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BOUNDEDNESS OF RIEMANN–LIOUVILLE OPERATOR FROM WEIGHTED SOBOLEV SPACE TO WEIGHTED LEBESGUE SPACE FOR $1<q<p<\infty$

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Abstract. In the paper, a criterion for the boundedness of the Riemann-Liouville fractional integration operator from a weighted Sobolev space to a weighted Lebesgue space is obtained for $1<q<p<\infty$.

1. Introduction

Let $I=(0,\infty)$, $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Suppose that υ, ρ are positive functions and ω is a nonnegative function on I such that υ^p, ρ^p, ω^q, $\rho^{-p'}$ and $\omega^{-q'}$ are locally summable on I.

Let $\overset{\circ}{AC}(I)$ be the set of all locally absolutely continuous functions with compact supports on I. Suppose that $L_{p,\upsilon}\equiv L_p(\upsilon,I)$ is the space of all functions measurable on I with the finite norm $\|f\|_{p,\upsilon}\equiv\|\upsilon f\|_p$, where $\|\cdot\|_p$ is the standard norm of the Lebesgue space $L_p(I)$.

Denote by $W_p^1(\rho,\upsilon)\equiv W_p^1(\rho,\upsilon,I)$ the space of all functions locally absolutely continuous on I with the finite norm

$$\|f\|_{W_p^1}=\|\rho f'\|_p+\|\upsilon f\|_p.$$

In addition, denote by $\overset{\circ}{W}{}_p^1(\rho,\upsilon)\equiv\overset{\circ}{W}{}_p^1(\rho,\upsilon,I)$ the closure of the set $\overset{\circ}{AC}(I)\cap W_p^1(\rho,\upsilon)$ with respect to the norm of the space $W_p^1(\rho,\upsilon)$.

Consider the Riemann-Liouville fractional integration operator I_α, $\alpha>0$:

$$I_\alpha f(x)=\int_0^x(x-s)^{\alpha-1}f(s)ds,\quad x\in I.$$

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The problem of boundedness of the Riemann-Liouville operator from $\overset{\circ}{W}{}^1_p(\rho,\upsilon)$ to $L_q(\omega,I)$ means the validity of the following inequality

$$\|\omega I_\alpha f\|_q \leqslant C(\|\rho f'\|_p + \|\upsilon f\|_p), \quad f \in \overset{\circ}{W}{}^1_p(\rho,\upsilon). \tag{1}$$

In the paper [8], inequality (1) is investigated for $1 < p \leqslant q < \infty$. Here we study the case $1 < q < p < \infty$.

For an arbitrary positive operator T inequality (1) can be rewritten in the form

$$\|\omega T f\|_q \leqslant C(\|\rho f'\|_p + \|\upsilon f\|_p), \quad f \in \overset{\circ}{W}{}^1_p(\rho,\upsilon). \tag{2}$$

When T is the unit operator, necessary and sufficient conditions for the validity of (2) for all values of the parameters $1 \leqslant q,\ p \leqslant \infty$ are given in [11] and [16]. The case $0 < q < 1,\ p > 1$, is investigated in [19]. In the paper [12], inequality (2) is studied for T when it is the integral operator

$$Kf(x) = \int_0^x K(x,s) f(s)ds, \quad x \in I, \tag{3}$$

with the kernel $K(x,s) \geqslant 0$, satisfying the condition: for some number $h \geqslant 1$ the inequality

$$\frac{1}{h}(K(x,t) + K(t,s)) \leqslant K(x,s) \leqslant h(K(x,t) + K(t,s)) \tag{4}$$

holds for all $x,t,s:\ 0 < s \leqslant t \leqslant x < \infty$. The papers [13] and [7] study inequality (2) for operator (3), whose kernel belongs to classes O_n^- and O_n^+, $n \geqslant 0$, introduced in [14]. These classes O_n^- and O_n^+, $n \geqslant 0$, are wider than the class of kernels satisfying (4); namely, such kernels belong to the class $O_1^- \cap O_1^+$.

In the case $\rho(\cdot) \equiv 0$, inequality (2) turns into the weighted Hardy inequality

$$\|\omega T f\|_q \leqslant C\|\upsilon f\|_p, \tag{5}$$

and its validity means the boundedness of the operator T from $L_{p,\upsilon}$ to $L_{q,\omega}$. The development of inequality (2) presented above repeats the same stages of the development of weighted Hardy inequality (5). Thus, inequality (5) was completely characterized for the Hardy operator $Hf(x) = \int_0^x f(s)ds$, for the Riemann-Liouville operator I_α, $\alpha \geqslant 1$, and for operators, whose kernels satisfy condition (4) (see the monograph [9] for more details). Then in the paper [14], inequality (5) was studied for operators with kernels from classes O_n^- and O_n^+, $n \geqslant 0$, in the case $1 < p \leqslant q < \infty$. There are also several results investigating inequality (5) for the Riemann-Liouville operator in the singular case $0 < \alpha < 1$. For example, in the works [10] and [17], for $0 < q < \infty$ and $1 < p < \infty$ criteria for the boundedness of the Riemann-Liouville operator I_α from L_p to $L_{q,\omega}$ are independently found in the case $\alpha > \frac{1}{p}$ and $\upsilon(\cdot) \equiv 1$. In the paper [18], for $1 < p \leqslant q < \infty$ inequality (5) for the Riemann-Liouville operator I_α was characterized under the assumption that one of the weight functions is increasing or decreasing. The most general results for the Riemann-Liouville operator in the singular case $0 < \alpha < 1$ are given in [1].

