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HARNACK INEQUALITIES FOR FUNCTIONAL SDES DRIVEN BY SUBORDINATE MULTIFRACTIONAL BROWNIAN MOTION

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Abstract. Being base on the Girsanov theorem for multifractional Brownian motion, which can be constructed by the multifractional derivative operator, we establish the Harnack inequalities for a class of stochastic functional differential equations driven by subordinate multifractional Brownian motion by an approximation technique.

1. Introduction

The dimension-free Harnack inequality with powers introduced in [37] and the log-Harnack inequality introduced in [31] have attracted more and more attentions because of its extensive applications in stochastic analysis, such as strong Feller property and contractivity properties (see [29, 30, 38]); heat kernel estimates (see [16, 34, 40, 42]); transportation-cost inequalities and properties of invariant measures (see [3, 25, 41]). Up to now, the dimension-free Harnack inequality and log-Harnack inequality have been intensively investigated for various stochastic (partial) differential equations driven by several different kinds of noise. We can refer to [2, 4, 18, 35, 39, 43, 48].

Very recently, by using the coupling argument, the Girsanov transformations and an approximation argument, Deng and Huang [11] established Harnack inequalities for the following stochastic differential equation driven by subordinate Brownian motion

$$X(t) = \xi + \int_0^t b(X(s))ds + \int_0^t F(X_s)ds + B_{S(t)}, \quad t \geqslant 0,$$

where $B = \{B_t\}_{t \geqslant 0}$ is a d-dimensional Brownian motion, $S = \{S(t)\}_{t \geqslant 0}$ is a subordinator and independent of B. The theory of subordinate Brownian motion recently received increasing attentions since they may describe some mathematical models in finance. There also exists several results on the Harnack inequality for subordinate Brownian and the time changed Brownian motion. For example, Rao et al, [28] and Mimica and Kim [23] studied the Harnack inequality for subordinate Brownian motion; Deng [13] established the Harnack inequalities for the inhomogeneous semigroup

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associated with a class of SDEs with Lévy noise containing a subordinate Brownian motion.

It is a natural question whether one can still establish the Harnack inequality when the driving noise is a more general, maybe non-Markovian process. As far as I know that the fractional Brownian motion (in short fBm) becomes the standard Brownian motion when $H = 1/2$, and the fBm W^H neither is a semimartingale nor a Markov process if $H \neq 1/2$. However, the fBm W^H, $H > 1/2$ is a long-memory process and presents an aggregation behavior. The long-memory property makes fBm as a potential candidate to model noise in mathematical finance (see [8]); in biology (see [6, 10]); in communication networks (see, for instance [44]); the analysis of global temperature anomaly [32] electricity markets [36] etc. There are several frontier works on the Harnack inequalities for stochastic (partial) differential equations driven by fractional Brownian motion, see [14, 15, 19, 45, 46, 47].

However, there is only a few result on the stochastic differential equations driven by subordinate fBm and we can only find that Deng and Schilling [12] established Harnack inequalities for stochastic differential equations driven by subordinat fBm with $H \in (0,1/2)$ and Li and Yan [21] established Harnack inequalities for stochastic differential equations driven by subordinat fBm with $H \in (1/2,1)$. On the other hand, the Hurst parameter H of the fractional Brownian motion can be dependent on time (e.g. [26]). The process was named multifractional Brownian motion, referring to the fact that the fractional parameter H was a function depending on time taking values between 0 and 1. As showed in the literature (see [20, 26]), multifractional Brownian motion seems to be a more flexible model than fractional Brownian motion. The multifractional Brownian motion possesses the good feature of fractional Brownian motion, such as Hölder continuity (see [7, 27]); self-similarity (see [24]) and long rang dependence (see [1]) etc. But, the multifractional Brownian motion is a non-stationary stochastic process which makes it more intricate to deal with stochastic differential equations driven by multifractional Brownian motion.

In connection with the above discussions, in this paper, we are interested in the following stochastic differential equation (SDE):

$$X(t) = \xi + \int_0^t b(X(s))ds + \int_0^t F(X_s)ds + B^h_{S(t)}, \quad t \geqslant 0, \tag{1}$$

where B^h is a multifractional Brownian motion with regularity function h valued on $(0,1)$, and $S = \{S(t)\}_{t\geqslant 0}$ is a subordinator and independent of B^h. We will discuss the multifractional derivative operator which acts as the inverse of the multifractional integral operator by using the variable order fractional calculus so that we can obtain the Girsanov theorem for the multifractional Brownian motion. As a result, we can establish the Harnack inequalities for a class of stochastic functional differential equations driven by subordinate multifractional Brownian motion by an approximation technique.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we devote ourselves to establish the Harnack inequalities for SDEs (1).

2. Preliminaries

2.1. Multifractional calculus

In this subsection, we will give meaning to the multifractional calculus. Many references in fractional analysis refer to this concept as fractional calculus of variable order, as the idea is to let the order of integration (or differentiation) be dependent on time (or possibly space, but for our application, time will be sufficient), see for example [33, 34] and the references therein. We will use the word multifractional calculus for the concept of fractional calculus of variable order, as this is coherent with the notion of multifractional stochastic processes.

DEFINITION 1. (Multifractional Riemann-Liouville integrals) For $0<c<d$, assume $f\in L^1([c,d])$, and $\alpha:[c,d]\to[a,b]\subset(0,\infty)$ is a differentiable function. We define the left multifractional Riemann Liouville integral operator I^{α}_{c+} by

$$(I^{\alpha}_{c+}f)(x)=\frac{1}{\Gamma(\alpha(x))}\int_c^x(x-y)^{\alpha(x)-1}f(y)dy.$$

And define the space $I^{\alpha}_{c+}L^p([0,T])$ as the image of $L^p([0,T])$ under the operator I^{α}_{c+}.

By the definition of the space $I^{\alpha}_{c+}L^p([0,T])$, we have that for all $g\in I^{\alpha}_{c+}L^p([0,T])$, $g(c)=0$. Indeed, since $g=I^{\alpha}_{c+}f$, we must have $I^{\alpha}_{c+}f(c)=0$ regardless of the function α. In the same way, some authors also proposed to generalize the fractional derivative i.e.

$$D^{\alpha}_{c+}f(x)=\frac{1}{\Gamma(1-\alpha(x))}\frac{d}{dx}\int_c^x\frac{f(t)}{(x-t)^{\alpha(x)}}dt.$$

However, by generalizing the fractional derivative in this way, the authors found that the derivative is no longer the inverse of the integral operator, but one rather finds that

$$D^{\alpha}_{c+}I^{\alpha}_{c+}=I+K,$$

where I is the identity, and under some conditions K is a compact operator. By solving Abel's integral equation, often used to motivate the definition of the fractional derivative in the case of constant regularity function, Harang et al. [17] defined the multifractional derivative of a function $g\in I^{\alpha}_{c+}L^p([0,T])$ as the inverse operation of I^{α}_{c+}, such that if $g\in I^{\alpha}_{c+}L^p([0,T])$ then there exists a unique $f\in L^p$ which satisfies $g=I^{\alpha}_{c+}f$, i.e, they defined the fractional derivative $D^{\alpha}_{c+}g=f$.

