

WEIGHTED BOUNDEDNESS OF THE HARDY–LITTLEWOOD MAXIMAL AND CALDERÓN–ZYGmund OPERATORS ON ORLICZ–MORREY AND WEAK ORLICZ–MORREY SPACES

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Abstract. For the Hardy–Littlewood maximal and Calderón–Zygmund operators, the weighted boundedness on the Lebesgue spaces are well known. We extend these to the Orlicz–Morrey spaces. Moreover, we prove the weighted boundedness on the weak Orlicz–Morrey spaces. To do this we show the weak–weak modular inequality. The Orlicz–Morrey space and its weak version contain weighted Orlicz, Morrey and Lebesgue spaces and their weak versions as special cases. Then we also get the boundedness for these function spaces as corollaries.

1. Introduction

Let $L^p(\mathbb{R}^n, w)$ and $wL^p(\mathbb{R}^n, w)$ be the weighted Lebesgue space and its weak version on the n -dimensional Euclidean space \mathbb{R}^n , respectively. Then it is well known that the Hardy–Littlewood maximal operator M is bounded from $L^1(\mathbb{R}^n, w)$ to $wL^1(\mathbb{R}^n, w)$ if $w \in A_1$, and, from $L^p(\mathbb{R}^n, w)$ to itself if $w \in A_p$, $p \in (1, \infty]$, where A_p is the Muckenhoupt class, see [26]. The Calderón–Zygmund operators have the same boundedness except the case $p = \infty$. It is also known that M is bounded from $wL^p(\mathbb{R}^n, w)$ to itself if $w \in A_p$, $p \in (1, \infty]$. This boundedness can be obtained by using the property of A_p -weights and the Marcinkiewicz interpolation theorem for the operators of restricted weak type, see [9, Theorem 1.4.19 (page 61)] for example. See also [20] for its simple proofs.

In this paper we extend these boundedness to the weighted Orlicz–Morrey space $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ and its weak version $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$, where Φ is a Young function and φ is a variable growth function. Namely, we prove the following boundedness:

$$\|Tf\|_{wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)} \leq C \|f\|_{L^{(\Phi, \varphi)}(\mathbb{R}^n, w)}, \quad (1.1)$$

$$\|Tf\|_{L^{(\Phi, \varphi)}(\mathbb{R}^n, w)} \leq C \|f\|_{L^{(\Phi, \varphi)}(\mathbb{R}^n, w)}, \quad (1.2)$$

$$\|Tf\|_{wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)} \leq C \|f\|_{wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)}, \quad (1.3)$$

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where T is the Hardy-Littlewood maximal operator or a Calderón-Zygmund operator. The function spaces $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ contain weighted Orlicz, Morrey and Lebesgue spaces and their weak versions as special cases. Then we also get the boundedness for these function spaces as corollaries.

For a measurable set $G \subset \mathbb{R}^n$, we denote its Lebesgue measure and characteristic function by $|G|$ and χ_G , respectively. A weight is a locally integrable function on \mathbb{R}^n which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set G , we define $w(G) = \int_G w(x) dx$. For $p \in (0, \infty]$, the weighted Lebesgue space and its weak version with respect to the measure $w(x) dx$ are denoted by $L^p(\mathbb{R}^n, w)$ and $wL^p(\mathbb{R}^n, w)$, respectively.

For a function $\varphi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$ and a ball $B = B(a, r)$, we denote $\varphi(a, r)$ by $\varphi(B)$. For a weight w , a measurable set G and a function f , let

$$w(G, f, t) = w(\{x \in G : |f(x)| > t\}), \quad t \in [0, \infty).$$

In the case $G = \mathbb{R}^n$, we briefly denote it by $w(f, t)$.

DEFINITION 1.1. (Orlicz-Morrey space and weak Orlicz-Morrey space) For a Young function $\Phi : [0, \infty) \rightarrow [0, \infty)$, a function $\varphi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$, a weight w and a ball B , let

$$\begin{aligned} \|f\|_{\Phi, \varphi, w, B} &= \inf \left\{ \lambda > 0 : \frac{1}{\varphi(B)w(B)} \int_B \Phi \left(\frac{|f(x)|}{\lambda} \right) w(x) dx \leq 1 \right\}, \\ \|f\|_{\Phi, \varphi, w, B, \text{weak}} &= \inf \left\{ \lambda > 0 : \frac{1}{\varphi(B)w(B)} \sup_{t \in (0, \infty)} \Phi(t) w \left(B, \frac{f}{\lambda}, t \right) \leq 1 \right\}. \end{aligned}$$

Let $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\begin{aligned} \|f\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n, w)} &= \sup_B \|f\|_{\Phi, \varphi, w, B}, \\ \|f\|_{wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)} &= \sup_B \|f\|_{\Phi, \varphi, w, B, \text{weak}}, \end{aligned}$$

where the suprema are taken over all balls B in \mathbb{R}^n . (For the definition of the Young function, see the next section.)

Then $\|f\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n, w)}$ is a norm and thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ is a Banach space, and $\|f\|_{wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)}$ is a quasi norm and thereby $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ is a quasi Banach space. The Orlicz-Morrey space $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ was first studied in [27]. The spaces $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n)$ were investigated in [12, 28, 29, 31], etc. For other kinds of Orlicz-Morrey spaces, see [4, 5, 7, 10, 32], etc. See also [13, 14] for Morrey-Banach spaces.

The function spaces $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ contain several function spaces as special cases. If $\varphi(B) = 1/w(B)$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ coincide with the weighted Orlicz space $L^\Phi(\mathbb{R}^n, w)$ and its weak version $wL^\Phi(\mathbb{R}^n, w)$, respectively. If $\Phi(t) = t^p$, $1 \leq p < \infty$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ are

denoted by $L^{(p,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(p,\varphi)}(\mathbb{R}^n, w)$, respectively, which are the generalized weighted Morrey space and its weak version. If $\varphi(B) = w(B)^{\kappa-1}$, $0 < \kappa < 1$, then $L^{(p,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(p,\varphi)}(\mathbb{R}^n, w)$ are denoted by $L^{p,\kappa}(\mathbb{R}^n, w)$ and $wL^{p,\kappa}(\mathbb{R}^n, w)$, respectively, which were introduced by Komori and Shirai [21]. If $\Phi(t) = t^p$, $1 \leq p < \infty$, and $\varphi(B) = 1/w(B)$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n, w) = L^p(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w) = wL^p(\mathbb{R}^n, w)$. Therefore, by (1.1), (1.2) and (1.3), we also have the norm inequalities for these function spaces as corollaries.

Let A_p be the Muckenhoupt class of weights, see Definition 2.5. Let $p \in [1, \infty)$ and w is a weight. Muckenhoupt [26] proved that the Hardy-Littlewood maximal operator M is bounded from $L^p(\mathbb{R}^n, w)$ to $wL^p(\mathbb{R}^n, w)$ if and only if $w \in A_p$. He also proved that, for $p \in (1, \infty)$, M is bounded from $L^p(\mathbb{R}^n, w)$ to itself if and only if $w \in A_p$. For the boundedness of the Hilbert transform, the same conclusions hold, see [15].

