q_j\varphi_j(x)\right)\right)$$

$$\leqslant \sum_{j=1}^{m} q_j f(\psi_j(x)), \quad x\in X.$$

Since $q_m > 0$, we obtain

$$f\left(\frac{1}{q_m}\left(\sum_{j=1}^{m} q_j\psi_j(x) - \sum_{j=1}^{m-1} q_j\varphi_j(x)\right)\right)$$

$$\leqslant \frac{1}{q_m}\left(\sum_{j=1}^{m} q_j f(\psi_j(x)) - \sum_{j=1}^{m-1} q_j f(\varphi_j(x))\right), \quad x\in X.$$

Integrating both sides of the previous inequality on X with respect to the measure μ (the μ-integrability of the last two functions in (29) is used), inequality (31) is obtained.

The proof is complete. □

4. From refinements of Jensen-Mercer type inequalities

A number of papers deal with refinements of either Jensen-Mercer's inequality or Niezgoda's inequality, see e.g., [2], [3], [8] and [9]. As we have seen in the introduction these inequalities consist of two parts: a Jensen-type inequality (either 5 or 8 or 10) and a majorization-type inequality (either 6 or 9 or 12). In most papers the Jensen-type inequality is refined, but it is difficult to realize this without analyzing the proofs of the results. This phenomenon can be illustrated by Theorem 1 of [3], Theorem 3 of [8] and Theorem 3 of [9]. All these results refine Jensen-Mercer's inequality by refining the discrete Jensen's inequality (5), however, with the exception of Theorem 1 of [3], neither the results nor their proofs show this. Similarly, Theorem 7 of [8] is a refinement of Niezgoda's inequality by refining the discrete Jensen's inequality (8), but this is not evident from the result.

Since the considered inequalities contain a Jensen-type inequality, it is natural to refine them by using a new or known refinement of either the discrete or the integral Jensen's inequalities. This is an easy and effective method because there are a lot of different refinements for Jensen-type inequalities. To illustrate this, we apply Theorem 9 to Niezgoda's inequality. Exactly the same way, Theorem 9 could be applied to Theorem 10, but for the sake of simplicity and clarity we consider only Niezgoda's inequality.

THEOREM 12. *Assume $(H_1–H_2)$ with $I = \{1,\dots,n\}$. Let $f : C \to \mathbb{R}$ be a continuous convex function on interval $C \subset \mathbb{R}$. Suppose $\mathbf{a} = (a_1,\dots,a_m)$ with $a_j \in C$, and $\mathbf{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in C$ for all $i,\ j$. If $\mathbf{a}$ majorizes each row of $\mathbf{X}$, that is*

$$\mathbf{x}_i = (x_{i1},\dots,x_{im}) \prec (a_1,\dots,a_m) = \mathbf{a} \text{ for each } i = 1,\dots,n,$$

then

$$f\left(\sum_{l=1}^{m} a_l - \sum_{l=1}^{m-1}\sum_{i=1}^{n} p_i x_{il}\right)$$

$$\leqslant \sum_{i=1}^{n}\left(\sum_{j=1}^{k}\lambda_j p_{\pi_j(i)}\right) f\left(\frac{\sum_{j=1}^{k}\lambda_j p_{\pi_j(i)}\left(\sum_{l=1}^{m} a_l - \sum_{l=1}^{m-1} x_{\pi_j(i)l}\right)}{\sum_{j=1}^{k}\lambda_j p_{\pi_j(i)}}\right)$$

$$\leqslant \sum_{i=1}^{n} p_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) \leqslant \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} p_i f(x_{ij}),$$

where $p_1,\dots,p_n$ represent a positive discrete probability distribution.

Proof. By applying Theorem 9 to (8), the assertion follows immediately. The proof is complete. □

It is more interesting but more difficult to refine the considered inequalities by refining their substantial part, the majorization-type inequality. There are only a few results in this direction, see e.g. Theorem 5, Theorem 2 of [3] and Theorem 4.1 of [15].