Let $\triangle^{(m)}[a,b]$ denote the m-simplex. That is, define $\triangle^{(m)}[a,b]$ to be given by

$$\triangle^{(m)}[a,b]=\{(s_1,\cdots,s_m)|a\leqslant s_1<\cdots<s_m\leqslant b\}.$$

Denote by $C^{\beta}([0,T];\mathbb{R})$ the space of β-Hölder continuous functions $f:[0,T]\to\mathbb{R}$, equipped with the norm

$$\|f\|_{\beta}=|f(0)|+\sup_{s\neq t\in[0,T]}\frac{|f(t)-f(s)|}{|t-s|^{\beta}}<\infty.$$

In addition, for $a \in [0,T]$ let us define $C^{\beta}_{\alpha}([0,T];\mathbb{R})$ to be the subspace of $C^{\beta}([0,T];\mathbb{R})$ such that $f \in C^{\beta}([0,T];\mathbb{R})$, $f(a) = 0$. Let α be a C^1 regularity function with values in $[a,b] \subset (0,1)$. We define the space of locally Hölder continuous functions $f : [0,T] \to \mathbb{R}$ by the norm

$$\|f\|_{\alpha(\cdot);[0,T]} := |f(0)| + \sup_{st\in[0,T]} \frac{|f(t)-f(s)|}{|t-s|^{\max(\alpha(x),\alpha(y))}} < \infty.$$

We denote this space by $C^{\alpha(\cdot)}([0,T];\mathbb{R})$. Moreover denote by $C_0^{\alpha(\cdot)}([0,T];\mathbb{R})$ the space of locally Hölder continuous functions which start in 0.

For $g \in C_0^{\alpha(\cdot)+\varepsilon}([0,T];\mathbb{R})$ with $\alpha \in C^1([0,T],[a,b])$ for $[a,b] \subset (0,1)$, $\varepsilon < 1-\alpha^*$ and $\alpha^* = \sup_{t\in[0,T]} \alpha(t)$, the functional G_0 evaluated in g defined by

$$G_0(g)(x) = \frac{1}{B(\alpha(x),1-\alpha(x)}\frac{d}{dx}\Big(\int_0^x \frac{\Gamma(\alpha(t))g(t)}{(x-t)^{\alpha(t)}}dt\Big),$$

where $B(\cdot,\cdot)$ is the Beta function. Let $\alpha \in C^1([0,T],(0,1))$. Define

$$F(s,x) := \int_0^1 U(s,x;\tau)d\tau,$$

where $U(s,x;\tau) = \alpha'(s+\tau(x-s)) \times (\ln(\tau)-\ln(1-\tau))\Big(\frac{\tau}{1-\tau}\Big)^{\alpha(s+\tau(x-s))}$.

Harang et al. [17] gave the following explicit representation of this multifractional derivative.

PROPOSITION 1. *The multifractional derivative can be represented in $L^p([0,T])$ as the infinite sequence of integrals*

$$D^{\alpha}_{0+}g(x) = G_0(g)(x) + \sum_{m=1}^{\infty}\int_{\triangle^{(m)}(0,x)} G_0(g)(s_{m+1})F(s_{m+1},s_m)\times\cdots\times F(s_1,x)ds_{m+1}\cdots ds_1,$$

for any $g \in C_0^{\alpha(\cdot)+\varepsilon}([0,T];\mathbb{R})$ with $\alpha \in C^1([0,T],[a,b])$ for $[a,b] \subset (0,1)$, $\varepsilon < 1-\alpha^*$. Furthermore, assume that for some $p > 1$, the regularity function α satisfies the inequality with $\alpha_* = \inf_{0\leqslant t\leqslant T}\alpha(t)$,

$$(\alpha_* + \varepsilon - \alpha(0)) \times p > -1.$$

Then we have the estimate

$$|D^{\alpha}_{0+}g(x)| \leqslant C(T,\varepsilon,\alpha,x)\|g\|_{\alpha(\cdot)+\varepsilon;[0,T]},$$

where $x \to C(T,\varepsilon,\alpha,x)$ is an L^p function for any p satisfying the inequality and hence $D^{\alpha}_{0+}g \in L^p([0,T])$.

2.2. Multifractional Brownian motion

In this subsection, we give some preliminary results on the multifractional Brownian motion with Riemann Liouville. The multifractional Brownian motion was first proposed in the 1990's by Peltier and Vehel in [26] and independently by Benassi et al. in [5]. The process is non-stationary and on very small time steps it behaves like a fractional Brownian motion. However, by letting the Hurst parameter in the fractional Brownian motion be a function of time, the Hölder regularity of the process is depending on time, and therefore it makes more sense to talk about local regularities rather than global. The process was initially proposed as a generalization with respect to the fBm representation given by Mandelbrot and Van-Ness, that is, the mBm was defined by

$$\tilde{B}_t^h = c(h_t)\int_{-\infty}^0 (t-s)^{h_t-\frac{1}{2}} - (-s)^{h_t-\frac{1}{2}} dB_s + c(h_t)\int_0^t (t-s)^{h_t-\frac{1}{2}} dB_s =: \tilde{B}_t^{(1),h} + \tilde{B}_t^{(2),h},$$

where $\{B_t\}_{t\in[0,T]}$ is a real valued Brownian motion, and $h:[0,T]\to(0,1)$ is a continuous function. Notice in the above representation that $\tilde{B}_t^{(1),h}$ is always measurable with respect to the filtration $\tilde{\mathscr{F}}_0$ (generated by the Brownian motion), as the stochastic process only "contributes" from $-\infty$ to 0. Therefore, we can think of $\tilde{B}_t^{(2),h}$ as the only part which contributes to the stochasticity of $\tilde{B}_t^h$ when $t>0$. The reason why one also considers the process $\tilde{B}_t^{(1),h}$ when analyzing regular fractional Brownian motions (in the case $h(t)=H$) is to ensure stationarity of the process. However, when we are considering the generalization $\tilde{B}_t^h$ above, when h is not constant, we do not get stationary of the process even though we consider the a representation as the one above. We are therefore inclined to choose $\tilde{B}_t^{(2),h}$ to be the multifractional noise we consider in this article due to its very simplistic nature. This multifractional process is often called in the literature the Riemann-Liouville multifractional Brownian motion, inspired by the original definition of the fractional Brownian motion defined by Lévy in the 1940's. The Riemann-Liouville multifractional Brownian motion was first analyzed by Lim in [22], and is well suited to the use of multifractional calculus, constructed above, in the analysis of differential equations driven by this process. In this subsection we will recite some of the basic properties of the Riemann-Liouville multifractional Brownian motion from [22].