Let $p \in (1, \infty)$ and w is a weight. It is also known that M is bounded from $wL^p(\mathbb{R}^n, w)$ to itself if and only if $w \in A_p$. We learned from [20] two kinds of simple proofs of this boundedness by Grafakos and by Yabuta. By our results, we see that any Calderón-Zygmund operator is bounded from $wL^p(\mathbb{R}^n, w)$ to itself if $w \in A_p$. In particular, the Riesz transforms are bounded from $wL^p(\mathbb{R}^n, w)$ to itself if and only if $w \in A_p$, see Corollary 5.3.

In the next section we state on the functions Φ , φ and w by which we define $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$. Then we state main results in Section 3. We recall the properties of Young functions and show a lemma in Section 4. To prove the norm inequalities (1.1), (1.2) and (1.3) we need the modular inequalities

$$\begin{aligned} \sup_{t \in (0, \infty)} \Phi(t)w(Tf, t) &\leq C \int_{\mathbb{R}^n} \Phi(c|f(x)|)w(x) dx, \\ \int_{\mathbb{R}^n} \Phi(|Tf(x)|)w(x) dx &\leq C \int_{\mathbb{R}^n} \Phi(c|f(x)|)w(x) dx, \\ \sup_{t \in (0, \infty)} \Phi(t)w(Tf, t) &\leq C \sup_{t \in (0, \infty)} \Phi(t)w(f, t), \end{aligned}$$

respectively, in which the first and the second are known. We prove the third in Section 5. Then, using the results in Sections 4 and 5, we prove the main results in Section 6. In the above, each modular inequality means that it holds for any function f such that the left-hand side is finite, and that the constant C is independent of f . We will make similar abbreviated statements involving other modular and (quasi-)norm inequalities; they will be always interpreted in the same way.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p , is dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. On the functions Φ , φ and w

In this section we state on the functions Φ , φ and w by which we define $L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$. We first recall the Young function and its generalization.

For an increasing (i.e. nondecreasing) function $\Phi : [0, \infty) \rightarrow [0, \infty]$, let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\},$$

with convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. Then $0 \leq a(\Phi) \leq b(\Phi) \leq \infty$.

Let $\overline{\Phi}$ be the set of all increasing functions $\Phi : [0, \infty) \rightarrow [0, \infty]$ such that

1. $0 \leq a(\Phi) < \infty, \quad 0 < b(\Phi) \leq \infty,$
2. $\lim_{t \rightarrow +0} \Phi(t) = \Phi(0) = 0,$
3. Φ is left continuous on $[0, b(\Phi)),$
4. if $b(\Phi) = \infty,$ then $\lim_{t \rightarrow \infty} \Phi(t) = \Phi(\infty) = \infty,$
5. if $b(\Phi) < \infty,$ then $\lim_{t \rightarrow b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) (\leq \infty).$

In what follows, if an increasing and left continuous function $\Phi : [0, \infty) \rightarrow [0, \infty)$ satisfies 2 and $\lim_{t \rightarrow \infty} \Phi(t) = \infty,$ then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \overline{\Phi}$.

For $\Phi, \Psi \in \overline{\Phi},$ we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in [0, \infty].$$

Now we recall the definition of the Young function and give its generalization.

DEFINITION 2.1. 1. A function $\Phi \in \overline{\Phi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi)).$ Let Φ_Y be the set of all Young functions, and let $\overline{\Phi}_Y$ be the set of all $\Phi \in \overline{\Phi}$ such that $\Phi \approx \Psi$ for some $\Psi \in \Phi_Y.$ (Each $\Phi \in \overline{\Phi}_Y$ is also called a quasi-convex function, see [19]).

2. Let \mathcal{Y} be the set of all Young functions such that $a(\Phi) = 0$ and $b(\Phi) = \infty,$ and let $\overline{\mathcal{Y}}$ be the set of all $\Phi \in \overline{\Phi}$ such that $\Phi \approx \Psi$ for some $\Psi \in \mathcal{Y}.$

By the convexity, any Young function Φ is continuous on $[0, b(\Phi))$ and strictly increasing on $[a(\Phi), b(\Phi)].$ Hence Φ is bijective from $[a(\Phi), b(\Phi)]$ to $[0, \Phi(b(\Phi))].$ If $\Phi \in \mathcal{Y},$ then Φ is continuous and bijective from $[0, \infty)$ to itself.

DEFINITION 2.2. 1. A function $\Phi \in \overline{\Phi}$ is said to satisfy the Δ_2 -condition, denoted by $\Phi \in \overline{\Delta}_2,$ if there exists a constant $C > 0$ such that

$$\Phi(2t) \leq C\Phi(t) \quad \text{for all } t > 0.$$

2. A function $\Phi \in \overline{\Phi}$ is said to satisfy the ∇_2 -condition, denoted by $\Phi \in \overline{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \leq \frac{1}{2k} \Phi(kt) \quad \text{for all } t > 0.$$

3. Let $\Delta_2 = \overline{\Phi}_Y \cap \overline{\Delta}_2$ and $\nabla_2 = \overline{\Phi}_Y \cap \overline{\nabla}_2$.

For $\Phi \in \overline{\Phi}$, we recall the dilation indices which are also called the Orlicz-Matuszewska-Maligranda indices:

DEFINITION 2.3. For $\Phi \in \overline{\Phi}$ with $a(\Phi) = 0$ and $b(\Phi) = \infty$, let

$$h_\Phi(\lambda) = \sup_{t \in (0, \infty)} \frac{\Phi(\lambda t)}{\Phi(t)}, \quad \lambda \in (0, \infty),$$

and define the lower and upper indices of Φ by

$$i(\Phi) = \lim_{\lambda \rightarrow +0} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \sup_{\lambda \in (0, 1)} \frac{\log h_\Phi(\lambda)}{\log \lambda},$$

$$I(\Phi) = \lim_{\lambda \rightarrow \infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \inf_{\lambda \in (1, \infty)} \frac{\log h_\Phi(\lambda)}{\log \lambda},$$

respectively, with convention $\log \infty = \infty$.

REMARK 2.1. By the definition we see that $h_\Phi(1) = 1$ and that h_Φ is increasing (i.e. non-decreasing) and submultiplicative which means that $h_\Phi(\lambda_1 \lambda_2) \leq h_\Phi(\lambda_1) h_\Phi(\lambda_2)$ for all $\lambda_1, \lambda_2 \in (0, \infty)$. In this case the above limits exist (permitting ∞) and $0 \leq i(\Phi) \leq I(\Phi) \leq \infty$, see [24] for example. If $\Phi \in \overline{\Delta}_2$, then $a(\Phi) = 0$ and $b(\Phi) = \infty$. In this case $0 < i(\Phi) \leq I(\Phi) < \infty$, see [11, 24] for example.

REMARK 2.2. Let $\Phi, \Psi \in \overline{\Phi}$ with $a(\Phi) = a(\Psi) = 0$ and $b(\Phi) = b(\Psi) = \infty$.

