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INEQUALITIES FOR QUERMASSINTEGRALS OF (NEW) p–PARALLEL BODIES

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Abstract. In this paper, we define a new family of convex bodies related to the family of p-parallel bodies, which is determined by the 0-extreme normal vectors, and establish some inequalities for their quermassintegrals.

1. Introduction

It is well-known that the volume of convex bodies and their Minkowski sum lead to the rich and powerful classical Brunn-Minkowski theory. The core elements of this theory such as mixed volumes are defined by the classical Steiner formula, which states that the volume of the Minkowski addition of a convex body K and a positive dilation λE of a convex body E is a polynomial of degree at most n in λ:

$$V(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}W_i(K;E)\lambda^i.$$

The coefficients $W_i(K;E)$ are called *relative quermassintegrals* of K, which are a special case of mixed volumes (see [11, Section 5.1]). In particular, we have $W_0(K;E)=V(K)$ and $W_n(K;E)=V(E)$.

In the early 1960s, Firey [1] introduced the concept of L_p *Minkowski addition* of convex bodies. If K and E are convex bodies containing the origin, λ is non-negative, and $1\leqslant p<\infty$, then the L_p Minkowski addition of K and λE (the so-called relative p-outer parallel body of K relative to E) is given by an intersection of half-spaces,

$$K+_p\lambda E=\bigcap_{u\in S^{n-1}}\left\{x\in\mathbb{R}^n : x\cdot u\leqslant\left[h(K,u)^p+\lambda^p h(E,u)^p\right]^{\frac{1}{p}}\right\},$$

where $h(K,\cdot)$ is the support function of K and $x\cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$. When $p=1$, the usual Minkowski addition is obtained.

It is known that there is no polynomial expression for the quermassintegrals of the p-outer parallel bodies when $p>1$ (see [2, Theorem 10.3]).

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Recently, A. R. Martínez Fernández et al. [8] introduced the following counterpart of the L_p Minkowski addition, the so-called p*-difference*. If K and E are convex bodies containing the origin, $-r(K;E) \leqslant \lambda \leqslant 0$ and $1 \leqslant p < \infty$, the p*-difference* of K and $|\lambda|E$ (i.e., the p-inner parallel body of K relative to E) is defined by

$$K \sim_p |\lambda|E = \bigcap_{u \in S^{n-1}} \left\{x \in \mathbb{R}^n : x \cdot u \leqslant [h(K,u)^p - |\lambda|^p h(E,u)^p]^{\frac{1}{p}}\right\}.$$

Here, $r(K;E) = \max\{r \geqslant 0 : \text{there is } x \in \mathbb{R}^n \text{ with } x + rE \subseteq K\}$ is the *relative inradius* of K with respect to E. In the above definition, the convex body K needs to satisfy the geometric assumption $h(K,u) - r(K;E)h(E,u) \geqslant 0$ (see [8] for further details). In addition, we note that the convex body $K +_p \lambda E$ has $[h(K,u)^p + \lambda^p h(E,u)^p]^{\frac{1}{p}}$ as its support function, but the convex body $K \sim_p |\lambda|E$ is the Wulff shape of the function $[h(K,u)^p - |\lambda|^p h(E,u)^p]^{\frac{1}{p}}$. We write K_λ^p to denote the p-inner and outer parallel bodies of K relative to E.

As it is well known, the smallest subset of vectors needed to determine the convex body is the set of 0-extreme normal vectors. More precisely, it was shown in [9, (2.9)] that

$$K = \bigcap_{u \in \mathscr{U}_0(K)} \{x \in \mathbb{R}^n : x \cdot u \leqslant h(K,u)\},$$

where the set $\mathscr{U}_0(K)$ denotes the set of 0-extreme normal vectors of K, i.e., those ones that cannot be written as a positive combination of two linearly independent normal vectors at one and the same boundary point of K.