In the following two results we give new refinements of the discrete and the integral version of Jensen-Mercer's inequality by refining inequality (6) and (12), respectively.

THEOREM 13. *Let the set I denote either $\{1,\ldots,n\}$ for some $n \geqslant 1$ or $\mathbb{N}_+$. Assume C is an interval, $f : C \to \mathbb{R}$ is a convex function, $(p_i)_{i\in I}$ represent a discrete probability distribution, and $(x_i)_{i\in I}$ is a sequence from $[a,b] \subset C$. For each $i \in I$ define*

$$y_i := \min\{x_i, a+b-x_i\}, \quad z_i := \max\{x_i, a+b-x_i\}.$$

Then for all $t \in [0,1]$ we have

$$f\left(a+b-\sum_{i\in I} p_i x_i\right) \leqslant \sum_{i\in I} p_i f(a+b-x_i)$$

$$\leqslant \sum_{i\in I} p_i \left(f(ta+(1-t)y_i) + f((1-t)z_i+tb) - f(x_i)\right)$$

$$\leqslant f(a)+f(b) - \frac{2(1-t)}{b-a}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right)\left(\sum_{i\in I} p_i y_i - a\right)$$

$$-\sum_{i\in I} p_i f(x_i) \leqslant f(a)+f(b) - \sum_{i\in I} p_i f(x_i)$$

Proof. Due to the compactness of the interval $[a,b]$, every series that occurs in the inequalities above is absolutely convergent.

Since for each $i \in I$ and for all $t \in [0,1]$

$$(x_i, a+b-x_i) \prec (ta+(1-t)y_i, (1-t)z_i+tb),$$

Theorem 6 implies that

$$f(x_i) + f(a+b-x_i) \leqslant f(ta+(1-t)y_i) + f((1-t)z_i+tb).$$

The second inequality follows from this by an elementary calculation.

Since

$$y_i \in \left[a, \frac{a+b}{2}\right], \quad z_i \in \left[\frac{a+b}{2}, b\right], \quad i \in I,$$

convexity of f yields that for each $i \in I$ and for all $t \in [0,1]$

$$f(ta+(1-t)y_i) \leqslant \frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{a+b}{2}-a}(ta+(1-t)y_i-a)+f(a)$$

and

$$f((1-t)z_i+tb) \leqslant \frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-\frac{a+b}{2}}((1-t)z_i+tb-b)+f(b).$$

By using these inequalities, a simple calculation confirms the third inequality.

The forth inequality is trivial.

The proof is complete. □

The integral version of the previous result can be found in the next assertion.

THEOREM 14. *Let* $(X,\mathscr{A},\mu)$ *be a probability space, let* $\varphi : X \to [a,b]$ *be a measurable function, and let the functions* $\chi,\ \psi : X \to [a,b]$ *define by*

$$\chi(x) := \min\{\varphi(x), a+b-\varphi(x)\}, \quad \psi(x) := \max\{\varphi(x), a+b-\varphi(x)\}.$$

Then for any convex function $f : [a,b] \to \mathbb{R}$ *and for all* $t \in [0,1]$ *we have*

$$\begin{aligned} f\left(a+b-\int_X \varphi d\mu\right) &\leqslant \int_X f\circ(a+b-\varphi)d\mu \\ &\leqslant \int_X \left(f(ta+(1-t)\chi(x))+f((1-t)\psi(x)+tb)-f(\varphi(x))\right)d\mu(x) \\ &\leqslant f(a)+f(b)-\frac{2(1-t)}{b-a}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right)\left(\int_X \chi d\mu - a\right) \\ &\quad -\int_X f\circ\varphi d\mu \leqslant f(a)+f(b)-\int_X f\circ\varphi d\mu. \end{aligned}$$

Proof. There is no problem defining the integrals, since f is Borel-measurable.

We can follow the process of proving Theorem 13.