DEFINITION 2. Let $\{B_t\}_{t\in[0,T]}$ be a one dimensional Brownian motion on a filtered probability space $(\Omega,\mathscr{F},\mathbb{P})$, and let $h:[0,T]\to[a,b]\subset(0,1)$ be a C^1 function. We define the Riemann Liouville multifractional Brownian motion (RLmBm) $\{B_t^h\}_{t\in[0,T]}$ by

$$B_t^h = \frac{1}{\Gamma(h_t+\frac{1}{2})}\int_0^t (t-s)^{h_t-\frac{1}{2}} dB_s; \quad t \geqslant 0,$$

where $\Gamma(\cdot)$ is the Gamma function. The function h is called the regularity function of $B_\cdot^h$.

The following proposition on the Girsanov theorem for RLmBm can be found in Harang et al. [17].

PROPOSITION 2. *Let $\{B_t^h\}_{t\in[0,T]}$ be a RLmBm with regularity function $h \in C^1([0,T],[a,b])$ for $[a,b] \subset (0,1/2)$, $0 < \varepsilon < 1-h^*$ and $h^* = \sup_{t\in[0,T]} h(t)$. Assume that*

(i) $\int_0^{\cdot} u_s ds \in C^{h(\cdot)+\frac{1}{2}+\varepsilon}([0,T]) \subseteq I^{h+\frac{1}{2}}L^2([0,T])$, *a.s.*

(ii) $\mathbb{E}[Z(T)] = 1$ *for* $Z(T) := \exp\big(\int_0^T D_{0+}^{h+\frac{1}{2}}\big(\int_0^{\cdot} u_s\big)(r)dB_r - \frac{1}{2}\int_0^T \big|D_{0+}^{h+\frac{1}{2}}\big(\int_0^{\cdot} u_s\big)(r)\big|^2 dr\big)$.

Then the stochastic process

$$\widetilde{B}_t^h = B_t^h + \int_0^t u_s ds$$

is an RLmBm under the measure $\widetilde{P}$ defined by

$$\frac{d\widetilde{P}}{dP} = Z(T).$$

Let $S = \{S(t)\}_{t\geqslant 0}$ be a subordinator (without killing), i.e. a nondecreasing Lévy process in $[0,\infty)$ starting at $S(0) = 0$. Due to the independent and stationary increments property, it is uniquely determined by the Laplace transform

$$\mathbb{E}e^{-uS(t)} = e^{-t\phi(u)}, \quad u > 0,\ t \geqslant 0,$$

where the characteristic (Laplace) exponent $\phi : (0,\infty) \to (0,\infty)$ is a Bernstein function with $\phi(0+) := \lim_{u\to 0}\phi(u) = 0$, i.e. a C^∞-function such that $(-1)^{n-1}\phi^{(n)} \geqslant 0$ for all $n \in \mathbb{N}$. Every such ϕ has a unique Lévy-Khintchine representation

$$\phi(u) = \kappa u + \int_{(0,\infty)} (1-e^{-ux})\nu(dx), \quad u > 0, \tag{2}$$

where $\kappa \geqslant 0$ is the drift parameter and ν is a Lévy measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1\wedge x)\nu(dx) < \infty.$$

It is clear that $\tilde{\phi} := \phi(u) - \kappa u$ is the Bernstein function of the subordinator $\tilde{S}(t) = S(t) - \kappa t$ having zero drift and Lévy measure ν.

3. Harnack inequality for (1)

Fix a constant $r > 0$. Denote by $\mathscr{L}$ the family of all right continuous functions $f : [-r,0] \to \mathbb{R}^d$ with left limits equipped with the norm $\|\cdot\|_2$

$$\|f\|_2^2 := \int_{-r}^0 |f(s)|^2 ds + |f(0)|^2.$$

For $f : [-r,\infty) \to \mathbb{R}^d$, we will denote $f_t \in \mathscr{L}$, $t \geqslant 0$, the corresponding segment process by

$$f_t(s) := f(t+s), \quad s \in [-r,0].$$

Throughout this paper we assume that the coefficients b, F satisfy the following Hypothesis:

(H) There exist constants $K \in \mathbb{R}$ and $K_1 \geqslant 0$ such that

$$\langle x-y, b(x)-b(y)\rangle \leqslant K|x-y|^2, \quad x,y \in \mathbb{R}^d,$$

and

$$|F(\xi)-F(\eta)| \leqslant K_1\|\xi-\eta\|_2, \quad \xi,\eta \in \mathscr{L}.$$

REMARK 1. The Hypothesis (H) ensures the existence, uniqueness and non-explosion of the solution to (1.1). Indeed, letting $L(t) = B^h_{S(t)}$, $\hat{b}(t,x) = b(x+L(t))$ and $\hat{F}(t,\xi) = F(\xi + L_t)$, one has

$$\langle x-y, \hat{b}(t,x)-\hat{b}(t,y)\rangle \leqslant K|x-y|^2, \quad x,y \in \mathbb{R}^d,\ t \geqslant 0$$

and

$$|\hat{F}(t,\xi)-\hat{F}(t,\eta)| \leqslant K_1\|\xi-\eta\|_2, \quad \xi,\eta \in \mathscr{L},\ t \geqslant 0.$$

Then the following ordinary functional differential equation

$$d\hat{X}(t) = \hat{b}(t,\hat{X}(t))dt + \hat{F}(t,\hat{X}_t)dt$$

has a unique solution which does not explode in finite time; setting $X(t) := \hat{X}(t) + L(t)$, we know that (1) has a unique non-explosive solution.

For $\xi \in \mathscr{L}$, let X_t^ξ be the solution to (1) with $X_0 = \xi$. Let P_t be the semigroup associated to X_t^ξ, i.e.

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi), \ t \geqslant 0, \ f \in \mathscr{B}_b(\mathscr{L}), \tag{3}$$

where $\mathscr{B}_b(\mathscr{L})$ denotes the set of all bounded measurable functions on $\mathscr{L}$.