The well known properties of the Wulff-shape (see e.g. [11, Section 7.5]) and the fact that $\mathscr{U}_0(K_\lambda^p) \subset \mathscr{U}_0(K)$ (see [7, Proposition 4.1.11]) allow to see that p-inner parallel bodies can be expressed by merely using the 0-extreme normal vectors, namely,

$$K_\lambda^p = \bigcap_{u \in \mathscr{U}_0(K)} \left\{x \in \mathbb{R}^n : x \cdot u \leqslant [h(K,u)^p - |\lambda|^p h(E,u)^p]^{\frac{1}{p}}\right\}.$$

An alternative proof of this fact will be given in Lemma 2.

It seems reasonable to define a full system of (new) p-parallel bodies just determined by the 0-extreme normal vectors.

DEFINITION 1. Let K and E be convex bodies containing the origin, $-r(K;E) \leqslant \lambda < \infty$ and $1 \leqslant p < \infty$. The full system of (new) p-parallel bodies is defined by

$$K^p(\lambda) = \bigcap_{u \in \mathscr{U}_0(K)} \left\{x \in \mathbb{R}^n : x \cdot u \leqslant [h(K,u)^p + \operatorname{sgn}(\lambda)|\lambda|^p h(E,u)^p]^{\frac{1}{p}}\right\}.$$

Here, sgn denotes the usual sign function. Note that $K_\lambda^p \subseteq K^p(\lambda)$ for $\lambda \geqslant 0$, and the inclusion may be strict (see Remark 1). Moreover, in the above definition, the convex body K also needs to satisfy the geometric assumption $h(K,u) - r(K;E)h(E,u) \geqslant 0$ for $-r(K;E) \leqslant \lambda \leqslant 0$.

In [5], the authors established upper and lower bounds for quermassintegrals of $K^p(\lambda)$ for $-r(K;E) \leqslant \lambda \leqslant 0$. In this paper, we give lower bounds for quermassintegrals of $K^p(\lambda)$ with the equality conditions for $\lambda \geqslant 0$. The equality case characterizes the *tangential bodies*. A convex body K containing a convex body E is called a *tangential body* of E, if each 0-extreme support plane (see Section 2 for the detailed definition) of K supports E. Given a convex body E, a special tangential body of E is the relative *form body* K^* (see Section 2 for the detailed definition) of a convex body K.

THEOREM 1. *Let K and E be convex bodies containing the origin, $1 \leqslant p < \infty$ and let q be the Hölder conjugate of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Then, for $\lambda \geqslant 0$ and all $i=0,\dots,n-1$,*

$$W_i(K^p(\lambda);E) \geqslant \left(\frac{1}{1+\lambda^p}\right)^{\frac{n-i}{q}} W_i(K;E) + \lambda^p \sum_{k=0}^{n-i-1} \left(\frac{1}{1+\lambda^p}\right)^{\frac{k+1}{q}} V(K[k],K^p(\lambda)[n-i-k-1],K^*,E[i]). \quad (1)$$

If K is a tangential body of E, then equality holds in (1) for all $i=0,\dots,n-1$ and $1 \leqslant p < \infty$. Conversely, suppose E is regular and strictly convex. If equality holds in (1) for some $i \in \{0,\dots,n-1\}$, $\lambda > 0$ and $1 < p < \infty$, then K is a tangential body of E.

When $p=1$, the coefficients $\left(\frac{1}{1+\lambda^p}\right)^{\frac{(n-i)}{q}}$ and $\left(\frac{1}{1+\lambda^p}\right)^{\frac{k+1}{q}}$ in (1) should be understood as 1.

In this theorem, there are some notions involved. $V(K_1,\dots,K_n)$ denotes the n-dimensional mixed volume of the convex bodies $K_1,\dots,K_n$. Note that $W_i(K;E) = V(K,\dots K,E,\dots,E)$, where K appears $n-i$ times and L appears i times. For the sake of brevity, we denote $(K_1[r_1],\dots,K_m[r_m]) \equiv (\underbrace{K_1,\dots,K_1}_{r_1},\dots,\underbrace{K_m,\dots,K_m}_{r_m})$.

A convex body K is called *strictly convex* if its boundary bd$K$ does not contain a segment, and *regular* if the supporting hyperplane (see Section 2 for the detailed definition) of K at any boundary point is unique.