The proof is complete. □

5. Applications

We demonstrate the applicability of our results by considering Fejér's inequality and its special case Hermite-Hadamard inequality.

The first application concerns Theorem 11.

THEOREM 15. *If* $f : [a,b] (\subset \mathbb{R}) \to \mathbb{R}$ *is a convex function, then for all* $t \in [0,1]$ *we have*

$$\int_a^b f(x)\,dx + \int_a^b f\left(\frac{t-1}{3}(x-a) + \frac{2a+b}{3}\right) dx \tag{32}$$

$$+\int_a^b f\left(\frac{2t-2}{3}(x-a) + \frac{t}{3}(a-b) + b\right) dx \leqslant \int_a^b f\left(\frac{t}{3}(x-a) + a\right) dx \tag{33}$$

$$+\int_a^b f\left(\frac{t}{3}(x-b) + b\right) dx + \int_a^b f\left(\frac{t}{3}(x-a) + \frac{2a+b}{3}\right) dx. \tag{34}$$

Proof. For each fixed $t \in [0,1]$ define the functions $\varphi_{t1}, \varphi_{t2}, \varphi_{t3}, \psi_{t1}, \psi_{t2}, \psi_{t3} : [a,b] \to \mathbb{R}$ by

$$\varphi_{t1}(x) = x, \quad \varphi_{t2}(x) = \frac{t-1}{3}(x-a) + \frac{2a+b}{3},$$

$$\varphi_{t3} = \frac{2t-2}{3}(x-a) + \frac{t}{3}(a-b) + b,$$

and

$$\psi_{t1}(x) = \frac{t}{3}(x-a) + a, \quad \psi_{t2}(x) = \frac{t}{3}(x-b) + b,$$

$$\psi_{t3}(x) = \frac{t}{3}(x-a) + \frac{2a+b}{3}.$$

It is easy to check that

$$(\varphi_{t1}(x), \varphi_{t2}(x), \varphi_{t3}(x)) \prec (\psi_{t1}(x), \psi_{t2}(x), \psi_{t3}(x)), \quad x \in [a,b]$$

for all $t \in [0,1]$, and therefore the result follows from Theorem 11.

The proof is complete. □

Next we derive some new refinements of Hermite-Hadamard inequality from the preceding result.

COROLLARY 1. *Let* $f : [a,b] (\subset \mathbb{R}) \to \mathbb{R}$ *be a convex function.*

(a) Then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{3}{8(b-a)} \int_{\frac{2a+b}{3}}^b f(x)\,dx + \frac{1}{4}\left(f(a) + f(b) + f\left(\frac{2a+b}{3}\right)\right) \tag{35}$$

$$\leqslant \frac{1}{8}\left(2f(a) + 3f(b) + 3f\left(\frac{2a+b}{3}\right)\right) \leqslant \frac{f(a)+f(b)}{2}. \tag{36}$$

(b) Define the functions $G,\ H:[0,1]\to\mathbb{R}$ *by*

$$G(t)=\frac{1}{3-t}\int_a^b f\left(\frac{1-t}{3}x+\frac{2a+tb}{3}\right)dx$$

$$+\frac{2-t}{3-t}\int_a^b f\left(\frac{2-2t}{3}x+\frac{ta+(t+1)b}{3}\right)dx$$

and

$$H(t)=\frac{1}{3-t}f\left(\frac{(5-t)a+(t+1)b}{6}\right)+\frac{2-t}{3-t}f\left(\frac{a+2b}{3}\right).$$

Then for all $t\in[0,1]$ *we have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx=G(0)\geqslant G(t)\geqslant H(t)\geqslant G(1)=H(1) \tag{37}$$

$$=\frac{1}{2}f\left(\frac{2a+b}{3}\right)+\frac{1}{2}f\left(\frac{a+2b}{3}\right)\geqslant f\left(\frac{a+b}{2}\right).$$