1. If $\Phi \approx \Psi$, then $i(\Phi) = i(\Psi)$ and $I(\Phi) = I(\Psi)$.
2. If $\Phi \in \overline{\mathcal{F}}$, then $1 \leq i(\Phi) \leq I(\Phi) \leq \infty$.
3. $\Phi \in \overline{\nabla}_2$ if and only if $1 < i(\Phi) \leq I(\Phi) \leq \infty$.
4. $\Phi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$ if and only if $1 < i(\Phi) \leq I(\Phi) < \infty$.
5. Let $\Phi \in \overline{\mathcal{F}}$. Then $\Phi \in \overline{\Delta}_2$ if and only if $1 \leq i(\Phi) \leq I(\Phi) < \infty$.
6. Let $0 < i(\Phi) \leq I(\Phi) < \infty$. If $0 < p < i(\Phi) \leq I(\Phi) < q < \infty$, then there exists a positive constant C such that, for all $t, \lambda \in (0, \infty)$,

$$\Phi(\lambda t) \leq C \max(\lambda^p, \lambda^q) \Phi(t),$$

that is, $t \mapsto \frac{\Phi(t)}{t^p}$ is almost increasing and $t \mapsto \frac{\Phi(t)}{t^q}$ is almost decreasing.

7. $\Phi \in \overline{\mathcal{F}}$ if and only if $t \mapsto \frac{\Phi(t)}{t}$ is almost increasing ([19, Lemma 1.1.1]).

Next, we say that a function $\theta : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(x, r)}{\theta(x, s)} \leq C, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2. \tag{2.1}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$\theta(x, r) \leq C\theta(x, s) \quad (\text{resp. } \theta(x, s) \leq C\theta(x, r)), \quad \text{if } r < s.$$

For two functions $\theta, \kappa : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$, we write $\theta \sim \kappa$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(x, r)}{\kappa(x, r)} \leq C.$$

As same as Definition 1.1 we also define $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ by using generalized Young functions $\Phi \in \overline{\Phi}_Y$ together with $\|\cdot\|_{\Phi, \varphi, w, B}$ and $\|\cdot\|_{\Phi, \varphi, w, B, \text{weak}}$, respectively. Then $\|\cdot\|_{L^{(\Phi, \varphi)}(\mathbb{R}^n, w)}$ and $\|\cdot\|_{wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)}$ are quasi norms and thereby $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ are quasi Banach spaces.

REMARK 2.3. Let $\Phi, \Psi \in \overline{\Phi}_Y$ and $\varphi, \psi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$. If $\Phi \approx \Psi$ and $\varphi \sim \psi$, then $L^{(\Phi, \varphi)}(\mathbb{R}^n, w) = L^{(\Psi, \psi)}(\mathbb{R}^n, w)$ and $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w) = wL^{(\Psi, \psi)}(\mathbb{R}^n, w)$ with equivalent quasi norms. It is also known by [16, Proposition 4.2] that, for $\Phi \in \Phi_Y$ and a measurable set G ,

$$\sup_{t \in (0, \infty)} \Phi(t)w(G, f, t) = \sup_{t \in (0, \infty)} t w(G, \Phi(|f|), t). \tag{2.2}$$

In this paper we consider the following classes of φ :

DEFINITION 2.4. For a weight w , let $\mathcal{G}_w^{\text{dec}}$ be the set of all functions $\varphi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$ such that φ is almost decreasing and that $r \mapsto \varphi(x, r)w(B(x, r))$ is almost increasing. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,

$$C\varphi(x, r) \geq \varphi(x, s), \quad \varphi(x, r)w(B(x, r)) \leq C\varphi(x, s)w(B(x, s)), \quad \text{if } r < s.$$

If $w(x) \equiv 1$, we denote $\mathcal{G}_w^{\text{dec}}$ by \mathcal{G}^{dec} simply.

On the weights we consider the following Muckenhoupt A_p classes:

DEFINITION 2.5. For $p \in [1, \infty)$, let A_p be the set of all weight functions w such that the following functional is finite:

$$[w]_{A_1} = \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \|w^{-1}\|_{L^\infty(B)}, \quad \text{if } p = 1,$$

$$[w]_{A_p} = \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1}, \quad \text{if } p \in (1, \infty),$$

where the suprema are taken over all balls B in \mathbb{R}^n . Let

$$A_\infty = \bigcup_{p \in [1, \infty)} A_p.$$

Then the following properties are known: Let w is a weight. Then $w \in A_\infty$ if and only if there exist positive constants δ and C such that, for any ball B and its subset E ,

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta. \quad (2.3)$$

If $1 \leq p < q \leq \infty$, then $A_p \subset A_q$. Let $p \in (1, \infty)$. If $w \in A_p$, then $w \in A_r$ for some $r \in [1, p)$.

Let $w \in A_p$ for some $p \in [1, \infty)$. Then, for any ball B ,

$$\left(\frac{1}{|B|} \int_B |f(x)| dx \right)^p \leq [w]_{A_p} \frac{1}{w(B)} \int_B |f(x)|^p w(x) dx. \quad (2.4)$$

Moreover, there exists a positive constant C such that, for any ball B and $k \in (1, \infty)$,

$$w(kB) \leq Ck^{np} [w]_{A_p} w(B). \quad (2.5)$$

If $w \in A_p$ for some $p \in [1, \infty)$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$, then φ satisfies the doubling condition (2.1), since w satisfies (2.5).

For the properties of A_p -weights, see [8, 9] for example.

3. Main results

The Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

for locally integrable functions f , where the supremum is taken over all balls B containing x . It is known that, if $\Phi \in \overline{\Phi}_V$ and $\varphi \in \mathcal{G}^{\text{dec}}$, then the Hardy-Littlewood maximal operator M is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$. Moreover, if $\Phi \in \overline{\nabla}_2$, then M is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to itself and from $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ to itself, see [17, 28].

Next we state known results for the boundedness of the Calderón-Zygmund operator. First we recall its definition following [36]. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions on \mathbb{R}^n and $\mathcal{S}'(\mathbb{R}^n)$ be the dual spaces of $\mathcal{S}(\mathbb{R}^n)$. Let Ω be the set of all increasing functions $\omega : (0, \infty) \rightarrow (0, \infty)$ such that $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

DEFINITION 3.1. (standard kernel) Let $\omega \in \Omega$. A continuous function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied;

$$|K(x, y)| \leq \frac{C}{|x - y|^n} \quad \text{for } x \neq y,$$

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq \frac{C}{|x - y|^n} \omega\left(\frac{|y - z|}{|x - y|}\right)$$

for $2|y - z| < |x - y|$.

DEFINITION 3.2. (Calderón-Zygmund operator) Let $\omega \in \Omega$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type ω , if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in C^\infty_{\text{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y) dy, \quad x \notin \text{supp } f. \tag{3.1}$$

REMARK 3.1. If $x \notin \text{supp } f$, then $K(x, y)$ is bounded on $\text{supp } f$ with respect to y . Therefore, if (3.1) holds for $f \in C^\infty_{\text{comp}}(\mathbb{R}^n)$, then (3.1) holds for $f \in L^1_{\text{comp}}(\mathbb{R}^n)$.