2. Background material

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets) in the Euclidean n-space $\mathbb{R}^n$. Let $\mathscr{K}_0^n$ be the subset of $\mathscr{K}^n$ consisting of all convex bodies containing the origin. We denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ and by B_n the n-dimensional unit ball. If $K \in \mathscr{K}^n$, then its support function, $h(K,\cdot):\mathbb{R}^n \to \mathbb{R}$, is defined by

$$h(K,u) = \max\{x \cdot u : x \in K\}, \quad u \in \mathbb{R}^n.$$

Obviously, for $K,L \in \mathscr{K}^n$,

$$K \subseteq L \quad \text{if and only if} \quad h(K,\cdot) \leqslant h(L,\cdot). \quad (2)$$

Let $K \in \mathscr{K}^n$. For each $u \in \mathbb{R}^n \setminus \{0\}$, the hyperplane

$$H_K(u) = \{x \in \mathbb{R}^n : x \cdot u = h(K,u)\}$$

is called the *supporting hyperplane* of K with outer normal u.

For $K_1, \dots, K_m \in \mathscr{K}^n$ and real numbers $\lambda_1, \dots, \lambda_m \geqslant 0$, the volume of the linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous polynomial. That is,

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

where $V(K_{i_1}, \dots, K_{i_n})$ are the *mixed volumes* of $K_{i_1}, \dots, K_{i_n} \in \mathscr{K}^n$. They are continuous, non-negative, symmetric, linear and monotone (with respect to set inclusion).

For $K, K_1, \dots, K_{n-1} \in \mathscr{K}^n$, the mixed volume has the integral representation (see e.g. [11, Theorem 5.1.7])

$$V(K, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h(K,u) dS(K_1, \dots, K_{n-1}; u), \tag{3}$$

where $S(K_1, \dots, K_{n-1}; \cdot)$ is the *mixed surface area measure* of $K_1, \dots, K_{n-1}$ on S^{n-1}.

An outer normal vector of K is called r-extreme normal vector, $r = 0, 1, \dots, n-1$, if it cannot be written as a positive combination of $r+2$ linearly independent normal vectors at one and the same boundary point of K. We denote the set of r-extreme normal vectors of K by $\mathscr{U}_r(K)$. Notice that each r-extreme normal vector is also an s-extreme one for $r < s \leqslant n-1$. When $r = 0$, we obtain the 0-extreme normal vectors. A support plane is said to be 0-extreme if its outer normal vector is 0-extreme.

The (relative) *form body* K^* of $K \in \mathscr{K}^n$ with respect to $E \in \mathscr{K}^n$ is defined as

$$K^* = \bigcap_{u \in \mathscr{U}_0(K)} \{x \in \mathbb{R}^n : x \cdot u \leqslant h(E,u)\}.$$

In the above equality, the set $\mathscr{U}_0(K)$ can be replaced by $\mathrm{cl}\mathscr{U}_0(K)$ because of the continuity of the support function.

3. Inequalities for quermassintegrals of $K^p(\lambda)$ for $\lambda \geqslant 0$

In order to establish some inequalities for the quermassintegrals of $K^p(\lambda)$ for $\lambda \geqslant 0$, we invoke the following binary operation which was introduced in [8]:

$$a +_p b = \begin{cases} \mathrm{sgn}_2(a,b)(|a|^p + |b|^p)^{\frac{1}{p}} & \text{if } ab \geqslant 0, \\ \mathrm{sgn}_2(a,b)(\max\{|a|,|b|\}^p - \min\{|a|,|b|\}^p)^{\frac{1}{p}} & \text{if } ab \leqslant 0, \end{cases}$$

where the function $\mathrm{sgn}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{sgn}_2(a,b) = \begin{cases} \mathrm{sgn}(a) = \mathrm{sgn}(b) & \text{if } ab > 0, \\ \mathrm{sgn}(a) & \text{if } ab \leqslant 0 \text{ and } |a| \geqslant |b|, \\ \mathrm{sgn}(b) & \text{if } ab \leqslant 0 \text{ and } |a| < |b|. \end{cases}$$

Next we list some properties of 0-extreme normal vectors, form bodies and p-difference which will be needed later on.