THEOREM 1. *Let $\{B_t^h\}_{t\in[0,T]}$ be a RLmBm with regularity function $h \in C^1([0,T],[a,b])$ for $[a,b] \subset (0,1/2)$, $0 < \varepsilon < 1-h^*$ and $h^* = \sup_{t\in[0,T]} h(t)$, and $T > r$. If the Hypothesis (H) holds. Then,*

(i) for any $\xi,\eta \in \mathscr{L}$ and $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned} P_T \log f(\eta) \leqslant \log P_T f(\xi) &+ C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\ &+ C_2(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2; \end{aligned}$$

(ii) for any $p > 1$, $\xi,\eta \in \mathscr{L}$ and non-negative $f \in \mathscr{B}_b(\mathscr{L})$,

$$\begin{aligned} \Big(P_T f(\eta)\Big)^p \leqslant P_T f^p(\xi) \exp\Big[\frac{p}{2(p-1)^2}\Big(&C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\ &+ C_2(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2\Big)\Big], \end{aligned}$$

where

$$C_1(T,K,K_1,\kappa,h,\varepsilon) = \frac{2K_1^2T^{1-2h^*-2\varepsilon}}{\kappa^2}\Big\|C\Big(T,\varepsilon,h+\frac{1}{2},\cdot\Big)\Big\|_{L_2}^2$$

and

$$\begin{aligned}&C_2(T,K,K_1,\kappa,h,\varepsilon,r)\\ &= T^{1-2h^*-2\varepsilon}\Big\|C\Big(T,\varepsilon,h+\frac{1}{2},\cdot\Big)\Big\|_{L_2}^2\Big(\frac{(e^{2K(T-r)}-1)K_1^2}{K\kappa^2}+2\Big(\int_0^{T-r}e^{-2Kt}dS(t)\Big)^{-2}\Big).\end{aligned}$$

For a measurable space $(E,\mathscr{F})$, let $\mathscr{P}(E)$ denote the family of all probability measures on $(E,\mathscr{F})$. For $\mu,\nu\in\mathscr{P}(E)$, the entropy $\mathrm{Ent}(\nu|\mu)$ is defined by

$$\mathrm{Ent}(\nu|\mu) := \begin{cases}\int \ln\frac{d\nu}{d\mu}d\nu, & \nu\ll\mu,\\ +\infty, & \text{otherwise.}\end{cases}$$

The total variation distance $\|\mu-\nu\|_{\mathrm{var}}$ is defined by

$$\|\mu-\nu\|_{\mathrm{var}} := \sup_{A\in\mathscr{F}}|\mu(A)-\nu(A)|.$$

By Pinsker's inequality (see [9]),

$$\|\mu-\nu\|_{\mathrm{var}}^2 \leqslant \frac{1}{2}\mathrm{Ent}(\nu|\mu), \quad \mu,\nu\in\mathscr{P}(E).$$

For $\xi\in\mathscr{L}$, let $P_T(\xi,\cdot)$ be the distribution of X_T^ξ. The following corollary is a direct consequence of Theorem 1, see [39] for the proof.

COROLLARY 1. *Let the assumptions in Theorem 1 hold. Then the following assertions hold.*

(i) For any $\xi,\eta\in\mathscr{L}$ and $P_T(\xi,\cdot)$ is equivalent to $P_T(\eta,\cdot)$ and

$$\begin{aligned}Ent(P_T(\xi,\cdot)|P_T(\eta,\cdot)) \leqslant{}& C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2\\ &+C_2(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2,\end{aligned}$$

which together with Pinsker's inequality implies that

$$\begin{aligned}2\|P_T(\xi,\cdot)-P_T(\eta,\cdot)\|_{var}^2 \leqslant{}& C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2\\ &+C_2(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2,\end{aligned}$$

(ii) For any $p>1$, $\xi,\eta\in\mathscr{L}$

$$\begin{aligned}P_T\Big\{\Big(\frac{dP_T(\xi,\cdot)}{dP_T(\eta,\cdot)}\Big)^{1/(p-1)}\Big\}(\xi) \leqslant \mathbb{E}\Big\{\exp\Big[\frac{p}{2(p-1)^2}\Big(&C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2\\ &+C_2(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2\Big)\Big]\Big\}.\end{aligned}$$

Let $\ell : [0,\infty) \to [0,\infty)$ be a sample path of S, which is a non-decreasing and càdlàg function with $\ell(0) = 0$. By (H) and the same explanation as in the Remark 1, for any $\xi \in \mathscr{L}$, the following functional SDE has a unique non-explosive solution with $X_0^\ell = \xi$:

$$dX^\ell(t) = b(X^\ell(t))dt + F(X_t^\ell)dt + dB^h_{\ell(t)}. \tag{4}$$

We denote the solution by $X_t^{\ell,\xi}$. Let

$$P_t^\ell f(\xi) = \mathbb{E} f(X_t^{\ell,\xi}),\ \ t \geqslant 0,\ \ f \in \mathscr{B}_b(\mathscr{L}). \tag{5}$$

PROPOSITION 3. *Let* $\{B_t^h\}_{t\in[0,T]}$ *be a RLmBm with regularity function* $h \in C^1([0,T],[a,b])$ *for* $[a,b] \subset (0,1/2)$, $0 < \varepsilon < 1 - h^*$ *and* $h^* = \sup_{t\in[0,T]} h(t)$, *and* $T > r$. *If the Hypothesis (H) holds. Then,*

(i) for any $\xi,\eta \in \mathscr{L}$ *and* $f \in \mathscr{B}_b(\mathscr{L})$ *with* $f \geqslant 1$,

$$\begin{aligned} P_T^\ell \log f(\eta) \leqslant \log P_T^\ell f(\xi) + \Big(&C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\ &+ C_3(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2\Big); \end{aligned}$$

(ii) for any $p > 1$, $\xi,\eta \in \mathscr{L}$ *and non-negative* $f \in \mathscr{B}_b(\mathscr{L})$,

$$\begin{aligned} \Big(P_T^\ell f(\eta)\Big)^p \leqslant P_T^\ell f^p(\xi) \exp\Big[\frac{p}{2(p-1)^2}\Big(&C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\ &+ C_3(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0)-\eta(0)|^2\Big)\Big], \end{aligned}$$

where

$$\begin{aligned} C_3(T,K,K_1,\kappa,h,\varepsilon,r) = &T^{1-2h^*-2\varepsilon}\Big\|C\Big(T,\varepsilon,h+\frac{1}{2},\cdot\Big)\Big\|_{L_2}^2 \\ &\cdot\Big(\frac{(e^{2K(T-r)}-1)K_1^2}{K\kappa^2} + 2\Big(\int_0^{T-r} e^{-2Kt}d\ell(t)\Big)^{-2}\Big). \end{aligned}$$

For $\varepsilon \in (0,1)$, consider the following regularization of ℓ:

$$\ell^\varepsilon := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \ell(s)ds + \varepsilon t = \int_0^t \ell(\varepsilon s + t)ds + \varepsilon t,\ \ t \geqslant 0.$$