It is known by [36] that any Calderón-Zygmund operator of type $\omega \in \Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This result was extended to Orlicz-Morrey spaces $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ by [29] as the following: Let $\varphi : (0, \infty) \rightarrow (0, \infty)$. Assume that $\varphi \in \mathcal{G}^{\text{dec}}$ and that there exists a positive constant C such that, for all $r \in (0, \infty)$,

$$\int_r^\infty \frac{\varphi(t)}{t} dt \leq C\varphi(r).$$

Let $\Phi \in \Delta_2 \cap \nabla_2$. For $f \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$, we define Tf on each ball B by

$$Tf(x) = T(f\chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} K(x, y)f(y) dy, \quad x \in B.$$

Then the first term in the right hand side is well defined, since $f\chi_{2B} \in L^\Phi_{\text{comp}}(\mathbb{R}^n) \subset L^1_{\text{comp}}(\mathbb{R}^n)$, and the integral of the second term converges absolutely. Moreover, $Tf(x)$ is independent of the choice of the ball B containing x . By this definition we can show that T is a bounded operator from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to itself. For the weighted boundedness, it is also known by [36] that, if $w \in A_1$, then T is bounded from $L^1(\mathbb{R}^n, w)$ to $wL^1(\mathbb{R}^n, w)$, and, if $w \in A_p$, $1 < p < \infty$, then T is bounded from $L^p(\mathbb{R}^n, w)$ to itself.

In this paper we extend the above results to the weighted Orlicz-Morrey space and its weak version. As a corollary we also get the boundedness of T from $wL^p(\mathbb{R}^n, w)$ to itself if $w \in A_p$, $1 < p < \infty$. The main result is the following:

THEOREM 3.1. *Let M be the Hardy-Littlewood maximal operator, and let T be a Calderón-Zygmund operator of type $\omega \in \Omega$. Let $\Phi \in \overline{\mathcal{T}}$, $w \in A_{l(\Phi)}$ and $\varphi \in \mathcal{G}^{\text{dec}}_w$.*

1. If $i(\Phi) = 1$, then M is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ to $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$. If $1 < i(\Phi) \leq \infty$, then M is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ to itself and from $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ to itself.
2. Assume that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\int_r^\infty \frac{\varphi(x, t)}{t} dt \leq C\varphi(x, r). \quad (3.2)$$

If $i(\Phi) = 1 \leq I(\Phi) < \infty$, then T is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ to $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$. If $1 < i(\Phi) \leq I(\Phi) < \infty$, then T is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ to itself and from $wL^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ to itself.

Ho [12] proved the boundedness of M on $L^{(\Phi, \varphi)}(\mathbb{R}^n, w)$ under stronger conditions. He treated the vector valued inequality.

To prove Theorem 3.1 we need the modular inequalities for which the assumption $w \in A_{i(\Phi)}$ is necessary, see Corollary 5.3.

From the theorem above, for the operators M and T , we get the following corollaries immediately:

COROLLARY 3.2. Let $\Phi \in \overline{\mathcal{Y}}$, $w \in A_{i(\Phi)}$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$.

1. If $i(\Phi) = 1$, then M is bounded from $L^\Phi(\mathbb{R}^n, w)$ to $wL^\Phi(\mathbb{R}^n, w)$. If $1 < i(\Phi) \leq \infty$, then M is bounded from $L^\Phi(\mathbb{R}^n, w)$ to itself and from $wL^\Phi(\mathbb{R}^n, w)$ to itself.
2. Assume that φ satisfies (3.2). If $i(\Phi) = 1 \leq I(\Phi) < \infty$, then T is bounded from $L^\Phi(\mathbb{R}^n, w)$ to $wL^\Phi(\mathbb{R}^n, w)$. If $1 < i(\Phi) \leq I(\Phi) < \infty$, then T is bounded from $L^\Phi(\mathbb{R}^n, w)$ to itself and from $wL^\Phi(\mathbb{R}^n, w)$ to itself.

COROLLARY 3.3. Let $p \in [1, \infty)$, $w \in A_p$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$.

1. If $p = 1$, then M is bounded from $L^{(1, \varphi)}(\mathbb{R}^n, w)$ to $wL^{(1, \varphi)}(\mathbb{R}^n, w)$. If $1 < p < \infty$, then M is bounded from $L^{(p, \varphi)}(\mathbb{R}^n, w)$ to itself and from $wL^{(p, \varphi)}(\mathbb{R}^n, w)$ to itself.
2. Assume that φ satisfies (3.2). Then T has the same boundedness as M .

Let $w \in A_p$ for some $p \in [1, \infty)$. If $\varphi(B) = w(B)^{\kappa-1}$ for some $\kappa \in [0, 1)$, then $\varphi(kB) \lesssim k^{-n\delta(1-\kappa)}\varphi(B)$ for some $\delta > 0$ and all $k \geq 1$ by (2.3). Hence, φ satisfies (3.2). Then we also have the following corollary:

COROLLARY 3.4. (cf. [21, Theorems 3.2 and 3.3]) If $w \in A_1$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$, then both M and T are bounded from $L^{1, \kappa}(\mathbb{R}^n, w)$ to $wL^{1, \kappa}(\mathbb{R}^n, w)$. If $1 < p < \infty$, $w \in A_p$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$, then both M and T are bounded from $L^{p, \kappa}(\mathbb{R}^n, w)$ to itself and from $wL^{p, \kappa}(\mathbb{R}^n, w)$ to itself.

4. Properties on Young functions

In this section we state the properties of Young functions and their generalization. For the theory of Orlicz spaces, see [18, 23, 25] for example.

For $\Phi \in \overline{\Phi}$, we recall the generalized inverse of Φ in the sense of O’Neil [30, Definition 1.2].

DEFINITION 4.1. For $\Phi \in \overline{\Phi}$ and $u \in [0, \infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0, \infty), \\ \infty, & u = \infty. \end{cases}$$

Let $\Phi \in \overline{\Phi}$. Then Φ^{-1} is finite, increasing and right continuous on $[0, \infty)$ and positive on $(0, \infty)$. If Φ is bijective from $[0, \infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ . In general, if $\Phi \in \overline{\Phi}$, then

$$\Phi(\Phi^{-1}(u)) \leq u \leq \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0, \infty],$$

which is a generalization of Property 1.3 in [30], see [33, Proposition 2.2]. Let $\Phi, \Psi \in \overline{\Phi}$. Then

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in [0, \infty],$$

if and only if

$$C^{-1}\Phi^{-1}(t) \leq \Psi^{-1}(t) \leq C\Phi^{-1}(t) \quad \text{for all } t \in [0, \infty],$$

see [33, Lemma 2.3]. That is, $\Phi \approx \Psi$ if and only if $\Phi^{-1} \sim \Psi^{-1}$.

DEFINITION 4.2. For a Young function Φ , its complementary function is defined by

$$\tilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0, \infty)\}, & t \in [0, \infty), \\ \infty, & t = \infty. \end{cases}$$

Then $\tilde{\Phi}$ is also a Young function, and $(\Phi, \tilde{\Phi})$ is called a complementary pair. For example, if $\Phi(t) = t^p/p$, then $\tilde{\Phi}(t) = t^{p'}/p'$ for $p, p' \in (1, \infty)$ and $1/p + 1/p' = 1$. If $\Phi(t) = t$, then

$$\tilde{\Phi}(t) = \begin{cases} 0, & t \in [0, 1], \\ \infty, & t \in (1, \infty]. \end{cases}$$

Namely, $\tilde{\Phi}$ is not necessary in \mathcal{Y} even if $\Phi \in \mathcal{Y}$.