LEMMA 1. *Let* $K,E,L,M \in \mathcal{K}_0^n$, *and let* $1 \leqslant p < \infty$. *The following properties hold:*

(*i*) $\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K +_p L) = \mathcal{U}_0(K +_p \lambda L)$ *for* $\lambda > 0$ *(see* [7, Proposition 4.1.8] *and* [5, Lemma 3.4]*).*

(*ii*) $\mathcal{U}_0(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_0(K)$. *If* E *is regular, then* $\mathcal{U}_0(K^*) = \mathrm{cl}\,\mathcal{U}_0(K)$ *(see* [4, Lemma 2.1]*).*

(*iii*) $K +_p L \subseteq M$ *if and only if* $K \subseteq M \sim_p L$ *(see* [8, Lemma 2.2 (*iv*)]*).*

In the following we prove that the definition of p-inner parallel bodies is equivalent to the definition of $K^p(\lambda)$ for $-r(K;E) \leqslant \lambda \leqslant 0$.

LEMMA 2. *Let* $K,E \in \mathcal{K}_0^n$ *with* $h(K,u) - r(K;E)h(E,u) \geqslant 0$, *and let* $1 \leqslant p < \infty$. *Then, for* $-r(K;E) \leqslant \lambda \leqslant 0$,

$$K_\lambda^p = K^p(\lambda).$$

Proof. From the definitions of p-inner parallel body and $K^p(\lambda)$, it is easy to obtain that $K_\lambda^p \subseteq K^p(\lambda)$.

Next we need to prove the reverse inclusion. Using the definition of $K^p(\lambda)$, it follows that, for all $u \in \mathcal{U}_0(K)$,

$$h(K^p(\lambda),u) \leqslant [h(K,u)^p - |\lambda|^p h(E,u)^p]^{\frac{1}{p}},$$

which implies $h(K^p(\lambda) +_p |\lambda| E, u) \leqslant h(K,u)$ for all $u \in \mathcal{U}_0(K)$. Thus

$$\begin{aligned} K^p(\lambda) +_p |\lambda| E &= \bigcap_{S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant h(K^p(\lambda) +_p |\lambda| E, u) \right\} \\ &\subseteq \bigcap_{\mathcal{U}_0(K)} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant h(K,u) \right\} \\ &= K. \end{aligned}$$

Lemma 1 (*iii*) ensures that $K^p(\lambda) \subseteq K \sim_p |\lambda| E = K_\lambda^p$. $\square$

LEMMA 3. *Let* $K,E \in \mathcal{K}_0^n$, *and let* $1 \leqslant p < \infty$. *Then, for* $\lambda \geqslant 0$,

$$h(K^p(\lambda),u)^p = h(K,u)^p + \lambda^p h(E,u)^p \text{ for all } u \in \mathcal{U}_0(K).$$

Proof. By the definition of $K^p(\lambda)$, we get, for $\lambda \geqslant 0$ and all $u \in \mathcal{U}_0(K)$,

$$h(K^p(\lambda),u)^p \leqslant h(K,u)^p + \lambda^p h(E,u)^p.$$

On the other hand, we have

$$K^p(\lambda) \supseteq K +_p \lambda E.$$

It yields that $h(K^p(\lambda),u)^p \geqslant h(K,u)^p + \lambda^p h(E,u)^p$ for all $u \in S^{n-1}$. Thus,

$$h(K^p(\lambda),u)^p = h(K,u)^p + \lambda^p h(E,u)^p \text{ for all } u \in \mathcal{U}_0(K). \quad \square$$

Now we consider some important properties of $K^p(\lambda)$ for $\lambda \geqslant 0$.

LEMMA 4. *Let $K, E \in \mathscr{K}_0^n$, and let $1 \leqslant p < \infty$. Then, for $\lambda \geqslant 0$,*
(i) $K +_p \lambda E \subseteq K +_p \lambda K^* \subseteq K^p(\lambda)$;
(ii) $r(K^p(\lambda); E) = r(K; E) +_p \lambda$;
(iii) $\mathscr{U}_0(K^p(\lambda)) \subseteq \mathrm{cl}\mathscr{U}_0(K)$. *If E is regular, then* $\mathscr{U}_0(K^p(\lambda)) = \mathrm{cl}\mathscr{U}_0(K)$.

