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A SIMPLE COUNTEREXAMPLE FOR THE PERMANENT–ON–TOP CONJECTURE

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Abstract. The permanent-on-top conjecture (POT) was an important conjecture on the largest eigenvalue of the Schur power matrix of a positive semi-definite Hermitian matrix, formulated by Soules. The conjecture claimed that for any positive semi-definite Hermitian matrix H, $\mathrm{per}(H)$ is the largest eigenvalue of the Schur power matrix of the matrix H. After half a century, the POT conjecture has been proven false by the existence of counterexamples which are checked with the help of computer. It raises concerns about a counterexample that can be checked by hand (without the need of computers). A new simple counterexample for the permanent-on-top conjecture is presented which is a complex matrix of dimension 5 and rank 2.

1. Introduction and notations

The symbol S_n denotes the symmetric group on n objects. The permanent of a square matrix is a vital function in linear algebra that is similar to the determinant. For an $n\times n$ matrix $A=(a_{ij})$ with complex coefficients, its permanent is defined as $\mathrm{per}(A)=\sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i,\sigma(i)}$. By $\mathscr{H}_n$ we mean the set of all $n\times n$ positive semi-definite Hermitian matrices. The Schur power matrix of a given $n\times n$ matrix $A=(a_{ij})$, denoted by $\pi(A)$, is a $n!\times n!$ matrix with the elements indexed by permutations $\sigma,\tau\in S_n$:

$$\pi_{\sigma\tau}(A)=\prod_{i=1}^{n} a_{\sigma(i)\tau(i)}.$$

CONJECTURE 1. *The permanent-on-top conjecture (POT)* [9]: Let H be an $n\times n$ positive semi-definite Hermitian matrix, then $\mathrm{per}(H)$ is the largest eigenvalue of $\pi(H)$.

In 2016, Shchesnovich provided a 5-square, rank 2 counterexample to the permanent-on-top conjecture with the help of computer [8].

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DEFINITION 1. For an $n\times n$ matrix $A=(a_{ij})$, let d_A be a function $S_n\to\mathbb{C}$ defined by

$$d_A(\sigma)=\prod_{i=1}^{n}a_{\sigma(i)i}.$$

This function is also called the "diagonal product" function [1]. Then we can define $\det(A)=\sum_{\sigma\in S_n}(-1)^{sign(\sigma)}d_A(\sigma)$ and $\operatorname{per}(A)=\sum_{\sigma\in S_n}d_A(\sigma)$.

For any n-square matrix A and $I,J\subset[n]$, $A[I,J]$ denotes the submatrix of A consisting of entries which are the intersections of i-th rows and j-th columns where $i\in I,\ j\in J$. We define $A(I,J)=A[I^c,J^c]$.

In this paper, we shall study the properties of the spectrum of the Schur power matrix by examining the spectra of the matrices $\mathscr{C}_k(A)$ which are defined in the manner:

For any $1\leqslant k\leqslant n$, the matrix $\mathscr{C}_k(A)$ is a matrix of size $\binom{n}{k}\times\binom{n}{k}$ with its (I,J) entry (I and J are k-element subsets of $[n]$) defined by $\operatorname{per}(A[I,J]).\operatorname{per}(A[I^c,J^c])$. There is another conjecture on these matrices $\mathscr{C}_k(A)$ which states that:

CONJECTURE 2. *Pate's conjecture* [7]: Let A be an $n\times n$ positive semi-definite Hermitian matrix and k be a positive integer number less than n, then the largest eigenvalue of $\mathscr{C}_k(A)$ is $\operatorname{per}(A)$.

Pate's conjecture is weaker than the permanent-on-top conjecture POT because it is well-known that every eigenvalue of $\mathscr{C}_k$ is also an eigenvalue of the Schur power matrix. In the case $k=1$, in [1], it was conjectured that $\operatorname{per}(A)$ is necessarily the largest eigenvalue of $\mathscr{C}_1(A)$ if $A\in\mathscr{H}_n$. Stephen W. Drury has provided an 8-square matrix as a counterexample for this case in the paper [2]. Besides, Bapat and Sunder raise a question as follows:

CONJECTURE 3. *Bapat & Sunder conjecture*: Let A and $B=(b_{ij})$ be $n\times n$ positive semi-definite Hermitian matrices, then

$$\operatorname{per}(A\circ B)\leqslant\operatorname{per}(A)\prod_{i=1}^{n}b_{ii}$$

where $A\circ B$ is the entrywise product (Hadamard product).

The Bapat & Sunder conjecture is weaker than the permanent-on-top conjecture and has been proved false by a counterexample which is a positive semi-definite Hermitian matrix of order 7 proposed by Drury [3]. In the present paper, a new simple counterexample for the permanent-on-top conjecture and Pate's conjecture is presented. It has size 5×5 and rank 2.

CONJECTURE 4. *The Lieb permanent dominance conjecture 1966* [4]: Let H be a subgroup of the symmetric group S_n and let χ be a character of degree m of H. Then

$$\frac{1}{m}\sum_{\sigma\in H}\chi(\sigma)\prod_{i=1}^{n}a_{i\sigma(i)}\leqslant\operatorname{per}(A)$$

holds for all $n\times n$ positive semi-definite Hermitian matrix A.

The permanent dominance conjecture is weaker than the permanent-on-top conjecture and still open. The POT conjecture was proposed by Soules in 1966 as a strategy to prove the permanent dominance conjecture.

DEFINITION 2. The elementary symmetric polynomials in n variables $x_1, x_2, \dots, x_n$ are e_k for $k = 0, 1, \dots, n$. In this paper, we define $e_k(x_i)$ for $i = 1, 2, \dots, n$ to be the elementary symmetric polynomial of degree k in $n-1$ variables obtained by erasing variable x_i from the set $\{x_1, x_2, \dots, x_n\}$ and, for any subset $I \subset [n]$, the notation $e_k[I]$ denote the elementary symmetric polynomial of degree k in $|I|$ variables x_i's, $i \in I$.