It is clear that for each $\varepsilon \in (0,1)$, the function ℓ^ε is a absolutely continuous, strictly increasing and satisfies for any $t \geqslant 0$

$$\ell^\varepsilon(t) \downarrow \ell \quad \text{as } \varepsilon \downarrow 0. \tag{6}$$

For $\xi \in \mathscr{L}$, let $X_t^{\ell^\varepsilon,\xi}$ be the solution to the following functional SDE with initial value ξ:

$$dX^{\ell^\varepsilon,\xi}(t) = b(X^{\ell^\varepsilon,\xi}(t))dt + F(X^{\ell^\varepsilon,\xi})dt + dB^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}. \tag{7}$$

The associated semigroup is denoted by $P_t^{\ell^\varepsilon}$. Note that this SDE is indeed driven by fBm and thus the method of coupling and Girsanov's transformation can be used to establish the dimension-free Harnack inequalities for $P_t^{\ell^\varepsilon}$.

PROPOSITION 4. *Let $\{B_t^h\}_{t\in[0,T]}$ be a RLmBm with regularity function $h \in C^1([0,T],[a,b])$ for $[a,b] \subset (0,1/2)$, $0<\varepsilon<1-h^*$ and $h^* = \sup_{t\in[0,T]} h(t)$, and $T>r$. If the Hypothesis (H) holds. Then,*

(i) for any $\xi,\eta \in \mathscr{L}$ and $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$P_T^{\ell^\varepsilon} \log f(\eta) \leqslant \log P_T^{\ell^\varepsilon} f(\xi) + \Big(C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\ + C_4(T,K,K_1,\kappa,h,\varepsilon,r,\varepsilon)|\xi(0)-\eta(0)|^2\Big);$$

(ii) for any $p>1$, $\xi,\eta\in\mathscr{L}$ and non-negative $f\in\mathscr{B}_b(\mathscr{L})$,

$$\Big(P_T^{\ell^\varepsilon} f(\eta)\Big)^p \leqslant P_T^{\ell^\varepsilon} f^p(\xi)\exp\Big[\frac{p}{2(p-1)^2}\Big(C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\ + C_4(T,K,K_1,\kappa,h,\varepsilon,r,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\Big],$$

where

$$C_4(T,K,K_1,\kappa,h,\varepsilon,r,\varepsilon) = T^{1-2h^*-2\varepsilon}\Big\|C\Big(T,\varepsilon,h+\frac{1}{2},\cdot\Big)\Big\|_{L_2}^2 \\ \cdot\Big(\frac{(e^{2K(T-r)}-1)K_1^2}{K\kappa^2} + 2\Big(\int_0^{T-r} e^{-2Kt}d\ell^\varepsilon(t)\Big)^{-2}\Big).$$

Proof. First of all, we will construct coupling as follows. Let Y_t solve the equation

$$dY(t) = b(Y(t))dt + F(X_t^{\ell^\varepsilon,\xi})dt + \lambda(t)\mathscr{I}_{[0,\tau)}(t)\frac{X^{\ell^\varepsilon,\xi}(t)-Y(t)}{|X^{\ell^\varepsilon,\xi}(t)-Y(t)|}|\xi(0)-\eta(0)|d\ell^\varepsilon(t) \\ + dB^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} \tag{8}$$

with $Y_0=\eta$, where

$$\lambda(t) := \frac{e^{-Kt}}{\int_0^{T-r} e^{-2Ks}d\ell^\varepsilon(s)},\ \ t\geqslant 0,$$

and

$$\tau := T\wedge \inf\{t\geqslant 0; X^{\ell^\varepsilon,\xi}(t)=Y(t)\}$$

is the coupling time. It is clear that $(X^{\ell^\varepsilon,\xi}(t),Y(t))$ is well defined for $t<\tau$. By (H), we have

$$d|X^{\ell^\varepsilon,\xi}(t)-Y(t)| \leqslant K|X^{\ell^\varepsilon,\xi}(t)-Y(t)|dt - \lambda(t)|\xi(0)-\eta(0)|d\ell^\varepsilon(t),\ \ t\in[0,\tau).$$

Thus, for $t\in[0,\tau)$,

$$\begin{aligned} |X^{\ell^\varepsilon,\xi}(t)-Y(t)| &\leqslant e^{Kt}|\xi(0)-\eta(0)|\Big(1-\int_0^t e^{-Ks}\lambda(s)d\ell^\varepsilon(s)\Big) \\ &\leqslant \frac{e^{Kt}\int_t^{T-r} e^{-2Ks}d\ell^\varepsilon(s)}{\int_0^{T-r} e^{-2Ks}d\ell^\varepsilon(s)}|\xi(0)-\eta(0)| \\ &=: \gamma(t)|\xi(0)-\eta(0)|. \end{aligned} \tag{9}$$

If $\tau(\omega) > T-r$ for some $\omega \in \Omega$, we can take $t = T-r$ in the above inequality to get

$$0 < |X^{\ell^\varepsilon,\xi}(t)(\omega) - Y(t)(\omega)| \leqslant 0,$$

which is absurd. Therefore, $\tau \leqslant T-r$. Letting $Y(t) = X^{\ell^\varepsilon,\xi}(t)$ for $t \in [\tau, T]$, $Y(t)$ solves (8) for $t \in [\tau, T]$. In particular, $X_T^{\ell^\varepsilon,\xi} = Y_T$. Moreover, by (9) and $\tau \leqslant T-r$, we have

$$|X^{\ell^\varepsilon,\xi}(t) - Y(t)|^2 \leqslant |\xi(0) - \eta(0)|^2 \gamma(t)^2 \mathscr{I}_{[0,T-r]}(t), \quad t \in [0,T]. \tag{10}$$

Denote by $\zeta^\varepsilon : [\ell^\varepsilon(0), \infty) \to [0,\infty)$ the inverse function of ℓ^ε. Then $\ell^\varepsilon(\zeta^\varepsilon(t)) = t$ for $t \geqslant \ell^\varepsilon(0)$, $\zeta^\varepsilon(\ell^\varepsilon(t)) = t$ for $t \geqslant 0$, and $t \to \zeta^\varepsilon(t)$ is absolutely continuous and strictly increasing. Let

$$\Psi(u) := \Phi \circ \zeta^\varepsilon(u + \ell^\varepsilon(0)),$$

where

$$\Phi(u) := [F(X_u^{\ell^\varepsilon,\xi}) - F(Y_u)] \frac{1}{(\ell^\varepsilon)'(u)} + \lambda(u) \mathscr{I}_{[0,\tau)}(u) \frac{X^{\ell^\varepsilon,\xi}(u) - Y(u)}{|X^{\ell^\varepsilon,\xi}(u) - Y(u)|} |\xi(0) - \eta(0)|.$$