Proof. (i) From the definition of form body one has $K +_p \lambda E \subseteq K +_p \lambda K^*$.

In the following we prove the right-hand inclusion. By the definition of $K^p(\lambda)$, we have

$$\begin{aligned} K +_p \lambda K^* &= \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant h(K +_p \lambda K^*, u) \right\} \\ &= \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant [h(K,u)^p + \lambda^p h(K^*,u)^p]^{\frac{1}{p}} \right\} \\ &\subseteq \bigcap_{u \in \mathscr{U}_0(K)} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant [h(K,u)^p + \lambda^p h(E,u)^p]^{\frac{1}{p}} \right\} \\ &= K^p(\lambda). \end{aligned}$$

(ii) Since $r(K;E)E \subseteq K$, and together with item (i), we get $(r(K;E) +_p \lambda)E \subseteq K +_p \lambda E \subseteq K^p(\lambda)$, which implies $r(K;E) +_p \lambda \leqslant r(K^p(\lambda); E)$.

On the other hand, we notice that $r(K^p(\lambda); E)E \subseteq K^p(\lambda)$ and hence for all $u \in \mathscr{U}_0(K)$

$$r(K^p(\lambda); E) h(E,u) \leqslant h(K^p(\lambda), u) = [h(K,u)^p + \lambda^p h(E,u)^p]^{\frac{1}{p}},$$

which yields

$$[r(K^p(\lambda); E)^p - \lambda^p]^{\frac{1}{p}} h(E,u) \leqslant h(K,u) \text{ for all } u \in \mathscr{U}_0(K).$$

Thus

$$\begin{aligned} [r(K^p(\lambda); E)^p - \lambda^p]^{\frac{1}{p}} E &= \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant [r(K^p(\lambda); E)^p - \lambda^p]^{\frac{1}{p}} h(E,u) \right\} \\ &\subseteq \bigcap_{u \in \mathscr{U}_0(K)} \left\{ x \in \mathbb{R}^n : x \cdot u \leqslant h(K,u) \right\} \\ &= K. \end{aligned}$$

Therefore we have $[r(K^p(\lambda); E)^p - \lambda^p]^{\frac{1}{p}} \leqslant r(K;E)$ and hence $r(K^p(\lambda); E) \leqslant r(K;E) +_p \lambda$.

(iii) From the definition of $K^p(\lambda)$, we know that $K^p(\lambda)$ is the form body of K with respect to $E' = K +_p \lambda E$. According to Lemma 1 (ii), we get $\mathscr{U}_0(K^p(\lambda)) \subseteq \mathrm{cl}\mathscr{U}_0(K)$. Since E is regular, we can use Lemma 1 (i) to get

$$\mathscr{U}_0(E') = \mathscr{U}_0(K +_p \lambda E) = \mathscr{U}_0(K +_p E) \supseteq \mathscr{U}_0(K) \cup \mathscr{U}_0(E) = S^{n-1},$$

which implies E' is regular. By Lemma 1 (ii), we deduce $\mathscr{U}_0(K^p(\lambda)) = \mathrm{cl}\mathscr{U}_0(K)$. $\square$

REMARK 1. Note that the inclusion $K_\lambda^p \subseteq K^p(\lambda)$ for $\lambda \geqslant 0$ may be strict. In fact, we consider $K=[0,1]^2$, $E=B_2$, and $\mathscr{U}_0(K)=\{(\pm 1,0),(0,\pm 1)\}$ in $\mathbb{R}^2$. If $u\in\mathscr{U}_0(K)$,

$$h(K+_p\lambda B_2,u)=(1+\lambda^p)^{\frac{1}{p}}=h(K^p(\lambda),u).$$

Let $u=(\cos\theta,\sin\theta)\in S^1\setminus\mathscr{U}_0(K)$ for $\theta\in(0,\frac{\pi}{4}]$. Then

$$\begin{aligned}h(K+_p\lambda B_2,u)&=\left(\left(\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)\right)^p+\lambda^p\right)^{\frac{1}{p}}\\&<\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)(1+\lambda^p)^{\frac{1}{p}}\\&=h(K^p(\lambda),u).\end{aligned}$$