2. Associated matrices

We define the associated matrix of a matrix representation $W : S_n \to GL_N(\mathbb{C})$ with respect to a $n \times n$ matrix A by:

$$M_W(A) = \sum_{\sigma \in S_n} d_A(\sigma) W(\sigma).$$

PROPOSITION 2.1. *The Schur power matrix of a given $n \times n$ Hermitian matrix A is the associated matrix of the left-regular representation with respect to A.*

Proof. Take a look at the (σ, τ) entry of $M_L(A)$ which is

$$\sum_{\eta \in S_n,\ \eta \circ \tau = \sigma} d_A(\eta) = d_A(\sigma \circ \tau^{-1}) = \prod_{i=1}^{n} a_{\sigma(i)\tau(i)}$$

the right side is the (σ, τ) entry of $\pi(A)$. $\square$

Let us now consider two important matrices $\mathscr{C}_1(A)$ and $\mathscr{C}_2(A)$ that shall appear frequently from now on.

DEFINITION 3. Let $\mathscr{N}_k : S_n \to GL_{\binom{n}{k}}(\mathbb{C})$ be the matrix representation given by the permutation action of S_n on $\binom{[n]}{k}$.

PROPOSITION 2.2. *For any $n \times n$ Hermitian matrix A, the matrix $\mathscr{C}_k(A)$ is the matrix $M_{\mathscr{N}_k}(A)$.*

We obtain directly the statement that every eigenvalue of matrix $M_{\mathscr{N}_k}(A)$ is an eigenvalue of the associated matrix of the left-regular representation which is the Schur power matrix. Consequently, Pate's conjecture is weaker than the permanent-on-top conjecture(POT).

3. Several properties of the Schur power matrix and $\mathscr{C}_1(A)$ in rank 2 case

The main object of this section is $n\times n$ positive semi-definite Hermitian matrices of rank 2. We know that every matrix $A\in\mathscr{H}_n$ of rank 2 can be written as the sum $v_1v_1^*+v_2v_2^*$ where v_1 and v_2 are two column vectors of order n.

DEFINITION 4. A matrix $A\in\mathscr{H}_n$ is called "formalizable" if A can be written in the form $v_1v_1^*+v_2v_2^*$ and every element of v_1 vector is non-zero.

DEFINITION 5. The formalized matrix A' of a given formalizable matrix A defined in the manner: if $A=v_1v_1^*+v_2v_2^*$ and $v_1=(a_1,\ldots,a_n)^T$, $a_i\neq 0\ \forall i=1,\ldots,n$; $v_2=(b_1,\ldots,b_n)^T$ then $A'=v_3v_3^*+v_4v_4^*$ where $v_3=(1,\ldots,1)^T$ and $v_4=(\frac{b_1}{a_1},\ldots,\frac{b_n}{a_n})^T$.

PROPOSITION 3.1. *Let $A\in\mathscr{H}_n$ be a formalizable matrix, then*

$$\pi(A)=\prod_{i=1}^{n}|a_i|^2\pi(A').$$

Proof. We compare the (σ,τ)-th entries of two matrices.

$$\pi_{\sigma\tau}(A)=\prod_{i=1}^{n}(a_{\sigma(i)}\overline{a_{\tau(i)}}+b_{\sigma(i)}\overline{b_{\tau(i)}})=\prod_{i=1}^{n}|a_i|^2\prod_{i=1}^{n}\left(1+\frac{b_{\sigma(i)}}{a_{\sigma(i)}}\frac{\overline{b_{\tau(i)}}}{\overline{a_{\tau(i)}}}\right)$$
$$=\prod_{i=1}^{n}|a_i|^2\pi_{\sigma\tau}(A'). \quad\square$$

REMARK 1. The same result will be obtained with the matrices $\mathscr{C}_k(A)$ and $\mathscr{C}_k(A')$. It is obvious to see that if the matrix A is a counterexample for the permanent-on-top conjecture and Pate's conjecture then so is A'. Assume that we have an unformalizable matrix $B\in\mathscr{H}$ of rank 2 that is a counterexample for the permanent-on-top conjecture and Pate's conjecture. That also implies that there is a column vector x such that the following inequality holds

$$\frac{x^*\pi(B)x}{\|x\|^2}>\mathrm{per}(B).$$

By continuity and $B=vv^*+uu^*$, we can change slightly the zero elements of the vector v such the the inequality remains. Therefore, if the permanent-on-top conjecture or Pate's conjecture is false for some positive semi-definite Hermitian matrix of rank 2 then so is the permanent-on-top conjecture and Pate's conjecture for some formalizable matrices. That draws our attention to the set of all formalizable matrices.

For any $n\times n$ positive semi-definite Hermitian matrix A of rank 2 there exist two eigenvectors of v and u of A such that $A=vv^*+uu^*$. Let u_i,v_i be the i-th row elements of v and u respectively for $i=\overline{1,n}$. In the case A has a zero row then $\mathrm{per}(A)=0$ and the Schur power matrix and matrices $\mathscr{C}_k(A)$ of A are all zero matrices, there is nothing to discuss. Otherwise, every row of A has a non-zero element (so does every column since A is a Hermitian matrix) which means that for any $i=\overline{1,n}$, the inequalities $|v_i|^2+|u_i|^2>0$ hold. Besides, A can be rewritten in the form

$$(\sin(x)v+\cos(x)u)(\sin(x)v+\cos(x)u)^*+(\cos(x)v-\sin(x)u)(\cos(x)v-\sin(x)u)^*$$

$$\forall x\in[0,2\pi]$$

and the system of n equations $\sin(x)v_i+\cos(x)u_i=0,\ i=\overline{1,n}$ takes finite solutions in the interval $[0,2\pi]$. Therefore, there exists $x\in[0,2\pi]$ satisfying that $(\sin(x)v+\cos(x)u)$ has every element different from 0. Hence, every rank 2 positive semi-definite Hermitian that has no zero-row is formalizable. Several properties about the formalized matrices are presented below.