Proof. (a) By substituting $t=0$ in (32–34), we obtain

$$\int_a^b f(x)\,dx+\int_a^b f\left(\frac{-1}{3}(x-a)+\frac{2a+b}{3}\right)dx+\int_a^b f\left(\frac{-2}{3}(x-a)+b\right)dx$$

$$\leqslant(b-a)\left(f(a)+f(b)+f\left(\frac{2a+b}{3}\right)\right).$$

Now integrations by substitution show that

$$\int_a^b f(x)\,dx-3\int_{\frac{2a+b}{3}}^{a} f(x)\,dx-\frac{3}{2}\int_b^{\frac{2a+b}{3}} f(x)\,dx\leqslant(b-a)\left(f(a)+f(b)+f\left(\frac{2a+b}{3}\right)\right),$$

which implies inequality (35).

By applying the Hermite-Hadamard inequality to the integral on the right hand side in (35), the first inequality in (36) follows. Finally, a simple application of the discrete Jensen's inequality gives the second inequality in (36).

(b) Since

$$G(0)=\frac{1}{3}\int_a^b f\left(\frac{1}{3}x+\frac{2a}{3}\right)dx+\frac{2}{3}\int_a^b f\left(\frac{2}{3}x+\frac{b}{3}\right)dx,$$

integrations by substitution show that

$$G(0) = \int_a^{\frac{2a+b}{3}} f(x)\,dx + \int_{\frac{2a+b}{3}}^{b} f(x)\,dx = \int_a^b f(x)\,dx.$$

It is obvious that

$$G(1) = H(1) = \frac{1}{2} f\left(\frac{2a+b}{3}\right) + \frac{1}{2} f\left(\frac{a+2b}{3}\right) \geqslant f\left(\frac{a+b}{2}\right).$$

Assume $t \in]0,1[$. It follows from inequality (32–34) by an easy but tedious calculation (integrations by substitution) that

$$\int_a^b f \geqslant \frac{3}{(3-t)(1-t)} \int_{\frac{t}{3}(b-a)+a}^{\frac{2a+b}{3}} f + \frac{6-3t}{2(3-t)(1-t)} \int_{\frac{t}{3}(b-a)+\frac{2a+b}{3}}^{b-\frac{t}{3}(b-a)} f. \tag{38}$$

With the help of integrations by substitution we can get that the right hand side of (38) is exactly $G(t)$, confirm the first inequality in (37).

By applying the Hermite-Hadamard inequality to the integrals in (38), it follows that $G(t) \geqslant H(t)$, ensuring the second inequality in (37).

The proof will be complete as soon as the third inequality in (37) is established. This is an immediate consequence of the discrete Jensen's inequality.

The proof is complete. □

REMARK 2. Theorem 15 and Corollary 1 typify in many respects the advantages offered by Theorem 11. It is not hard to construct functions $\varphi_i,\ \psi_i : [a,b] \to [a,b]$ $(i = 1,\ldots,m)$ depending on one or more parameters such that

$$(\varphi_1(x),\ldots,\varphi_m(x)) \prec (\psi_1(x),\ldots,\psi_m(x)) \quad (x \in [a,b]).$$

Then by applying Theorem 11, we can obtain a general inequality for convex functions defined on $[a,b]$ as in (32-34). From this inequality interesting and useful expressions and functions (like the expression in (35) and the function G) can be derived which are associated with Hermite-Hadamard inequality. Functions similar to G and H are defined in many papers (see e.g. [18]), but the methods are different from the technique used here.

The following result corresponds to Theorem 14 and it is a generalization of Fejér's inequality.