By the symmetry, the same argument shows that $h(K+_p\lambda B_2,u)<h(K^p(\lambda),u)$ for all $u=(\cos\theta,\sin\theta)\in S^1\setminus\mathscr{U}_0(K)$ with $\theta\in[\frac{\pi}{4},\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{4}]\cup[\frac{3\pi}{4},\pi)\cup(\pi,\frac{5\pi}{4}]\cup[\frac{5\pi}{4},\frac{3\pi}{2})\cup(\frac{3\pi}{2},\frac{7\pi}{4}]\cup[\frac{7\pi}{4},2\pi)$. Thus the p-outer parallel body $K+_p\lambda B_2$ is strictly contained in $K^p(\lambda)$ (see Figure 1).

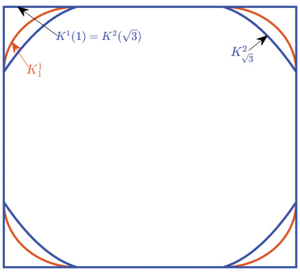

Figure 1: $K^2_{\sqrt{3}}\subset K^1_1\subset K^1(1)=K^2(\sqrt{3})$

The following lemma states the relationship between tangential bodies and $K^p(\lambda)$ for $\lambda\geqslant 0$.

LEMMA 5. *Let $K,E\in\mathscr{K}_0^n$, and let $1\leqslant p<\infty$. Then K is a tangential body of E if and only if*

$$K^p(\lambda)=(1+\lambda^p)^{\frac{1}{p}}K \text{ for } \lambda\geqslant 0.$$

Proof. If K is a tangential body of E, then $h(K,u)=h(E,u)$ for all $u\in\mathscr{U}_0(K)$. Therefore, we get, for $\lambda\geqslant 0$,

$$\begin{aligned}K^p(\lambda)&=\bigcap_{u\in\mathscr{U}_0(K)}\left\{x\in\mathbb{R}^n: x\cdot u\leqslant[h(K,u)^p+\lambda^p h(E,u)^p]^{\frac{1}{p}}\right\}\\&=\bigcap_{u\in\mathscr{U}_0(K)}\left\{x\in\mathbb{R}^n: x\cdot u\leqslant[h(K,u)^p+\lambda^p h(K,u)^p]^{\frac{1}{p}}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \bigcap_{u\in\mathscr{U}_0(K)} \left\{x\in\mathbb{R}^n : x\cdot u \leqslant (1+\lambda^p)^{\frac{1}{p}} h(K,u)\right\} \\
&= (1+\lambda^p)^{\frac{1}{p}} K.
\end{aligned}
$$

Conversely, if $K^p(\lambda) = (1+\lambda^p)^{\frac{1}{p}} K$ for $\lambda \geqslant 0$, then, by Lemma 3, we get, for all $u\in\mathscr{U}_0(K)$,

$$
h(K^p(\lambda),u) = [h(K,u)^p + \lambda^p h(E,u)^p]^{\frac{1}{p}} = (1+\lambda^p)^{\frac{1}{p}} h(K,u),
$$

which leads to $h(K,u) = h(E,u)$ for all $u\in\mathscr{U}_0(K)$. This implies that K is a tangential body of E. $\square$

REMARK 2. If K is a tangential body of E, and together with the fact that K^* is a tangential body of E, we deduce $K = K^*$. Notice that it implies $K^p(\lambda) = K +_p \lambda K^*$ for $\lambda \geqslant 0$.

Now we are in a position to prove Theorem 1.1.