Let $H\in\mathscr{H}_n$ be a formalizable matrix of the form $H=vv^*+uu^*$ where $v=(1,\dots,1)^T$ and $u=(x_1,x_2,\dots,x_n)^T$. We recall quickly the Kronecker product [10].

DEFINITION 6. The Kronecker product (also known as tensor product or direct product) of two matrices A and B of sizes $m\times n$ and $s\times t$, respectively, is defined to be the $(ms)\times(nt)$ matrix

$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \dots a_{1n}B\\ a_{21}B & a_{22}B & \dots a_{2n}B\\ \vdots & \vdots & \vdots\\ a_{n1}B & a_{n2}B & \dots a_{nn}B\end{pmatrix}.$$

LEMMA 1. The upper bound of rank of the Schur power matrix of rank 2: *If A is $n\times n$ of rank 2 then rank of $\pi(A)$ is not larger than 2^n-n.*

Proof. We observe that $\operatorname{rank}(A)=2$ implies that $\dim(\operatorname{Im}(A))=2$ and $\dim(\operatorname{Ker}(A))=n-2$. Let $\langle w,t\rangle$ be an orthonormal basis of the orthogonal complement of $\operatorname{Ker}(A)$ in $\mathbb{C}^n$, then denote $v=Aw,\ u=At$. Thus, A can be rewritten in the form vw^*+ut^* where $v=(a_1,\dots,a_n)^T$, $u=(b_1,\dots,b_n)^T$. It is obvious that $\operatorname{Im}(A)=\langle v,u\rangle$. Let us denote the Kronecker product of n copies of the matrix A by $\otimes^n A$. The mixed-product property of Kronecker product implies that $\operatorname{Im}(\otimes^n A)=\langle\{\otimes_{i=1}^n t_i,\ t_i\in\{v,u\}\}\rangle$. Furthermore, the Schur power matrix of A is a diagonal submatrix of $\otimes^n A$ obtained by deleting all entries of $\otimes^n A$ that are products of entries of A having two entries in the same row or column. Let define the function f in the manner that

$$f:\{\otimes_{i=1}^n t_i,\ t_i\in\{v,u\}\}\to\widetilde{V}$$

and the σ-th element of $f(\otimes_{i=1}^n\alpha_i)$ vector of order $n!$ is $\prod_{i=1}^n t_i(\sigma(i))$ where $t_i(j)$ is the j-th row element of the column vector t_i. Let $\mathscr{B}=\{f(\otimes_{i=1}^n t_i),\ t_i\in\{v,u\}\}$ then $\mathscr{B}$ is a generator of $\operatorname{Im}(\pi(A))$ since $\pi(A)$ is a principal matrix of $\otimes^n A$ and $\operatorname{Im}(\otimes^n A)=\langle\{\otimes_{i=1}^n t_i,\ t_i\in\{v,u\}\}\rangle$. We partition $\mathscr{B}$ into disjoint sets S_k

$$k=0,1,\dots,n,\ S_k=\{f(\otimes_{i=1}^n t_i),\ t_i\in\{v,u\},\text{ v appears k times in the Kronecker product}\}.$$

Hence, for any $k=1,2,\dots,n$ the σ-th row element of the sum vector $\sum_{w\in S_k} w$ is

$$\sum_{\substack{1\leqslant i_1<\dots<i_k\leqslant n\\ 1\leqslant i_{k+1}<\dots<i_n\leqslant n}} \prod_{j=1}^{k} a_{\sigma(i_j)} \prod_{t=k+1}^{n} b_{\sigma(i_t)} = \sum_{\substack{1\leqslant i_1<\dots<i_k\leqslant n\\ 1\leqslant i_{k+1}<\dots<i_n\leqslant n}} \prod_{j=1}^{k} a_{i_j} \prod_{t=k+1}^{n} b_{i_t}$$

and $S_0=\{(1,,1,\dots,1)^T\}$. Therefore, for any $k=1,\dots,n$ then $S_0\cup S_k$ is linearly dependent. Hence, by deleting an arbitrary element of each set S_k , $k=1,\dots,n$, then it still remains a generator of $\mathrm{Im}(\pi(H))$. Thus

$$\operatorname{rank}(\pi(A))=\dim(\mathrm{Im}(\pi(A)))\leqslant|\mathscr{B}|-n=2^n-n. \qquad \square$$

LEMMA 2. The permanent of a formalized matrix [5]:

$$\operatorname{per}(H)=\sum_{k=0}^{n} k!(n-k)!|e_k|^2.$$

Proof. We show that

$$\begin{aligned}
\operatorname{per}(H) &= \sum_{\sigma\in S_n}\prod_{i=1}^{n}(1+x_i\overline{x_{\sigma(i)}})\\
&= n!+\sum_{\sigma\in S_n}\sum_{k=1}^{n}\sum_{1\leqslant i_1<\dots<i_k\leqslant n} x_{i_1}\dots x_{i_k}\overline{x_{\sigma(i_1)}\dots x_{\sigma(i_k)}}\\
&= n!+\sum_{k=1}^{n}\sum_{1\leqslant i_1<\dots<i_k\leqslant n} x_{i_1}\dots x_{i_k}\overline{\sum_{\sigma\in S_n} x_{\sigma(i_1)}\dots x_{\sigma(i_k)}}\\
&= n!+\sum_{k=1}^{n}\sum_{1\leqslant i_1<\dots<i_k\leqslant n} k!(n-k)!x_{i_1}\dots x_{i_l}\overline{e_k}\\
&= \sum_{k=0}^{n} k!(n-k)!|e_k|^2. \qquad \square
\end{aligned}$$