Let

$$v_t = D_{0+}^{h+\frac{1}{2}} \Big(\int_0^{\cdot} \Psi(u) du \Big)(t).$$

Next, we will check that the process $\int_0^{\cdot} \Psi(u) du$ satisfies condition (i) and (ii) in the Proposition 2. First we will show (i), i.e. $\int_0^{\cdot} \Psi(u) du \in I_{0+}^{h+\frac{1}{2}} L^2([0,T])$, which is equivalent to showing $v \in L^2([0,T])$. By the Proposition 1, we know that

$$|v_t| \leqslant C\Big(T, \varepsilon, h + \frac{1}{2}, t\Big) \Big\| \int_0^{\cdot} \Psi(u) du \Big\|_{h(\cdot)+\frac{1}{2}+\varepsilon;[0,T]},$$

where $t \to C(T, \varepsilon, h+\frac{1}{2}, t) \in L^2([0,T])$ since $(h_* + \varepsilon - h(0)) \times 2 > -1$ for some small $\varepsilon > 0$. By (10) and a simple calculation we can also shows $\Big\| \int_0^{\cdot} \Psi(u) du \Big\|_{h(\cdot)+\frac{1}{2}+\varepsilon;[0,T]} \in L^2([0,T])$. A simple calculation shows $\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} |\Psi(u)|^2 du < \infty$. Then, by using $h \in (0, 1/2)$ we have $\int_0^{\cdot} \Phi(u) du \in I_{0+}^{H+1/2}(L^2([0, \ell^\varepsilon(T) - \ell^\varepsilon(0)]; \mathbb{R}^d))$. Therefore, the following stochastic integral defines a martingale

$$M_t := -\int_0^t \langle v_s, dB_s \rangle, \quad t \geqslant 0,$$

where $B = \{B_t\}_{t \geqslant 0}$ is a d-dimensional standard Brownian motion. By the Proposition 1, we know that

$$|v_s| \leqslant C\Big(T, \varepsilon, h + \frac{1}{2}, s\Big) \Big\| \int_0^{\cdot} \Psi(u) du \Big\|_{h(\cdot)+\frac{1}{2}+\varepsilon;[0,T]},$$

and this yields for any $s \in [0, \ell^\varepsilon(T) - \ell^\varepsilon(0)]$,

$$|v_s| \leqslant C\Big(T, \varepsilon, h + \frac{1}{2}, s\Big) T^{\frac{1}{2} - h^* - \varepsilon} \max_{u \in [0,T]} |\Phi(u)|.$$

Recalling that ℓ is an sample path of the subordinator S with drift parameter $\kappa \geqslant 0$, one have

$$(\ell^{\varepsilon})'(t) = \frac{\ell(t+\varepsilon)-\ell(t)}{\varepsilon} + \varepsilon > \kappa,$$

and therefore

$$|v_s| \leqslant C\Big(T,\varepsilon,h+\frac{1}{2},s\Big)T^{\frac{1}{2}-h^*-\varepsilon} \max_{u\in[0,T]}\Big(\frac{K}{\kappa}\|X_u^{\ell^{\varepsilon},\xi}-Y_u\|_2 + \lambda(u)|\xi(0)-\eta(0)|\Big).$$

On the other hand, by view of the definition of $\|\cdot\|_2$ we have for all $t \geqslant 0$

$$\begin{aligned}
\|X_t^{\ell^{\varepsilon},\xi}-Y_t\|_2^2 &= \int_{-r}^{0}|X^{\ell^{\varepsilon},\xi}(t+s)-Y(t+s)|^2ds + |\xi(0)-\eta(0)|^2\\
&= \int_{t-r}^{t}|X^{\ell^{\varepsilon},\xi}(s)-Y(s)|^2ds + |\xi(0)-\eta(0)|^2\\
&\leqslant \int_{-r}^{0}|\xi(s)-\eta(s)|^2ds + \int_0^t |X^{\ell^{\varepsilon},\xi}(s)-Y(s)|^2ds + |\xi(0)-\eta(0)|^2\\
&= \|\xi-\eta\|_2^2 + \int_0^t |X^{\ell^{\varepsilon},\xi}(s)-Y(s)|^2ds.
\end{aligned}$$

Then, by (10) we have for all $t \geqslant 0$

$$\begin{aligned}
\|X_t^{\ell^{\varepsilon},\xi}-Y_t\|_2^2 &\leqslant \|\xi-\eta\|_2^2 + |\xi(0)-\eta(0)|^2\int_0^{T-r}\gamma(s)^2ds\\
&\leqslant \|\xi-\eta\|_2^2 + \frac{e^{2K(T-r)}-1}{2K}|\xi(0)-\eta(0)|^2,
\end{aligned} \tag{11}$$

where in the last inequality we have used $\gamma(s) \leqslant e^{Ks}$ for $s \in [0,T-r]$. By the definition of $\lambda(t)$, it is easy to see that for all $t \geqslant 0$

$$|\lambda(t)| \leqslant \Big(\int_0^{T-r} e^{-2Kt}d\ell^{\varepsilon}(t)\Big)^{-1}. \tag{12}$$

Thus, by (11) and (12) the compensator of the martingale M_t satisfies for all $t \geqslant 0$,

$$\begin{aligned}
\langle M\rangle_t &= \int_0^t |v_s|^2ds\\
&\leqslant \Bigg(\frac{2K_1^2\|\xi-\eta\|_2^2}{\kappa^2} + \Bigg(\frac{(e^{2K(T-r)}-1)K_1^2}{K\kappa^2} + 2\Big(\int_0^{T-r}e^{-2Kt}d\ell^{\varepsilon}(t)\Big)^{-2}\Bigg)\\
&\quad\cdot|\xi(0)-\eta(0)|^2\Bigg)T^{1-2h^*-2\varepsilon}\int_0^t C^2\Big(T,\varepsilon,h+\frac{1}{2},s\Big)ds\\
&\leqslant \frac{2K_1^2T^{1-2h^*-2\varepsilon}}{\kappa^2}\Big\|C\Big(T,\varepsilon,h+\frac{1}{2},\cdot\Big)\Big\|_{L_2}^2\|\xi-\eta\|_2^2 + T^{1-2h^*-2\varepsilon}\Big\|C\Big(T,\varepsilon,h+\frac{1}{2},\cdot\Big)\Big\|_{L_2}^2\\
&\quad\cdot\Bigg(\frac{(e^{2K(T-r)}-1)K_1^2}{K\kappa^2} + 2\Big(\int_0^{T-r}e^{-2Kt}d\ell^{\varepsilon}(t)\Big)^{-2}\Bigg)|\xi(0)-\eta(0)|^2\\
&= C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 + C_4(T,K,K_1,\kappa,h,\varepsilon,r,\varepsilon)|\xi(0)-\eta(0)|^2.
\end{aligned} \tag{13}$$