Let $(\Phi, \tilde{\Phi})$ be a complementary pair of Young functions. Then the following inequality holds ([35, (1.3)]):

$$t \leq \Phi^{-1}(t)\tilde{\Phi}^{-1}(t) \leq 2t \quad \text{for } t \in [0, \infty]. \tag{4.1}$$

Let Φ be a Young function and (X, μ) a measure space, and let $L^\Phi(X, \mu)$ be the Orlicz space with the norm $\|\cdot\|_{L^\Phi(X, \mu)}$. Then a simple calculation shows that, for any measurable subset $G \subset X$ with $\mu(G) > 0$,

$$\|\chi_G\|_{L^\Phi(X, \mu)} = \frac{1}{\Phi^{-1}(1/\mu(G))}. \quad (4.2)$$

Let $(\Phi, \tilde{\Phi})$ be a complementary pair of Young functions. Then the following generalized Hölder's inequality holds (see [30]):

$$\int_X |f(x)g(x)| d\mu(x) \leq 2\|f\|_{L^\Phi(X, \mu)}\|g\|_{L^{\tilde{\Phi}}(X, \mu)}. \quad (4.3)$$

Let $\Phi \in \Phi_Y$, $\varphi: \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$ and $B = B(a, r) \subset \mathbb{R}^n$, and let $\mu_B = w dx / (\varphi(B)w(B))$. Then by the definition of $\|\cdot\|_{\Phi, \varphi, w, B}$ and (4.2) we have

$$\|\chi_B\|_{\Phi, \varphi, w, B} = \|\chi_B\|_{L^\Phi(B, \mu_B)} = \frac{1}{\Phi^{-1}(1/\mu_B(B))} = \frac{1}{\Phi^{-1}(\varphi(B))}. \quad (4.4)$$

Moreover, by (4.3) we have

$$\frac{1}{\varphi(B)w(B)} \int_B |f(x)g(x)|w(x) dx \leq 2\|f\|_{\Phi, \varphi, w, B}\|g\|_{\tilde{\Phi}, \varphi, w, B}. \quad (4.5)$$

Here we show the following lemma:

LEMMA 4.1. *Let w be a weight, $\Phi \in \overline{\mathcal{Y}}$ and $\varphi: \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$. Then there exists a positive constant C such that, for all balls B ,*

$$\frac{1}{w(B)} \int_B |f(x)|w(x) dx \leq C\Phi^{-1}(\varphi(B))\|f\|_{\Phi, \varphi, w, B}. \quad (4.6)$$

Moreover, assume that $t \mapsto \Phi(t)/t^p$ is almost increasing for some $p \in (1, \infty)$. Then there exists a positive constant C_p such that

$$\left(\frac{1}{w(B)} \int_B |f(x)|^p w(x) dx\right)^{1/p} \leq C_p \Phi^{-1}(\varphi(B))\|f\|_{\Phi, \varphi, w, B}, \quad (4.7)$$

and, for all $q \in [1, p)$, there exists a positive constant $C_{p,q}$ such that

$$\left(\frac{1}{w(B)} \int_B |f(x)|^q w(x) dx\right)^{1/q} \leq C_{p,q} \Phi^{-1}(\varphi(B))\|f\|_{\Phi, \varphi, w, B, \text{weak}}. \quad (4.8)$$

Proof. We may assume that $\Phi \in \mathcal{Y}$. By (4.5), (4.4) and (4.1) we have

$$\begin{aligned} \frac{1}{w(B)} \int_B |f(x)|w(x) dx &\leq 2\varphi(B)\|f\|_{\Phi, \varphi, w, B}\|\chi_B\|_{\tilde{\Phi}, \varphi, w, B} \\ &= \frac{2\varphi(B)}{\tilde{\Phi}^{-1}(\varphi(B))}\|f\|_{\Phi, \varphi, w, B} \\ &\leq 2\Phi^{-1}(\varphi(B))\|f\|_{\Phi, \varphi, w, B}. \end{aligned}$$

Next, we assume that $t \mapsto \Phi(t)/t^p$ is almost increasing for some $p \in (1, \infty)$. Then $t \mapsto \Phi(t^{1/p})/t$ is almost increasing, which implies $\Phi(\cdot)^{1/p} \in \overline{\mathcal{Y}}$, see Remark 2.2. Let $\Phi_p \in \mathcal{Y}$ such that $\Phi_p \approx \Phi(\cdot)^{1/p}$. Then $\Phi_p^{-1} \sim (\Phi^{-1})^p$ and $\| |f|^p \|_{\Phi_p, \varphi, w, B} \sim (\|f\|_{\Phi, \varphi, w, B})^p$. Using (4.6), we have

$$\begin{aligned} \left(\frac{1}{w(B)} \int_B |f(x)|^p w(x) dx \right)^{1/p} &\lesssim (\Phi_p^{-1}(\varphi(B)) \| |f|^p \|_{\Phi_p, \varphi, w, B})^{1/p} \\ &\sim \Phi^{-1}(\varphi(B)) \|f\|_{\Phi, \varphi, w, B}. \end{aligned}$$

Finally, we show (4.8). We may assume that $\|f\|_{\Phi, \varphi, w, B, \text{weak}} = 1$. Then

$$w(B, f, t) \leq \frac{\varphi(B)w(B)}{\Phi(t)} \quad \text{for all } t \in (0, \infty).$$

Let $q \in [1, p)$ and $t_0 = \Phi^{-1}(\varphi(B))$. Then $\Phi(t_0) = \varphi(B)$. Since $t \mapsto \Phi(t)/t^p$ is almost increasing,

$$\begin{aligned} \int_B |f(x)|^q w(x) dx &= q \int_0^{t_0} t^{q-1} w(B, f, t) dt + q \int_{t_0}^\infty t^{q-1} w(B, f, t) dt \\ &\leq t_0^q w(B) + q \int_{t_0}^\infty t^{q-1} \frac{\varphi(B)w(B)}{\Phi(t)} dt \\ &= t_0^q w(B) + q\varphi(B)w(B) \int_{t_0}^\infty \frac{t^p}{\Phi(t)} t^{-p+q-1} dt \\ &\lesssim t_0^q w(B) + q\varphi(B)w(B) \frac{t_0^p}{\Phi(t_0)} \int_{t_0}^\infty t^{-p+q-1} dt \\ &= t_0^q w(B) + \frac{q}{p-q} t_0^q w(B). \end{aligned}$$

This shows the conclusion. \square

At the end of this section we state another lemma.

LEMMA 4.2. ([34, Lemma 4.4]) *Let $\Phi \in \Delta_2$ and $\varphi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$. If φ satisfies (3.2), then there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,*

$$\int_r^\infty \frac{\Phi^{-1}(\varphi(x, t))}{t} dt \leq C\Phi^{-1}(\varphi(x, r)).$$

Note that [34, Lemma 4.4] is the case $\varphi : (0, \infty) \rightarrow (0, \infty)$. However the proof is the same.

5. Modular inequalities

In this section we show the modular inequalities with $\Phi \in \overline{\mathcal{Y}}$ by using the indices $i(\Phi)$ and $I(\Phi)$.