We use the elementary symmetric polynomials to examine entries of $\mathscr{C}_1(H)$ with the (i,j)-th entry defined by $(1+x_i\overline{x_j}).\operatorname{per}(H(i|j))$ and

$$\begin{aligned}
\operatorname{per}(H(i|j)) &= \sum_{\sigma\in S_n;\ \sigma(i)=j}\prod_{l\neq i}(1+x_l\overline{x_{\sigma(l)}})\\
&= \sum_{\sigma\in S_n;\ \sigma(i)=j}\sum_{k=0}^{n-1}\sum_{1\leqslant i_1<\dots<i_k\leqslant n;\ i_m\neq i\ \forall m=1,\dots,k} x_{i_1}\dots x_{i_k}\overline{x_{\sigma(i_1)}\dots x_{\sigma(i_k)}}\\
&= \sum_{k=0}^{n}\sum_{1\leqslant i_1<\dots<i_k\leqslant n,\ i_m\neq i} k!(n-1-k)!x_{i_1}\dots x_{i_l}\overline{e_k(x_j)}\\
&= \sum_{k=0}^{n-1} k!(n-1-k)!e_k(x_i)\overline{e_k(x_j)}.
\end{aligned}$$

And notice that

$$e_k = x_i e_{k-1}(x_i) + e_k(x_i)\ \forall k = 1,\dots,n.$$

Then

$$\begin{aligned}\frac{\operatorname{per}(H)}{n} &= \frac{1}{n}\sum_{k=0}^{n} k!(n-k)!|e_k|^2\\ &= (n-1)!(|e_0|^2+|e_n|^2)\\ &\quad + \sum_{k=1}^{n-1}\frac{k!(n-k)!}{n}(x_i e_{k-1}(x_i)+e_k(x_i))\overline{(x_j e_{k-1}(x_j)+e_k(x_j))}.\end{aligned}$$

Hence

$$\begin{aligned}&(1+x_i\overline{x_j}).\operatorname{per}(H(i|j)) - \frac{\operatorname{per}(H)}{n}\\ &= \sum_{k=1}^{n-1}\left(k!(n-1-k)! - \frac{k!(n-k)!}{n}\right)e_k(x_i)\overline{e_k(x_j)}\\ &\quad + \left((k-1)!(n-k)! - \frac{k!(n-k)!}{n}\right)x_i e_{k-1}(x_i)\overline{x_j e_{k-1}(x_j)}\\ &\quad - \frac{k!(n-k)!}{n}(x_i e_{k-1}(x_i)\overline{e_k(x_j)} + \overline{x_j e_{k-1}(x_j)}e_k(x_i))\\ &= \sum_{k=1}^{n-1}\frac{(k-1)!(n-1-k)!}{n}(ke_k(x_i)-(n-k)x_i e_{k-1}(x_i))(k\overline{e_k(x_j)}-(n-k)\overline{x_j e_{k-1}(x_j)})\\ &= \sum_{k=1}^{n-1}\frac{(k-1)!(n-1-k)!}{n}(ne_k(x_i)-(n-k)e_k)(\overline{ne_k(x_j)-(n-k)e_k}).\end{aligned}$$

Therefore, we have the following proposition.

PROPOSITION 3.2. *The matrix $\mathscr{C}_1(H)$ can be rewritten in the form*

$$\mathscr{C}_1(H) = \frac{\operatorname{per}(H)}{n}vv^* + \sum_{k=1}^{n-1}\frac{(k-1)!(n-1-k)!}{n}v_k v_k^*$$

where $v = (1,\dots,1)^T$ of order n, for $k = 1,\dots,n-1$, $v_k = (\dots,\underbrace{ne_k(x_i)-(n-k)e_k}_{i\text{-th element}},\dots)^T$.

PROPOSITION 3.3. *For any $k = 1,\dots,n-1$, $\langle v, v_k\rangle = 0$.*

Proof.

$$\begin{aligned}\langle v, v_k\rangle &= \sum_{i=1}^{n}(ne_k(x_i)-(n-k)e_k)\\ &= n\sum_{i=1}^{n}e_k(x_i) - n(n-k)e_k\\ &= 0. \quad \square\end{aligned}$$

PROPOSITION 3.4. *The rank of* $\mathscr{C}_1(H)$ *is the cardinality of the set* $\{x_i, i=\overline{1,n}\}$. *In formula,* $\operatorname{rank}(\mathscr{C}_1(H)) = |\{x_i, i=\overline{1,n}\}|$.

Proof. For the i-th element of v_k, we have

$$ne_k(x_i) - (n-k)e_k = ke_k - nx_i e_{k-1}(x_i) = ke_k + n\sum_{j=1}^{k}(-1)^j e_{k-j}x_i^j$$

which leads us to a conclusion that $\langle v, v_1, \ldots, v_{n-1}\rangle = \langle p_0, \ldots, p_{n-1}\rangle$ where

$$p_j = (\ldots, \underbrace{x_i^j}_{\text{i-th element}}, \ldots)^T$$

which is equal to $|\{x_i, i=\overline{1,n}\}|$ by the determinantal formula of Vandermonde matrices. $\square$

PROPOSITION 3.5. *The determinant of* $\mathscr{C}_1(H)$ *is given by*

$$\det(\mathscr{C}_1(H)) = \frac{\operatorname{per}(H)}{n}\prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot \prod_{i<j}|x_i - x_j|^2.$$

Proof. Case 1: There are indices i and j such that $x_i = x_j$ then $\operatorname{rank}(\mathscr{C}_1(H)) < n$ that is equivalent to $\det(\mathscr{C}_1(H)) = 0$.