Let

$$R := \exp\left[M(\ell^\varepsilon(T)) - \frac{1}{2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right].$$

By Novikov's criterion, we have $\mathbb{E}R = 1$. According to Girsanov's theorem, $\widetilde{B}_t := \int_0^t \Psi(u)du + B_t$ is a d-dimensional Brownian motion and $\widetilde{B}_t^h := \int_0^t \Psi(u)du + B_t^h$ is a d-dimensional multifractional Brownian motion with $h \in (0,1/2)$ under the new probability measure $R\mathbb{P}$. Rewrite (8) as

$$dY(t) = b(Y(t))dt + F(Y_t)dt + d\widetilde{B}^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}.$$

Thus, the distribution of $\{Y_t\}_{0\leqslant t\leqslant T}$ under $R\mathbb{P}$ coincides with that of $\{X_t^{\ell^\varepsilon,\eta}\}$ under $\mathbb{P}$; in particular, it holds that for any $f \in \mathscr{B}_b(\mathscr{L})$,

$$\mathbb{E}f(X_T^{\ell^\varepsilon,\eta}) = \mathbb{E}_{R\mathbb{P}}f(Y_T) = \mathbb{E}[Rf(Y_T)] = \mathbb{E}[Rf(X_t^{\ell^\varepsilon,\xi})]. \tag{14}$$

By (14) and the Young inequality, and the observation that

$$\begin{aligned}\log R &= -\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} \langle v_s, dB_s\rangle - \frac{1}{2}\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} |v_s|^2 ds \\ &= -\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} \langle v_s, d\widetilde{B}_s\rangle + \frac{1}{2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)},\end{aligned}$$

we get that, for any $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned}P_T^{\ell^\varepsilon}\log f(\eta) &= \mathbb{E}\log f(X_T^{\ell^\varepsilon,\eta}) \\ &= \mathbb{E}[R\log f(X_T^{\ell^\varepsilon,\xi})] \\ &\leqslant \log\mathbb{E}f(X_T^{\ell^\varepsilon,\xi}) + \mathbb{E}[R\log R] \\ &= \log P_T^{\ell^\varepsilon} f(\xi) + \mathbb{E}_{R\mathbb{P}}\log R \\ &= \log P_T^{\ell^\varepsilon} f(\xi) + \frac{1}{2}\mathbb{E}_{R\mathbb{P}}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}.\end{aligned}$$

Combining this with (13), we obtain the desired log-Harnack inequality.

Next, we prove the second assertion of the proposition. For any non-negative $f \in \mathscr{B}_b(\mathscr{L})$ we can obtain from (14) and the Hölder's inequality

$$\begin{aligned}(P_T^{\ell^\varepsilon} f)^p(\eta) &= (\mathbb{E}f(X_T^{\ell^\varepsilon,\eta}))^p \\ &= (\mathbb{E}[Rf(X_T^{\ell^\varepsilon,\xi})])^p \\ &\leqslant P_T^{\ell^\varepsilon} f^p(\xi)\cdot(\mathbb{E}[R^{p/(p-1)}])^{p-1}.\end{aligned} \tag{15}$$

Furthermore, by (13) we get

$$
\begin{aligned}
R^{p/(p-1)} &= \exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p}{2(p-1)}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right] \\
&= \exp\left[\frac{p}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right] \\
&\quad \cdot \exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right] \\
&\leqslant \exp\left[\frac{p}{2(p-1)^2}\Big(C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2\right. \\
&\qquad \left. + C_4(T,K,K_1,\kappa,h,\varepsilon,r,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\right] \\
&\qquad \cdot \exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right],
\end{aligned}
$$

and noting the fact that $\exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right]$, $0\leqslant t\leqslant T$ is a martingale with mean 1 due to Novikov's criterion. Then, we have

$$
\begin{aligned}
\mathbb{E}\left[R^{p/(p-1)}\right] \leqslant \exp\Big[\frac{p}{2(p-1)^2}\Big(&C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi-\eta\|_2^2 \\
&+ C_4(T,K,K_1,\kappa,h,\varepsilon,r,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\Big].
\end{aligned}
$$

Inserting this estimate into (15), we get the power-Harnack inequality. □

Proof of Proposition 3. Fix $T>r$. By a standard approximation argument, we may assume that $f\in C_b(\mathscr{L})$.

Step 1: First, we assume that b is globally Lipschitzian: there exists a constant $C>0$ such that

$$|b(x)-b(y)|\leqslant C|x-y|,\ \ x,y\in\mathbb{R}^d.$$

By the Lipschitz continuity of b and F, and noting that $|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)|\leqslant\|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)\|_2$, we have for $t\geqslant 0$

$$
\begin{aligned}
|X^{\ell^\varepsilon,,\xi}(t)-X^{\ell,\xi}(t)| &\leqslant C\int_0^t |X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)|du + K_1\int_0^t \|X_u^{\ell^\varepsilon,\xi}-X_u^{\ell,\xi}\|_2 du \\
&\quad + |B^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}-B^h_{\ell(t)}| \\
&\leqslant (C+K_1)\int_0^t \|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)\|_2 du + |B^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}-B^h_{\ell(t)}|.
\end{aligned}
$$

By the Hölder's inequality we have for $t\in[0,T]$

$$
\begin{aligned}
|X^{\ell^\varepsilon,\xi}(t)-X^{\ell,\xi}(t)|^2 &\leqslant 2(C+K_1)t\int_0^t \|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)\|_2^2 du + 2|B^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}-B^h_{\ell(t)}|^2 \\
&\leqslant 2(C+K_1)T\int_0^t \|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)\|_2^2 du + 2|B^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}-B^h_{\ell(t)}|^2.
\end{aligned}
$$

Applying the Lemma 3.3 of [11] with $g^{(\varepsilon)}(t) = |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|$ and $h^{(\varepsilon)}(t) = 2|B^h_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} - B^h_{\ell(t)}|^2$, we conclude that $X_T^{\ell^\varepsilon,\xi} \to X_T^{\ell,\xi}$ in $\mathscr{L}$ as $\varepsilon \downarrow 0$, and so

$$\lim_{\varepsilon\downarrow 0} P_T^{\ell^\varepsilon} f = P_T^\ell f, \quad f \in C_b(\mathscr{L}).$$

Since ℓ is of bounded variation, it is easy to get from (6) that

$$\lim_{\varepsilon\downarrow 0} \int_0^{T-r} e^{-2Kt} d\ell^\varepsilon(t) = \int_0^{T-r} e^{-2Kt} d\ell(t).$$

Letting $\varepsilon \downarrow 0$ in the Proposition 4, we obtain the desired results.