Proof. From (3), (2), Lemma 4 (i) and Hölder's inequality, we get, for $\lambda \geqslant 0$,

$$
\begin{aligned}
&W_i(K^p(\lambda);E) = V(K^p(\lambda)[n-i],E[i]) \\
&= \frac{1}{n}\int_{S^{n-1}} h(K^p(\lambda),u)\,dS(K^p(\lambda)[n-i-1],E[i];u) \\
&\geqslant \frac{1}{n}\int_{S^{n-1}} h(K+_p\lambda K^*,u)\,dS(K^p(\lambda)[n-i-1],E[i];u) \\
&= \frac{1}{n}\int_{S^{n-1}} [h(K,u)^p + \lambda^p h(K^*,u)^p]^{\frac{1}{p}}\,dS(K^p(\lambda)[n-i-1],E[i];u) \\
&\geqslant \frac{1}{n}\int_{S^{n-1}} \left[\left(\frac{1}{1+\lambda^p}\right)^{\frac{1}{q}} h(K,u) + \left(\frac{\lambda^p}{1+\lambda^p}\right)^{\frac{1}{q}} \lambda h(K^*,u)\right] dS(K^p(\lambda)[n-i-1],E[i];u) \\
&= \left(\frac{1}{1+\lambda^p}\right)^{\frac{1}{q}} V(K,K^p(\lambda)[n-i-1],E[i]) + \frac{\lambda^p}{(1+\lambda^p)^{\frac{1}{q}}} V(K^p(\lambda)[n-i-1],K^*,E[i]) \\
&\geqslant \left(\frac{1}{1+\lambda^p}\right)^{\frac{2}{q}} V(K[2],K^p(\lambda)[n-i-2],E[i]) \\
&\qquad + \left(\frac{1}{1+\lambda^p}\right)^{\frac{1}{q}} \frac{\lambda^p}{(1+\lambda^p)^{\frac{1}{q}}} V(K,K^p(\lambda)[n-i-2],K^*,E[i]) \\
&\qquad + \frac{\lambda^p}{(1+\lambda^p)^{\frac{1}{q}}} V(K^p(\lambda)[n-i-1],K^*,E[i]) \\
&\quad \geqslant \cdots \geqslant \left(\frac{1}{1+\lambda^p}\right)^{\frac{n-i}{q}} W_i(K;E) \\
&\qquad + \lambda^p \sum_{k=0}^{n-i-1} \left(\frac{1}{1+\lambda^p}\right)^{\frac{k+1}{q}} V(K[k],K^p(\lambda)[n-i-k-1],K^*,E[i]).
\end{aligned} \tag{4}
$$

Next we consider the equality conditions in (1). We assume that $\lambda > 0$, otherwise the result is trivial. If equality holds in (1) for some $i \in \{0,\dots,n-1\}$ and $1 < p < \infty$, we must have

$$\begin{aligned}\int_{S^{n-1}} & h(K^p(\lambda),u)dS(K^p(\lambda)[n-i-1],E[i];u)\\ &= \int_{S^{n-1}} h(K+_p\lambda K^*,u)dS(K^p(\lambda)[n-i-1],E[i];u),\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\int_{S^{n-1}} [h(K,u)^p+\lambda^p h(K^*,u)^p]^{\frac{1}{p}} dS(K^p(\lambda)[n-i-1],E[i];u) \qquad (6)\\ &= \int_{S^{n-1}} \Big[\Big(\frac{1}{1+\lambda^p}\Big)^{\frac{1}{q}} h(K,u) + \Big(\frac{\lambda^p}{1+\lambda^p}\Big)^{\frac{1}{q}} \lambda h(K^*,u)\Big] dS(K^p(\lambda)[n-i-1],E[i];u).\end{aligned}$$

Firstly, we assume (6) holds. From Hölder's inequality, we know that

$$\big[h(K,u)^p+\lambda^p h(K^*,u)^p\big]^{\frac{1}{p}} \geqslant \Big(\frac{1}{1+\lambda^p}\Big)^{\frac{1}{q}} h(K,u) + \Big(\frac{\lambda^p}{1+\lambda^p}\Big)^{\frac{1}{q}} \lambda h(K^*,u).$$

Then we get $\big[h(K,u)^p+\lambda^p h(K^*,u)^p\big]^{\frac{1}{p}} = \big(\frac{1}{1+\lambda^p}\big)^{\frac{1}{q}} h(K,u) + \big(\frac{\lambda^p}{1+\lambda^p}\big)^{\frac{1}{q}} \lambda h(K^*,u)$ for all $u \in \operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];u)$. By the equality conditions of Hölder's inequality, we obtain $\frac{\frac{1}{1+\lambda^p}}{\frac{\lambda^p}{1+\lambda^p}} = \frac{h(K,u)^p}{\lambda^p h(K^*,u)^p}$ for all $u \in \operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];u)$. So we get $h(K,u) = h(K^*,u)$ for all $u \in \operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];u)$.