Case 2: x_i's are distinct then $\{v, v_1, \ldots, v_{n-1}\}$ makes a basis of $\mathbb{C}^n$. Therefore, $\mathscr{C}_1(H)$ is similar to the Gramian matrix of n vectors $\left\{\sqrt{\frac{\operatorname{per}(H)}{n}}v; \sqrt{\frac{(k-1)!(n-1-k)!}{n}}v_k,\ k=\overline{1,n-1}\right\}$. Thus

$$\begin{aligned}\det(\mathscr{C}_1(H)) &= \det\left(G\left(\sqrt{\frac{\operatorname{per}(H)}{n}}v; \sqrt{\frac{(k-1)!(n-1-k)!}{n}}v_k,\ k=\overline{1,n-1}\right)\right)\\ &= \frac{\operatorname{per}(H)}{n}\prod_{k=1}^{n-1}\frac{(k-1)!(n-1-k)!}{n}\cdot\det(G(v, v_1, \ldots, v_{n-1})).\end{aligned}$$

And from the proof of proposition 3.4, we obtain that

$$(v, v_1, \ldots, v_{n-1}) = (p_0, p_1, \ldots, p_{n-1})\begin{pmatrix} 1 & \ldots & ke_k & \ldots & (n-1)e_{n-1}\\ 0 & \ldots & (-1)^2 ne_{k-1} & \ldots & (-1)^2 ne_{n-2}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & \ldots & \ldots & (-1)^j ne_{k-j} & \ldots\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & \ldots & \ldots & \ldots & (-1)^{n-1}n \end{pmatrix}.$$

The matrix in the right side is the transition matrix given by

$$\text{The } (i,j)\text{-th entry} = \begin{cases} (-1)^i n e_{j-i} & \text{if } i > 1 \\ (j-1)e_{j-1} & \text{if } i = 1 \text{ and } j > 1 \\ 1 & \text{if } (i,j) = (1,1) \end{cases}$$

with convention that $e_0 = 1$; $e_t = 0$ if $t < 0$. Moreover, we observe that the transition matrix is an upper triangular matrix with the absolute value of diagonal entries equal to n except the $(1,1)$-th entry equal to 1 and $(p_0, p_1, \ldots, p_{n-1})$ is a Vandermonde matrix. Hence

$$\begin{aligned} \det(\mathscr{C}_1(H)) &= \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot \det(G(p_0, p_1, \ldots, p_{n-1})) \\ &= \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot |\det(p_0, p_1, \ldots, p_{n-1})|^2 \\ &= \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot \prod_{i<j} |x_i - x_j|^2. \end{aligned}$$

The right side is also equal to 0 if there are indices $i \neq j$ such that $x_i = x_j$. Hence the equality holds in both cases. □

REMARK 2. From the proposition 3.5, we are able to calculate the determinant of $\mathscr{C}_1(H)$ of any positive semi-definite Hermitian matrix H of rank 2 in the way:

Let A be an $n \times n$ positive semi-definite Hermitian matrix of rank 2 then A can be written in the form $vv^* + uu^*$ with v_i, u_i are the i-th elements of v and u respectively. Then the following formula for the determinant of $\mathscr{C}_1(H)$ is achieved.

THEOREM 1. *Let $H = vv^* + uu^*$ be an $n \times n$ positive semi-definite Hermitian matrix then:*

$$\det(\mathscr{C}_1(H)) = \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot \prod_{i<j} |v_i u_j - v_j u_i|^2$$

where v_i and u_i are i-th elements of the vector v and u respectively.

4. A counterexample for the conjectures 1 and 2 in the case $n = 5$

Let us take the values of u_i's and v_i's, $a \in \mathbb{R}$

$$u_1 = ai,\ u_2 = -a,\ u_3 = -ai,\ u_4 = a,\ u_5 = 0,\ v_i = 1\ \forall i = 1, \ldots, 5$$

then $e_1 = e_2 = e_3 = e_5 = 0$, $e_4 = -a^4$.

For any matrix of the form, the spectrum of $\mathscr{C}_1(H)$ is determined clearly by the mentioned above properties and theorems.

By lemma 3.1, $\operatorname{rank}(\pi(H)) \leqslant 2^5 - 5 = 27$ which means that there are at most 27 positive eigenvalues.

By lemma 3.2,

$$\begin{aligned}\operatorname{per}(H) &= 120 + 24|e_1|^2 + 12|e_2|^2 + 12|e_3|^2 + 24|e_4|^2 + 120|e_5|^2 \\ &= 120 + 24a^8\end{aligned}$$

and the proposition 3.2 implies that

$$\mathscr{C}_1(H) = \frac{\operatorname{per}(H)}{5} vv^* + \frac{6}{5} v_1 v_1^* + \frac{2}{5} v_2 v_2^* + \frac{2}{5} v_3 v_3^* + \frac{6}{5} v_4 v_4^*$$

where

$$v_1 = \begin{pmatrix} -5ai \\ 5a \\ 5ai \\ -5a \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -5a^2 \\ 5a^2 \\ -5a^2 \\ 5a^2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 5a^3 i \\ 5a^3 \\ -5a^3 i \\ -5a^3 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} a^4 \\ a^4 \\ a^4 \\ a^4 \\ -4a^4 \end{pmatrix}.$$

Notice that $\{v, v_1, v_2, v_3, v_4\}$ is orthogonal, thus those vectors are eigenvectors of $\mathscr{C}_1(H)$ corresponding to the eigenvalues

$$\operatorname{per}(H) = 120 + 24a^8, \quad \frac{6}{5}\|v_1\|^2 = 120a^2, \quad \frac{2}{5}\|v_2\|^2 = 40a^4,$$

$$\frac{2}{5}\|v_3\|^2 = 40a^6, \quad \frac{6}{5}\|v_4\|^2 = 24a^8.$$

We replace $a^2 = c$, then $\operatorname{tr}(\pi(H)) = 120(1+c)^4$. The spectrum of $\mathscr{C}_1(H)$ is

$$\{120 + 24c^4, 120c, 40c^2, 40c^3, 24c^4\}.$$

Moreover, every eigenvalue of $\mathscr{C}_1(H)$ except $\operatorname{per}(H)$ is an eigenvalue of $\pi(H)$ with multiplicity at least 4 and, every eigenvalue of $\mathscr{C}_2(H)$ except eigenvalues of $\mathscr{C}_1(H)$ is an eigenvalue of $\pi(H)$ with multiplicity at least 5. Therefore, if we can calculate the sum and the sum of squares of at most 2 unknown positive eigenvalues of $\pi(H)$, then the spectrum is determined. We compute the trace of $\mathscr{C}_2(H)$. The $(i,j)(i,j)$-th diagonal entry of $\mathscr{C}_2(H)$ is given by

$$\begin{aligned}&\operatorname{per}(H[\{i,j\},\{i,j\}]).\operatorname{per}(H(\{i,j\},\{i,j\})) \\ &= (2 + |e_1[\{i,j\}]|^2 + 2\,|e_2[\{i,j\}]|^2) \\ &\quad \times (6 + 2\,|e_1[\{i,j\}^c])|^2 + 2\,|e_2[\{i,j\}^c]|^2 + 6\,|e_3[\{i,j\}^c]|^2).\end{aligned}$$

Hence, we use the table to represent all the diagonal entries of $\mathscr{C}_2(H)$.