We first state known weighted inequalities.

THEOREM 5.1. ([1, 2, 15, 26, 36]) *Let M be the Hardy-Littlewood maximal operator, and let T be a Calderón-Zygmund operator of type $\omega \in \Omega$. Let $w \in A_p$, $1 \leq p \leq \infty$.*

1. *If $1 < p \leq \infty$, then*

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx.$$

If $p = 1$, then

$$\sup_{t \in (0, \infty)} tw(Mf, t) \leq C \int_{\mathbb{R}^n} |f(x)| w(x) dx.$$

2. *If $1 < p < \infty$, then*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx.$$

If $p = 1$, then

$$\sup_{t \in (0, \infty)} tw(Tf, t) \leq C \int_{\mathbb{R}^n} |f(x)| w(x) dx.$$

Coifman and Fefferman [2] prove the inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} (Mf(x))^p w(x) dx, \quad (5.1)$$

for any $w \in A_\infty$ and any Calderón-Zygmund operator with standard kernel (the case $\omega(t) = t$ in Definition 3.1). By the kernel estimates in [36] we see that the inequality (5.1) holds for any Calderón-Zygmund operator of type $\omega \in \Omega$. From the inequality (5.1) Curbera, Garcia-Cuerva, Martell and Perez [3] proved the following inequalities:

$$\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) dx \leq C \int_{\mathbb{R}^n} \Phi(Mf(x)) w(x) dx, \quad (5.2)$$

$$\sup_{t \in (0, \infty)} \Phi(t) w(Tf, t) \leq C \sup_{t \in (0, \infty)} \Phi(t) w(Mf, t). \quad (5.3)$$

Then they proved the following modular inequalities except (5.5) and (5.8), see [3, Theorem 3.7]. In this section we prove (5.5) and then (5.8). That is, we have the following theorem:

THEOREM 5.2. *Let M be the Hardy-Littlewood maximal operator, and let T be a Calderón-Zygmund operator of type $\omega \in \Omega$. Let $\Phi \in \overline{\mathcal{A}}$, and let $w \in A_{i(\Phi)}$.*

1. *If $1 < i(\Phi) \leq \infty$, then*

$$\int_{\mathbb{R}^n} \Phi(Mf(x)) w(x) dx \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|) w(x) dx, \quad (5.4)$$

$$\sup_{t \in (0, \infty)} \Phi(t) w(Mf, t) \leq C \sup_{t \in (0, \infty)} \Phi(t) w(Cf, t). \quad (5.5)$$

If $i(\Phi) = 1$, then

$$\sup_{t \in (0, \infty)} \Phi(t)w(Mf, t) \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx. \tag{5.6}$$

2. If $1 < i(\Phi) \leq I(\Phi) < \infty$, then

$$\int_{\mathbb{R}^n} \Phi(|Tf(x)|)w(x) dx \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx, \tag{5.7}$$

$$\sup_{t \in (0, \infty)} \Phi(t)w(Tf, t) \leq C \sup_{t \in (0, \infty)} \Phi(t)w(Cf, t). \tag{5.8}$$

If $i(\Phi) = 1 \leq I(\Phi) < \infty$, then

$$\sup_{t \in (0, \infty)} \Phi(t)w(Tf, t) \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx. \tag{5.9}$$

Kokilashvili and Krbec [19] also investigated the modular inequalities except (5.5) and (5.8). If $1 < i(\Phi) \leq I(\Phi) < \infty$ and w is a weight, then the modular inequality (5.4) implies $w \in A_{i(\Phi)}$. see [19, Theorem 2.1.1]. If $T = R_j$, $i = 1, \dots, n$, which are the Riesz transforms, then (5.9) also implies $w \in A_{i(\Phi)}$, see [19, Theorem 3.1.1]. From this fact, (5.2) and (5.3) we have the following corollary:

COROLLARY 5.3. *Let M be the Hardy-Littlewood maximal operator, and let R_j , $i = 1, \dots, n$, be the Riesz transforms. Let w be a weight and $\Phi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$, i.e., $1 < i(\Phi) \leq I(\Phi) < \infty$. Then the following are equivalent:*

1. $\int_{\mathbb{R}^n} \Phi(Mf(x))w(x) dx \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx,$
2. $\sup_{t \in (0, \infty)} \Phi(t)w(Mf, t) \leq C \sup_{t \in (0, \infty)} \Phi(t)w(Cf, t),$
3. $\sup_{t \in (0, \infty)} \Phi(t)w(Mf, t) \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx,$
4. $\int_{\mathbb{R}^n} \Phi(|R_j f(x)|)w(x) dx \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx,$
5. $\sup_{t \in (0, \infty)} \Phi(t)w(R_j f, t) \leq C \sup_{t \in (0, \infty)} \Phi(t)w(Cf, t),$
6. $\sup_{t \in (0, \infty)} \Phi(t)w(R_j f, t) \leq C \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x) dx,$
7. $w \in A_{i(\Phi)}.$

Note that another pair of indices a_Φ and b_Φ are defined by

$$a_\Phi = \inf_{t \in (0, \infty)} \frac{t\Phi'(t)}{\Phi(t)}, \quad b_\Phi = \sup_{t \in (0, \infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Then $t \mapsto \Phi(t)/t^{a_\Phi}$ is increasing and $t \mapsto \Phi(t)/t^{b_\Phi}$ is decreasing (not almost), see [6, Proposition 2.1 (ii) and (iii)] for example. However, these indices a_Φ and b_Φ are not sharp for the modular inequalities, see the following example.

EXAMPLE 5.1. Let

$$\Phi(t) = \begin{cases} t^2, & t \in [0, 1/4], \\ t/2 - 1/16, & t \in (1/4, 1/2], \\ t^2/2 + 1/16, & t \in (1/2, \infty). \end{cases}$$

Then

$$i(\Phi) = I(\Phi) = 2, \quad \text{but} \quad a_\Phi = 4/3, \quad b_\Phi = 2.$$

Liu and Wang [22] also considered the weighted Orlicz spaces and they showed the modular inequality (5.5) by using the Marcinkiewicz-type interpolation theorem, see the proof of [22, Theorem 5.1]. However, they used indices a_Φ and b_Φ . Therefore, (5.5) in Theorem 5.2 sharpens their result.