Moreover, we assume (5) holds. By Lemma 4 (i), we obtain $h(K^p(\lambda),u) = h(K+_p\lambda K^*,u)$ for all $u \in \operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];u)$. According to (6), we get $h(K,u) = h(K^*,u)$ for all $u \in \operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];u)$, and hence $h(K+_p \lambda K^*,u) = (1+\lambda^p)^{\frac{1}{p}} h(K,u)$ for all $u \in \operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];u)$. Since $E \in \mathscr{K}_0^n$ is regular and strictly convex, then $\operatorname{supp} S(K^p(\lambda)[n-i-1],E[i];\cdot) = \operatorname{cl}\mathscr{U}_i(K^p(\lambda)) \supseteq \operatorname{cl}\mathscr{U}_0(K^p(\lambda)) = \operatorname{cl}\mathscr{U}_0(K)$ (see [10, p. 135–136]), where the last equality is due to Lemma 4 (iii). From Lemma 3, one has $h(K^p(\lambda),u) = [h(K,u)^p+\lambda^p h(E,u)^p]^{\frac{1}{p}} = (1+\lambda^p)^{\frac{1}{p}} h(K,u)$ for all $u \in \mathscr{U}_0(K)$, and hence $h(K,u) = h(E,u)$ for all $u \in \mathscr{U}_0(K)$. This implies that K is a tangential body of E.

Conversely, if K is a tangential body of E, then $K = K^*$. By Remark 2, we get $K^p(\lambda) = K+_p\lambda K^*$, which means that equality holds in the first inequality of (4). In addition, since $K = K^*$ then $\frac{\frac{1}{1+\lambda^p}}{\frac{\lambda^p}{1+\lambda^p}} = \frac{h(K,u)^p}{\lambda^p h(K^*,u)^p}$ for all $u \in S^{n-1}$, and by the equality conditions of Hölder's inequality, equality holds in the second inequality of (4). In conclusion, equality holds in all the inequalities of (4). Thus equality holds in (1) for all $i = 0,\dots,n-1$ and $1 \leqslant p < \infty$. $\square$

We note that $h(K_\lambda^p,u) = [h(K,u)^p+\lambda^p h(E,u)^p]^{\frac{1}{p}}$ for $\lambda \geqslant 0$ and all $u \in S^{n-1}$. By Hölder's inequality, we get

$$\big[h(K,u)^p+\lambda^p h(E,u)^p\big]^{\frac{1}{p}} \geqslant \Big(\frac{1}{1+\lambda^p}\Big)^{\frac{1}{q}} h(K,u) + \Big(\frac{\lambda^p}{1+\lambda^p}\Big)^{\frac{1}{q}} \lambda h(E,u),$$

which implies $K_\lambda^p \supseteq \left(\frac{1}{1+\lambda^p}\right)^{\frac{1}{q}} K + \frac{\lambda^{\frac{p}{q}}}{(1+\lambda^p)^{\frac{1}{q}}} E$. Then, an analogous argument to the proof of inequality (1) shows that

$$W_i(K_\lambda^p;E) \geqslant \left(\frac{1}{1+\lambda^p}\right)^{\frac{n-i}{q}} W_i(K;E) + \lambda^p \sum_{k=0}^{n-i-1} \left(\frac{1}{1+\lambda^p}\right)^{\frac{k+1}{q}} V(K[k], K_\lambda^p[n-i-k-1], E[i+1]). \tag{7}$$

Note: For $1 < p < \infty$, the left and right-hand sides of inequality (1) are larger than those of inequality (7), respectively, which means that there is no inclusion relationship between the two inequalities. If $p = 1$ and $i = 0$, then (7) and (1) reduce to the relative Steiner formula.

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