Coordinates	Values
$(1,2)(1,2)$	$(2+2c+2c^2)(6+4c+2c^2)$
$(1,3)(1,3)$	$(2+2c^2)(6+2c^2)$
$(1,4)(1,4)$	$(2+2c+2c^2)(6+4c+2c^2)$
$(1,5)(1,5)$	$(2+c)(6+2c+2c^2+6c^3)$
$(2,3)(2,3)$	$(2+2c+2c^2)(6+4c+2c^2)$
$(2,4)(2,4)$	$(2+2c^2)(6+2c^2)$
$(2,5)(2,5)$	$(2+c)(6+2c+2c^2+6c^3)$
$(3,4)(3,4)$	$(2+2c+2c^2)(6+4c+2c^2)$
$(4,5)(4,5)$	$(2+c)(6+2c+2c^2+6c^3)$
$\mathrm{tr}(\mathscr{C}_2(H))$	$120+48c^4+104c^3+152c^2+120c$

Furthermore, we use the symmetric polynomials to calculate the sum of all squares of eigenvalues.

$$\begin{aligned}\mathrm{tr}(\pi(H)^2) &= \sum_{\sigma\in S_5}\sum_{\tau\in S_5}\left|\prod_{i=1}^{5}(1+u_{\sigma(i)}\overline{u_{\tau(i)}})\right|^2\\ &= 120\sum_{\sigma\in S_5}\left|\prod_{i=1}^{5}(1+u_i\overline{u_{\sigma(i)}})\right|^2\end{aligned}$$

We know that $u_5=0$, and for $k=1,\dots,4$ we have $u_k=a.i^k$ with $a^2=c$ then

$$\begin{aligned}&\mathrm{tr}(\pi(H)^2)\\ &= 120\sum_{\sigma\in S_5}\left|\prod_{i=1}^{5}(1+u_i\overline{u_{\sigma(i)}})\right|^2\\ &= 120\left(\sum_{k=1}^{4}\sum_{\sigma\in S_5,\ \sigma(k)=5}\left|\prod_{j\neq k,5}(1+u_j\overline{u_{\sigma(j)}})\right|^2+\sum_{\sigma\in S_5,\ \sigma(5)=5}\left|\prod_{i=1}^{4}(1+u_i\overline{u_{\sigma(i)}})\right|^2\right)\\ &= 120\left(\sum_{k=1}^{4}\sum_{\sigma\in S_5,\ \sigma(k)=5}\left|\prod_{j\neq k,5}(1+c.i^{j-\sigma(j)})\right|^2+\sum_{\sigma\in S_5,\ \sigma(5)=5}\left|\prod_{j=1}^{4}(1+c.i^{j-\sigma(j)})\right|^2\right).\end{aligned}$$

LEMMA 3. *By the fundamental theorem of symmetric polynomials and* $e_1=e_2=e_3=e_5=0$ *then every monomial symmetric polynomial in 5 variables of degree non-divisible by 4 takes* (u_1,u_2,u_3,u_4,u_5) *as a root.*

The lemma 4.1 reduces the sums

$$\begin{aligned}
&\sum_{k=1}^{4}\sum_{\sigma\in S_5,\sigma(k)=5}\left|\prod_{j\neq k,5}(1+c\cdot i^{j-\sigma(j)})\right|^2\\
&=\sum_{k=1}^{4}\sum_{\sigma\in S_5,\sigma(k)=5}(1+c^2)^3+(1+c^2)c\sum_{j\neq k,5}2\,\mathrm{Re}(i^{j-\sigma(j)})\\
&\quad+(1+c^2)c^2\sum_{i_1<i_2\neq k,5}(i^{i_1-\sigma(i_1)}+i^{\sigma(i_1)-i_1})(i^{i_2-\sigma(i_2)}+i^{\sigma(i_2)-i_2})\\
&\quad+c^3\prod_{j\neq k,5}(i^{j-\sigma(j)}+i^{\sigma(j)-j})\\
&=96(1+c^2)^3+\sum_{k=1}^{4}\sum_{\sigma\in S_5,\sigma(k)=5}c^2(1+c^2)2\,\mathrm{Re}\left(\sum_{i_1<i_2\neq k,5}i^{i_1-i_2+\sigma(i_2)-\sigma(i_1)}\right)\\
&=96(1+c^2)^3+\sum_{k=1}^{4}c^2(1+c^2)\,\mathrm{Re}\left(\sum_{i_1\neq i_2\neq k,5}e^{i_1-i_2}\sum_{\sigma\in S_5,\ \sigma(k)=5}i^{\sigma(i_2)-\sigma(i_1)}\right)
\end{aligned}$$

combine with

$$\sum_{\sigma\in S_5,\sigma(k)=5}i^{\sigma(i_2)-\sigma(i_1)}=2\sum_{\alpha=1}^{4}i^{\alpha}\sum_{\beta\neq\alpha}i^{\beta}=-2.4=-8.$$

We attain

$$\begin{aligned}
&\sum_{k=1}^{4}\sum_{\sigma\in S_5,\sigma(k)=5}\left|\prod_{j\neq k,5}(1+c\cdot i^{j-\sigma(j)})\right|^2\\
&=96(1+c^2)^3-8c^2(1+c^2)\sum_{k=1}^{4}\mathrm{Re}\left(\sum_{i_1\neq i_2\neq k,5}e^{i_1-i_2}\right)\\
&=96(1+c^2)^3+64c^2(1+c^2).
\end{aligned}$$