PROPOSITION 1. *Let $(X,\mathscr{A},\mu)$ be a probability space, let $f : [a,b] \to \mathbb{R}$ be a convex function, and let $\varphi : X \to [a,b]$ be a measurable function. Define the functions $\chi,\ \psi : X \to [a,b]$ by*

$$\chi(x) := \min\{\varphi(x), a+b-\varphi(x)\}, \quad \psi(x) := \max\{\varphi(x), a+b-\varphi(x)\}.$$

If

$$\int_X f\circ\varphi d\mu = \int_X f\circ(a+b-\varphi)\,d\mu, \tag{39}$$

then for all $t\in[0,1]$ *we have*

$$f\left(a+b-\int_X \varphi d\mu\right) \leqslant \int_X f\circ\varphi d\mu$$

$$\leqslant \frac{1}{2}\int_X \left(f\left(ta+(1-t)\chi(x)\right)+f\left((1-t)\psi(x)+tb\right)\right)d\mu(x)$$

$$\leqslant \frac{1}{2}\left(f(a)+f(b)-\frac{2(1-t)}{b-a}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right)\left(\int_X \chi d\mu - a\right)\right)$$

$$\leqslant \frac{f(a)+f(b)}{2}.$$

Proof. It is an immediate consequence of Theorem 14. □

This result contains Fejér's and Hermite-Hadamard inequalities with some refinements of their right hand side.

COROLLARY 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function.*

(a) *Assume* $g:[a,b]\to\mathbb{R}$ *is a nonnegative and Lebesgue-integrable function satisfying* $g(x)=g(a+b-x)$ *for all* $x\in[a,b]$*. Then for all* $t\in[0,1]$ *we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leqslant \int_a^b f(x)\,g(x)\,dx \leqslant N_g(t)$$

$$\leqslant P_g(t) \leqslant \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx,$$

where

$$N_g(t) := \int_a^{\frac{a+b}{2}}\left(f\left(ta+(1-t)x\right)g(x)\,dx + \int_{\frac{a+b}{2}}^b f\left(tb+(1-t)x\right)g(x)\,dx\right),$$

and

$$P_g(t) := \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - \frac{(1-t)}{b-a}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right)$$

$$\cdot\int_a^{\frac{a+b}{2}} g(x)\,(x-a)\,dx.$$

(b) *Then for all* $t \in [0,1]$ *we have*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\leqslant \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(ta+(1-t)x)\,dx + \int_{\frac{a+b}{2}}^b f(tb+(1-t)x)\,dx\right)$$

$$\leqslant \frac{f(a)+f(b)}{2} - \frac{1-t}{4}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right) \leqslant \frac{f(a)+f(b)}{2}.$$

Proof. (a) Let $X := [a,b]$, let $\mathscr{A}$ be the σ-algebra of Lebesgue-measurable subsets of $[a,b]$, let the probability measure μ be defined on $\mathscr{A}$ by

$$\mu(A) := \int_A g(x)\,dx,$$

and let $\varphi : X \to [a,b]$ be given by $\varphi(x) := x$. Under these choices of parameters in Proposition 1, it is easy to check that condition (39) is satisfied for every convex function on $[a,b]$, and therefore Proposition 1 can be applied.

(b) Let $g(x) := 1 \ \ (x \in [a,b])$ in (a).

The proof is complete. □

REMARK 3. (a) Proposition 1 shows that it is worth to study the following problem: find probability spaces $(X,\mathscr{A},\mu)$ and measurable functions $\varphi : X \to [a,b]$ for which (39) holds for all convex functions $f : [a,b] \to \mathbb{R}$.

(b) The refinement of the right hand side of Fejér's inequality by $N_g(t)$ is proved in Theorem 2.7 of [17], by applying a totally different method. It follows from our new refinement by $P_g(t)$ that

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - \int_a^b f(x)g(x)\,dx \tag{40}$$

$$\geqslant \frac{1}{b-a}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right)\cdot\int_a^{\frac{a+b}{2}} g(x)(x-a)\,dx,$$

which is a positive lower bound for (40) if f is strictly convex and $\int_a^b g(x)\,dx > 0$.

(c) The following special case is proved in Theorem 2.1 of [16]:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{4} + \frac{1}{2}f\left(\frac{a+b}{2}\right) = P_1(0).$$

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