Step 2: For the general case, we shall make use of the approximation argument proposed in [39]. Let

$$\tilde{b}(x) := b(x) - Kx, \quad x \in \mathbb{R}^d.$$

Then $\tilde{b}$ satisfies the dissipative condition:

$$\langle \tilde{b}(x) - \tilde{b}(y), x-y \rangle \leqslant 0, \quad x, y \in \mathbb{R}^d,$$

and it is easy to see that the mapping $id - \varepsilon b : \mathbb{R}^d \to \mathbb{R}^d$ is injective for any $\varepsilon > 0$. For $\varepsilon > 0$, let $\tilde{b}^{(\varepsilon)}$ be the Yoshida approximation of $\tilde{b}$, i.e.

$$\tilde{b}^{(\varepsilon)}(x) := \frac{1}{\varepsilon}\left[\Big(id - \varepsilon b\Big)^{-1}(x) - x\right], \quad x \in \mathbb{R}^d.$$

Then $\tilde{b}^{(\varepsilon)}$ is dissipative and globally Lipschitzian, $|\tilde{b}^{(\varepsilon)}| \leqslant |\tilde{b}|$ and $\lim_{\varepsilon\downarrow 0} \tilde{b}^{(\varepsilon)} = \tilde{b}$. Let $b^{(\varepsilon)}(x) := \tilde{b}^{(\varepsilon)}(x) + Kx$. Then $b^{(\varepsilon)}$ is also Lipschitzian and

$$\langle x-y, b^{(\varepsilon)}(x) - b^{(\varepsilon)}(y) \rangle \leqslant K|x-y|^2, \quad x, y \in \mathbb{R}^d.$$

Let $X_t^{\ell,(\varepsilon),\xi}$ solve the SDE (1.1) with b replaced by $b^{(\varepsilon)}$ and $X_0^{\ell,(\varepsilon),\xi} = \xi \in \mathscr{L}$. Denote by $P_t^{\ell,(\varepsilon)}$ the associated semigroup. Due to the first step of the proof, the statements of the Proposition 3.1 hold with P_t^ℓ replaced by $P_t^{\ell,(\varepsilon)}$. If

$$\lim_{\varepsilon\downarrow 0} P_T^{\ell,(\varepsilon)} f = P_T^\ell f, \quad f \in C_b(\mathscr{L}), \tag{16}$$

we complete the proof by applying the Proposition 3 with P_t^ℓ replaced by $P_t^{\ell,(\varepsilon)}$ and letting $\varepsilon \downarrow 0$. Indeed, noticing that

$$\begin{aligned}
&d|X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 \\
=&2\langle X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t), b^{(\varepsilon)}(X^{\ell^\varepsilon,\xi}(t)) - b^{(\varepsilon)}(X^{\ell,\xi}(t))\rangle dt \\
&+ 2\langle X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t), b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))\rangle dt \\
&+ 2\langle X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t), F^{(\varepsilon)}(X_t^{\ell^\varepsilon,\xi}) - F^{(\varepsilon)}(X_t^{\ell,\xi})\rangle dt \\
\leqslant& (2K+1)|X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 dt + |b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))|^2 dt \\
&+ 2K_1\|X_t^{\ell^\varepsilon,\xi} - X_t^{\ell,\xi}\|_2^2 dt \\
\leqslant& (2|K| + 2K_1 + 1)\|X_t^{\ell^\varepsilon,\xi} - X_t^{\ell,\xi}\|_2^2 dt + |b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))|^2 dt,
\end{aligned}$$

one has for $t \in [0,T]$

$$\begin{aligned}
&|X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 \\
\leqslant &(2|K| + 2K_1 + 1)\int_0^t \|X_s^{\ell^\varepsilon,\xi} - X_s^{\ell,\xi}\|_2^2 ds + \int_0^t |b^{(\varepsilon)}(X^{\ell,\xi}(s)) - b(X^{\ell,\xi}(s))|^2 ds \\
= &(2|K| + 2K_1 + 1)\int_0^t \|X_s^{\ell^\varepsilon,\xi} - X_s^{\ell,\xi}\|_2^2 ds + \int_0^t |\tilde{b}^{(\varepsilon)}(X^{\ell,\xi}(s)) - \tilde{b}(X^{\ell,\xi}(s))|^2 ds.
\end{aligned}$$

Applying the Lemma 3.3 of [11] with $g^{(\varepsilon)}(t) = |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|$ and $h^{(\varepsilon)}(t) = \int_0^t |\tilde{b}^{(\varepsilon)}(X^{\ell,\xi}(s)) - \tilde{b}(X^{\ell,\xi}(s))|^2 ds$, we know that $X_T^{\ell^\varepsilon,\xi} \to X_T^{\ell,\xi}$ in $\mathscr{L}$ as $\varepsilon \downarrow 0$, and thus (16) follows. $\square$

Proof of Theorem 3.1. Since the processes S and B^h are independent, we have

$$P_T f(\cdot) = \mathbb{E}\Big[P_T^\ell f(\cdot)|\ell = S\Big], f \in \mathscr{B}_b(\mathscr{L}). \tag{17}$$

By the first assertion of the Proposition 3, for all $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned}
P_T \log f(\eta) = &\mathbb{E}\Big[P_T^\ell \log f(\eta)|\ell = S\Big] \\
\leqslant &\mathbb{E}\Big[P_T^\ell \log f(\xi)|\ell = S\Big] + C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi - \eta\|_2^2 \\
&+ C_2(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0) - \eta(0)|^2,
\end{aligned}$$

which, together with the Jensen's inequality and (17), implies the log-Harnack inequality. Analogously, by the second assertion of Proposition 3, for all non-negative $f \in \mathscr{B}_b(\mathscr{L})$

$$\begin{aligned}
P_T f(\eta) = &\mathbb{E}\Big[P_T^\ell f(\eta)|\ell = S\Big] \\
\leqslant &\mathbb{E}\Big[(P_T^\ell f^p(\xi))^{1/p} \exp\Big[\frac{1}{2(p-1)}\Big(C_1(T,K,K_1,\kappa,h,\varepsilon)\|\xi - \eta\|_2^2 \\
&+ C_3(T,K,K_1,\kappa,h,\varepsilon,r)|\xi(0) - \eta(0)|^2\Big|_{\ell=S}\Big)\Big]\Big].
\end{aligned}$$

It remains to use the Hölder inequality and (17) to derive the power-Harnack inequality. $\square$

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