The lemma 4.1 also reduces the sum

$$\begin{aligned}
&\sum_{\sigma\in S_5,\ \sigma(5)=5}\left|\prod_{i=1}^{4}(1+c\cdot i^{j-\sigma(j)})\right|^2=\sum_{\sigma\in S_4}\left|\prod_{i=1}^{4}(1+c\cdot i^{j-\sigma(j)})\right|^2\\
&=\sum_{\sigma\in S_4}\left|1+c^4+c^3\sum_{i=1}^{4}i^{\sigma(j)-j}+c\sum_{i=1}^{4}i^{j-\sigma(j)}+c^2\sum_{j_1<j_2}i^{j_1+j_2-\sigma(j_1)-\sigma(j_2)}\right|^2\\
&=24(1+c^4)^2+(c^6+c^2)\sum_{\sigma\in S_4}\left|\sum_{i=1}^{4}i^{j-\sigma(j)}\right|^2+c^4\sum_{\sigma\in S_4}\left|\sum_{j_1<j_2}i^{j_1+j_2-\sigma(j_1)-\sigma(j_2)}\right|^2.
\end{aligned}$$

We compute each part separately by the lemma 4.1

$$\sum_{\sigma\in S_4}\left|\sum_{i=1}^{4} i^{j-\sigma(j)}\right|^2 = 24\cdot 4-8\sum_{j_1\neq j_2} i^{j_1-j_2} = 96+32=128$$

$$\sum_{\sigma\in S_4}\left|\sum_{j_1<j_2} i^{j_1+j_2-\sigma(j_1)-\sigma(j_2)}\right|^2$$

$$= \sum_{\sigma\in S_4}\left(\binom{4}{2}+\frac{1}{4}\sum_{\{i_1,i_2,i_3,i_4\}=\{1,2,3,4\}} i^{\sigma(i_1)+\sigma(i_2)-\sigma(i_3)-\sigma(i_4)+i_3+i_4-i_1-i_2}\right.$$

$$\left. +2\sum_{j_1\neq j_2} i^{j_1-j_2+\sigma(j_2)-\sigma(j_1)}\right)$$

$$= 144+2\sum_{(i_1,i_2,i_3,i_4)} i^{i_3+i_4-i_1-i_2}-16\sum_{j_1\neq j_2} i^{j_1-j_2} = 208-4\sum_{j_1\neq j_2} i^{2j_1+2j_2} = 224.$$

Thus, we obtain $\mathrm{tr}(\pi(H)^2) = 120(24(1+c^4)^2+128(c^6+c^2)+224c^4+96(1+c^2)^3+64c^2(1+c^2))$.

Hence, the spectrum of $\pi(H)$ is

- $\mathrm{per}(H)=120+24c^4$ of multiplicity 1
- $120c, 40c^2, 40c^3, 24c^4$ of multiplicity 4
- $64c^3, 112c^2$ of multiplicity 5
- 0 of multiplicity 93.

We observe that $c=2$ is a solution of the inequality $120+24c^4-64c^3<0$. Therefore, the matrix $H=vv^*+uu^*$ where $v=(1,\ldots,1)^T$, $u=2(i,-1,-i,1,0)^T$ is a counterexample to the permanent-on-top conjecture (POT).

$$H=\begin{pmatrix} 3 & 1-2i & -1 & 1+2i & 1\\ 1+2i & 3 & 1-2i & -1 & 1\\ -1 & 1+2i & 3 & 1-2i & 1\\ 1-2i & -1 & 1+2i & 3 & 1\\ 1 & 1 & 1 & 1 & 1\end{pmatrix}$$

The spectrum of this counterexample is also given by above calculations:

- $\mathrm{per}(H)=504$ of multiplicity 1
- 240, 160, 320, 384 of multiplicity 4
- 512 and 448 of multiplicity 5
- 0 of multiplicity 93

Once, I have the counterexample, a shorter way to prove the matrix H is a counterexample for Pate's conjecture in the case $n=5$ and $k=2$ is available by Tensor product. For the purposes of this paper let us describe the tensor product of vector spaces in terms of bases:

DEFINITION 7. Let V and W be vector spaces over $\mathbb{C}$ with bases $\{v_i\}$ and $\{w_i\}$, respectively. Then $V\otimes W$ is the vector space spanned by $\{v_i\otimes w_j\}$ subject to the rules:

$$(\alpha v+\alpha' v')\otimes w=\alpha(v\otimes w)+\alpha'(v'\otimes w)$$

$$v\otimes(\alpha w+\alpha' w')=\alpha(v\otimes w)+\alpha'(v\otimes w')$$

for all $v,v'\in V$ and $w,w'\in W$ and all scalars α,α'.

If $\langle,\rangle$ is an inner product on V then we can define an inner product $\langle,\rangle$ on $V\otimes V$ in the manner:

$$\langle v_{i_1}\otimes v_{i_2}, v_{i_3}\otimes v_{i_4}\rangle=\langle v_{i_1},v_{i_3}\rangle\langle v_{i_2},v_{i_4}\rangle$$

for any $v_{i_1},v_{i_2},v_{i_3},v_{i_4}$ vectors.

On $\mathbb{C}[x,y]$, we consider the inner product, and the resulting Euclidean norm $|\cdot|$, such that monomials are orthogonal and $|x^ny^k|^2=n!k!$.

PROPOSITION 4.1. *The permanent of the Gram matrix of any 1-forms* $f_j\in\mathbb{C}x\oplus\mathbb{C}y$ *is* $\left|\prod f_j\right|^2$.

Proof. We prove the generalization of the statement which states that if $f_1,f_2,\dots,f_n$, $g_1,g_2,\dots,g_n$ be $2n$ 1-forms and A be an $n\times n$ matrix with (i,j)-th entry $\langle f_i,g_j\rangle$, then

$$\operatorname{per}(A)=\left\langle\prod_{i=1}^n f_i,\prod_{i=1}^n g_i\right\rangle$$

Let $f_i=\alpha_i x+\beta_i y, g_i=\alpha_i' x+\beta_i' y$ for any $i\in\{1,2,\dots,n\}$.