To prove (5.5) we prepare the following lemma:

LEMMA 5.4. For $w \in A_\infty$, let

$$M_w f(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f(y)| w(y) dy.$$

Let $\Phi \in \overline{\mathcal{Y}}$. If $i(\Phi) > 1$, then there exists a positive constant c_1 such that

$$\sup_{t \in (0, \infty)} \Phi(t) w(M_w f, t) \leq c_1 \sup_{t \in (0, \infty)} \Phi(t) w(c_1 f, t).$$

Proof. We may assume that $\Phi \in \mathcal{Y}$. First note that M_w is bounded from $wL^p(\mathbb{R}^n, w)$ to itself as same as M is bounded from $wL^p(\mathbb{R}^n)$ to itself if $p \in (1, \infty]$. If $i(\Phi) > 1$, then $\Phi^\theta \in \overline{\mathcal{F}}_Y$ for some $\theta \in (0, 1)$. In this case we have the inequality

$$\Phi(M_w f(x)) \leq (cM_w(\Phi(c|f|)^\theta)(x))^{1/\theta},$$

for some constant c by the same way as [3, Proof of Proposition 5.1]. Then

$$\begin{aligned} \sup_{t \in (0, \infty)} t w(\Phi(M_w f), t) &\leq \sup_{t \in (0, \infty)} t w((cM_w(\Phi(c|f|)^\theta))^{1/\theta}, t) \\ &= \sup_{t \in (0, \infty)} t^{1/\theta} w(cM_w(\Phi(c|f|)^\theta), t) \\ &\lesssim \sup_{t \in (0, \infty)} t^{1/\theta} w(\Phi(c|f|)^\theta, t) \\ &= \sup_{t \in (0, \infty)} t w(\Phi(c|f|), t). \end{aligned}$$

By (2.2) we have the conclusion. \square

Proof of (5.5). We may assume that $\Phi \in \mathcal{Y}$. We use a similar way to the proof of (5.4) in [3]. Let $w \in A_i(\Phi)$. In both cases $1 < i(\Phi) < \infty$ and $i(\Phi) = \infty$, there exists $r \in (1, i(\Phi))$ such that $w \in A_r$. Set $\Phi_r(t) = \Phi(t^{1/r})$. Then $i(\Phi_r) = i(\Phi)/r > 1$. By (2.4) we have $Mf(x) \leq ([w]_{A_r} M_w(|f|^r)(x))^{1/r}$ and then

$$\Phi(Mf(x)) \leq \Phi_r(M_w \tilde{f}(x)),$$

where $\tilde{f} = [w]_{A_r} |f(x)|^r$. By Lemma 5.4 and (2.2) we have

$$\begin{aligned} \sup_{t \in (0, \infty)} tw(\Phi(Mf), t) &\leq \sup_{t \in (0, \infty)} tw(\Phi_r(M_w \tilde{f}), t) \\ &\lesssim \sup_{t \in (0, \infty)} tw(\Phi_r(c_1 \tilde{f}), t) \\ &= \sup_{t \in (0, \infty)} tw(\Phi(Cf), t), \end{aligned}$$

which shows the conclusion. \square

6. Proofs

To prove Theorem 3.1, we prepare three lemmas.

LEMMA 6.1. *Let $\Phi \in \overline{\mathcal{Y}}$, $w \in A_{i(\Phi)}$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$. Let B be a ball. If $i(\Phi) = 1$ and $\|f\|_{L(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, then*

$$\|M(f\chi_{2B})\|_{\Phi, \varphi, w, B, \text{weak}} \leq C \quad \text{and} \quad \|T(f\chi_{2B})\|_{\Phi, \varphi, w, B, \text{weak}} \leq C.$$

If $1 < i(\Phi) \leq I(\Phi) \leq \infty$ and $\|f\|_{L(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$ or $\|f\|_{\text{wL}(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, then

$$\|M(f\chi_{2B})\|_{\Phi, \varphi, w, B} \leq C \quad \text{or} \quad \|M(f\chi_{2B})\|_{\Phi, \varphi, w, B, \text{weak}} \leq C,$$

respectively. If $1 < i(\Phi) \leq I(\Phi) < \infty$ and $\|f\|_{L(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$ or $\|f\|_{\text{wL}(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, then

$$\|T(f\chi_{2B})\|_{\Phi, \varphi, w, B} \leq C \quad \text{or} \quad \|T(f\chi_{2B})\|_{\Phi, \varphi, w, B, \text{weak}} \leq C,$$

respectively. In the above the constant C is independent of f and B .

Proof. We use Theorem 5.2. We only prove the case $i(\Phi) = 1$ and M , since the other cases are similar. If $i(\Phi) = 1$ and $\|f\|_{L(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, then by (5.6) we have

$$\begin{aligned} \sup_{t \in (0, \infty)} \Phi(t) w(B, M(f\chi_{2B})/C, t) &\leq \sup_{t \in (0, \infty)} \Phi(t) w(M(f\chi_{2B})/C, t) \\ &\leq C \int_{2B} \Phi(|f|) w(x) dx \\ &\leq C \varphi(2B) w(2B) \leq C' \varphi(B) w(B). \end{aligned}$$

We may assume that $C' \geq 1$. Then

$$\sup_{t \in (0, \infty)} \Phi(t) w(B, M(f\chi_{2B}) / (C't), t) \leq \varphi(B)w(B),$$

which shows the conclusion. \square

LEMMA 6.2. Let $\Phi \in \overline{\mathcal{Y}}$, $w \in A_{i(\Phi)}$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$. Let B be a ball. If one of the following three conditions holds; (1) $i(\Phi) = 1$ and $\|f\|_{L(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, (2) $1 < i(\Phi) \leq \infty$ and $\|f\|_{L(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, (3) $1 < i(\Phi) \leq \infty$ and $\|f\|_{wL(\Phi, \varphi)(\mathbb{R}^n, w)} = 1$, then

$$M(f\chi_{(2B)^c})(x) \leq C_0 \Phi^{-1}(\varphi(B)), \quad x \in B, \quad (6.1)$$

where the constant C_0 is independent of f and B .

Proof. Let $f_2 = f\chi_{(2B)^c}$, $B = B(a, r)$ and $x \in B$. We show that, for all balls $B' \ni x$,

$$\frac{1}{|B'|} \int_{B'} |f_2(x)| dx \lesssim \Phi^{-1}(\varphi(B)).$$

Let $B' = B(z, r')$. If $r' \leq r/2$, then $\int_{B'} |f_2(y)| dy = 0$, since $B' \subset 2B$. If $r' > r/2$, then $B' \subset B(a, 3r')$. Setting $B'' = B(a, 3r')$, we have

$$\frac{1}{|B'|} \int_{B'} |f_2(x)| dx \lesssim \frac{1}{|B''|} \int_{B''} |f_2(x)| dx.$$

If we show

$$\frac{1}{|B''|} \int_{B''} |f_2(x)| dx \lesssim \Phi^{-1}(\varphi(B'')), \quad (6.2)$$

then we have (6.1), since φ is almost decreasing and Φ^{-1} satisfies the doubling condition.