We compute each side of the equality:

The left side is

$$\begin{aligned}\operatorname{per}(A)&=\sum_{\sigma\in S_n}\prod_{i=1}^n\langle f_i,g_{\sigma(i)}\rangle=\sum_{\sigma\in S_n}\prod_{i=1}^n\langle\alpha_i x+\beta_i y,\alpha'_{\sigma(i)}x+\beta'_{\sigma(i)}y\rangle\\&=\sum_{\sigma\in S_n}\prod_{i=1}^n(\alpha_i\cdot\overline{\alpha_{\sigma(i)}}+\beta_i\cdot\overline{\beta'_{\sigma(i)}})\\&=\sum_{\sigma\in S_n}\sum_{k=0}^n\sum_{\substack{1\leqslant i_1<\dots<i_k\leqslant n\\1\leqslant i_{k+1}<\dots<i_n\leqslant n}}\alpha_{i_1}\dots\alpha_{i_k}\beta_{i_{k+1}}\dots\beta_{i_n}\overline{\alpha'_{\sigma(i_1)}\dots\alpha'_{\sigma i_k}\beta'_{\sigma(i_{k+1})}\dots\beta'_{\sigma(i_n)}}\end{aligned}$$

$$= \sum_{k=0}^{n} k!(n-k)! \left(\sum_{\substack{1\leqslant i_1<\ldots<i_k\leqslant n \\ 1\leqslant i_{k+1}<\ldots<i_n\leqslant n}} \alpha_{i_1}\ldots\alpha_{i_k}\beta_{i_{k+1}}\ldots\beta_{i_n} \right)$$
$$\times \left(\sum_{\substack{1\leqslant i_1<\ldots<i_k\leqslant n \\ 1\leqslant i_{k+1}<\ldots<i_n\leqslant n}} \overline{\alpha'_1\ldots\alpha'_{i_k}\beta'_{i_{k+1}}\ldots\beta'_{i_n}} \right)$$

and the right side is

$$\left\langle \prod_{i=1}^{n} f_i, \prod_{i=1}^{n} g_i \right\rangle$$
$$= \left\langle \sum_{k=0}^{n} x^k y^{n-k} \sum_{\substack{1\leqslant i_1<\ldots<i_k\leqslant n \\ 1\leqslant i_{k+1}<\ldots<i_n\leqslant n}} \alpha_{i_1}\ldots\alpha_{i_k}\beta_{i_{k+1}}\ldots\beta_{i_n}, \right.$$
$$\left. \sum_{k=0}^{n} x^k y^{n-k} \sum_{\substack{1\leqslant i_1<\ldots<i_k\leqslant n \\ 1\leqslant i_{k+1}<\ldots<i_n\leqslant n}} \alpha'_{i_1}\ldots\alpha'_{i_k}\beta'_{i_{k+1}}\ldots\beta'_{i_n} \right\rangle$$
$$= \sum_{k=0}^{n} k!(n-k)! \left(\sum_{\substack{1\leqslant i_1<\ldots<i_k\leqslant n \\ 1\leqslant i_{k+1}<\ldots<i_n\leqslant n}} \alpha_{i_1}\ldots\alpha_{i_k}\beta_{i_{k+1}}\ldots\beta_{i_n} \right)$$
$$\times \left(\sum_{\substack{1\leqslant i_1<\ldots<i_k\leqslant n \\ 1\leqslant i_{k+1}<\ldots<i_n\leqslant n}} \overline{\alpha'_1\ldots\alpha'_{i_k}\beta'_{i_{k+1}}\ldots\beta'_{i_n}} \right). \quad \square$$

Let $f_j = x + yi^j\sqrt{2}$ $(j = 1,2,3,4)$ and $f_5 = x$. Their Gram matrix is the given matrix H with $perH = |f_1 f_2 f_3 f_4 f_5|^2 = |x^5 - 4xy^4|^2 = 5! + 16\cdot 4! = 504$ (according to the proposition 4.1). When $\{p,q,r,s,t\} = \{1,2,3,4,5\}$, define $F_{p,q} = f_p f_q \otimes f_r f_s f_t$ and an inner product on $\mathbb{C}[x,y]\otimes\mathbb{C}[x,y]$ as the definition 4.1. It is obvious that $\mathscr{C}_2(H)$ of H is the Gram matrix of the ten tensors $F_{p,q}$ with $\{p,q,r,s,t\} = \{1,2,3,4,5\}$, $p<q$, and $r<s<t$. We observe that

$$(1+i)F_{41} + (-1+i)F_{12} + (-1-i)F_{23} + (1-i)F_{34} - 2iF_{51} + 2F_{52} + 2iF_{53} - 2F_{54}$$
$$= 16\sqrt{2}x^2\otimes y^3 - 32\sqrt{2}xy\otimes xy^2 + 16\sqrt{2}y^2\otimes x^2y,$$

whose norm squared is

$$2^9\cdot 2!3! + 2^{11}\cdot 2! + 2^9\cdot 2!\cdot 2! = 512\cdot 24,$$

while the norm squared of the coefficient vector is

$$|1+i|^2 + |-1+i|^2 + |-1-i|^2 + |1-i|^2 + |-2i|^2 + 2^2 + |2i|^2 + |-2|^2 = 24.$$

Therefore, a linear operator mapping eight orthonormal vectors to F_{12}, F_{23}, F_{34}, F_{41}, F_{51}, F_{52}, F_{53}, F_{54} has norm at least $\sqrt{512}$, so the Gram matrix of these eight tensors, which is an 8-square diagonal submatrix of $\mathscr{C}_2(H)$, has norm (=largest eigenvalue) at least 512, whence so does $\mathscr{C}_2(H)$ itself. In fact, the norm of $\mathscr{C}_2(H)$ is 512.

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