Case (1): We use (2.4) and (4.6). Since $w \in A_1$, we have

$$\frac{1}{|B''|} \int_{B''} |f_2(x)| dx \lesssim \frac{1}{w(B'')} \int_{B''} |f_2(x)| w(x) dx \lesssim \Phi^{-1}(\varphi(B'')).$$

Case (2): We use (2.4) and (4.7). Since $i(\Phi) > 1$ and $w \in A_{i(\Phi)}$, we can take $p \in (1, i(\Phi))$ such that $w \in A_p$. In this case $t \mapsto \Phi(t)/t^p$ is almost increasing and

$$\frac{1}{|B''|} \int_{B''} |f_2(x)| dx \lesssim \left(\frac{1}{w(B'')} \int_{B''} |f_2(x)|^p w(x) dx \right)^{1/p} \lesssim \Phi^{-1}(\varphi(B'')).$$

Case (3): We use (2.4) and (4.8). Since $i(\Phi) > 1$ and $w \in A_{i(\Phi)}$, we can take $q \in (1, i(\Phi))$ such that $w \in A_q$. In this case $t \mapsto \Phi(t)/t^p$ is almost increasing for $p \in (q, i(\Phi))$ and

$$\frac{1}{|B''|} \int_{B''} |f_2(x)| dx \lesssim \left(\frac{1}{w(B'')} \int_{B''} |f_2(x)|^q w(x) dx \right)^{1/q} \lesssim \Phi^{-1}(\varphi(B'')).$$

Therefore, we have the conclusion. \square

LEMMA 6.3. Let $\Phi \in \overline{\mathcal{T}}$, $w \in A_{i(\Phi)}$ and $\varphi \in \mathcal{G}_w^{\text{dec}}$. Assume that φ satisfies (3.2). Let B be a ball. If one of the following three conditions holds; (1) $i(\Phi) = 1$ and $\|f\|_{L(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, (2) $1 < i(\Phi) \leq I(\Phi) < \infty$ and $\|f\|_{L(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, (3) $1 < i(\Phi) \leq I(\Phi) < \infty$ and $\|f\|_{wL(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, then

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)| dy \leq C_0 \Phi^{-1}(\varphi(B)), \quad x \in B,$$

where the constant C_0 is independent of f and B .

Proof. By Item 5 in Remark 2.2 we may assume that $\Phi \in \Delta_2$. Let $B = B(a, r)$ and $B_k = B(a, 2^k r)$, $k = 1, 2, \dots$. Then

$$\begin{aligned} \int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)| dy &= \sum_{k=2}^{\infty} \int_{B_k \setminus B_{k-1}} |K(x,y)f(y)| dy \\ &\lesssim \sum_{k=2}^{\infty} \frac{1}{|B_k|} \int_{B_k} |f(y)| dy. \end{aligned}$$

For each case of (1), (2) and (3), by the same way as in the proof of the previous lemma, we have

$$\frac{1}{|B_k|} \int_{B_k} |f(y)| dy \lesssim \Phi^{-1}(\varphi(B_k)),$$

instead of (6.2). By the doubling condition of $\Phi^{-1}(\varphi(\cdot))$ and Lemma 4.2 we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)| dy &\lesssim \sum_{k=2}^{\infty} \Phi^{-1}(\varphi(B_k)) \\ &\sim \sum_{k=2}^{\infty} \int_{2^{k-1}r}^{2^k r} \frac{\Phi^{-1}(\varphi(a,t))}{t} dt \\ &\leq \int_r^{\infty} \frac{\Phi^{-1}(\varphi(a,t))}{t} dt \lesssim \Phi^{-1}(\varphi(B)), \end{aligned}$$

which shows the conclusion. \square

Now we prove Theorem 3.1.

Proof of Theorem 3.1, Part 1. Let $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ or $f \in wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$. We may assume that $\|f\|_{L(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$ or $\|f\|_{wL(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, respectively. We will show that $\|Mf\|_{\Phi,\varphi,w,B} \leq C$ or $\|Mf\|_{\Phi,\varphi,w,B,\text{weak}} \leq C$ for any ball $B = B(a, r)$. Let $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. If $i(\Phi) = 1$ and $\|f\|_{L(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, or, if $1 < i(\Phi) \leq \infty$ and $\|f\|_{wL(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, then $\|Mf_1\|_{\Phi,\varphi,w,B,\text{weak}} \leq C$ by Lemma 6.1. If $1 < i(\Phi) \leq \infty$ and $\|f\|_{L(\Phi,\varphi)(\mathbb{R}^n,w)} = 1$, then $\|Mf_1\|_{\Phi,\varphi,w,B} \leq C$ by Lemma 6.1. Moreover, by Lemma 6.2 we have

$$\|Mf_2\|_{\Phi,\varphi,w,B,\text{weak}} \leq \|Mf_2\|_{\Phi,\varphi,w,B} \leq C_0.$$

The proof is complete. \square

Proof of Theorem 3.1, Part 2. Let $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n, w)$ or $f \in wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)$. We may assume that $\|f\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n, w)} = 1$ or $\|f\|_{wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)} = 1$, respectively. For any ball $B = B(a, r)$, let $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$, and let

$$Tf(x) = Tf_1(x) + \int_{\mathbb{R}^n} K(x, y)f_2(y) dy, \quad x \in B. \tag{6.3}$$

We will show that $Tf(x)$ in (6.3) is well defined and independent of the choice of B containing x and that T is bounded.

For the part Tf_1 , by Lemma 6.1, if $i(\Phi) = 1$ and $\|f\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n, w)} = 1$, or, if $1 < i(\Phi) \leq I(\Phi) < \infty$ and $\|f\|_{wL^{(\Phi,\varphi)}(\mathbb{R}^n, w)} = 1$, then

$$\|Tf_1\|_{\Phi, \varphi, w, B, \text{weak}} \leq C. \tag{6.4}$$

If $1 < i(\Phi) \leq I(\Phi) < \infty$ and $\|f\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n, w)} = 1$, then

$$\|Tf_1\|_{\Phi, \varphi, w, B} \leq C. \tag{6.5}$$

Moreover, by Lemma 6.3 we have

$$\left| \int_{\mathbb{R}^n} K(x, y)f_2(y) dy \right| \leq \int_{\mathbb{R}^n \setminus 2B} |K(x, y)f(y)| dy \leq C_0\Phi^{-1}(\varphi(B)), \quad x \in B.$$

Then

$$\begin{aligned} \int_B \Phi \left(\frac{|\int_{\mathbb{R}^n} K(x, y)f_2(y) dy|}{C_0} \right) w(x) dx &\leq \int_B \Phi(\Phi^{-1}(\varphi(B))) w(x) dx \\ &= \varphi(B)w(B), \end{aligned}$$

that is,

$$\left\| \int_{\mathbb{R}^n} K(\cdot, y)f_2(y) dy \right\|_{\Phi, \varphi, w, B, \text{weak}} \leq \left\| \int_{\mathbb{R}^n} K(\cdot, y)f_2(y) dy \right\|_{\Phi, \varphi, w, B} \leq C_0. \tag{6.6}$$

Moreover, if $x \in B \cap B'$ and

$$f = f_1 + f_2 = g_1 + g_2, \quad f_1 = f\chi_{2B}, \quad g_1 = g\chi_{2B'}$$

then $\text{supp}(f_2 - g_2)$ is compact and $x \notin \text{supp}(f_2 - g_2)$. From (3.1), it follows that

$$\int_{\mathbb{R}^n} K(x, y)(f_2(y) - g_2(y)) dy = T(f_2 - g_2)(x).$$

Hence

$$\left(Tf_1(x) + \int_{\mathbb{R}^n} K(x, y)f_2(y) dy \right) - \left(Tg_1(x) + \int_{\mathbb{R}^n} K(x, y)g_2(y) dy \right) = 0.$$

Therefore, $Tf(x)$ in (6.3) is well defined and independent of the choice of B containing x . Further, by (6.4), (6.5) and (6.6) we have

$$\|Tf\|_{\Phi, \varphi, w, B, \text{weak}} \leq C \quad \text{or} \quad \|Tf\|_{\Phi, \varphi, w, B} \leq C, \quad \text{for all balls } B,$$

which shows the conclusion. \square

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