One more important direction in the development of weighted Hardy inequality (5) is its investigation for operators with variable limits of integration, which has been intensively studied in many works (see, e.g., [3], [4], [5], [6], [19] and [20]) in recent years. This direction is of independent interest, but, as it turned out, it has a connection with inequality (2).

The main goal of this paper is to characterize inequality (1) for $1 < q < p < \infty$ and $\alpha > \frac{1}{p}$. To achieve this goal, we apply a technique based on equivalence of inequality (2) and a certain weighted inequality. Then we combine this technique with the conditions for the boundedness of operators with variable limits of integration on Lebesgue spaces.

Let us finally note that the boundedness of the operator T from a weighted Sobolev space $\overset{\circ}{W}{}^1_p(\rho,\upsilon)$ to a weighted Lebesgue space $L_q(\omega,I)$ can be used to establish the boundedness of this operator T from a weighted Sobolev space $\overset{\circ}{W}{}^1_p(\rho,\upsilon)$ to a weighted Sobolev space $W^1_{q,q_1}(\nu,\omega)$ with the norm

$$\|f\|_{W^1_{q,q_1}} = \|\nu f'\|_{q_1} + \|\omega f\|_q,$$

where $1 < q_1 < \infty$ and ν is a positive function on I such that ν^{q_1} is locally summable on I. This means that characterizations of the inequality

$$\|Tf\|_{W^1_{q,q_1}(\nu,\omega)} \leqslant \|f\|_{\overset{\circ}{W}{}^1_p(\rho,\upsilon)},$$

follow from characterizations of inequality (2). Consequently, using inequality (1) we can find conditions for the boundedness of the Riemann-Liouville operator I_α between Sobolev spaces. In the paper [13] (see Section 5), this application is described in detail.

2. Preliminaries

In the sequel, the relation $A \ll B$ means $A \leqslant CB$ with a constant C depending only on the parameters p and q. Moreover, if $A \ll B \ll A$ we write $A \approx B$.

As in [11], we introduce the following function

$$\delta(x,y) = \sup\left\{ d>0 : \int_{x-d}^{x} \rho^{-p'}(t)dt \leqslant \int_{x}^{x+y} \rho^{-p'}(t)dt, (x-d,x] \subset I \right\},$$

with the domain $D(\delta) = \{(x,y) : x \in I, y > 0, [x,x+y) \in I\}$. If we fix $x \in I$, then at least for a sufficiently small $y > 0$ we have

$$\int_{x-\delta(x,y)}^{x} \rho^{-p'}(t)dt = \int_{x}^{x+y} \rho^{-p'}(t)dt. \tag{6}$$

Let $x \in I$ and D_x be a set of $y > 0$ such that $x + y \in I$ and (6) holds. For all $x \in I$ we define

$$d^+(x) = \sup\{d : \|\rho^{-1}\|_{p',(x-\delta(x,d),x+d)} \|\upsilon\|_{p,(x-\delta(x,d),x+d)} \leqslant 1,\ d \in D_x\},$$

and $d^-(x) = \delta(x, d^+(x))$. Assume that $\mu^-(x) = x - d^-(x)$ and $\mu^+(x) = x + d^+(x)$.

Let for some $c \in I$ we have

$$\|\rho^{-1}\|_{p',(0,c)} + \|\upsilon\|_{p,(0,c)} = \infty,\ \ \|\rho^{-1}\|_{p',(c,\infty)} + \|\upsilon\|_{p,(c,\infty)} = \infty. \tag{7}$$

For simplicity, we assume that (7) holds. The validity of (7) is equivalent to the condition $\overset{\circ}{W}{}^1_p(\rho,\upsilon) = W^1_p(\rho,\upsilon)$ (see [11]). How to overcome the difficulties that arise when the condition (7) does not hold is also given in [11].

On the basis of Lemmas 1.1-1.3 of [11] the functions $\mu^-(x) = x - d^-(x)$ and $\mu^+(x) = x + d^+(x)$ are strictly increasing functions continuous on I. Moreover,

$$\lim_{x\to 0^+} \mu^\pm(x) = 0,\ \lim_{x\to\infty} \mu^\pm(x) = \infty.$$

This gives that $0 < \mu^\pm(x) < \infty$ for any $x \in I$. We need the following statement from [12] and [13].

LEMMA A. *Let condition (7) hold. Then the functions $\mu^-(x)$ and $\mu^+(x)$ are locally absolutely continuous on I.*

Denote by φ^+ and φ^- the inverses of the functions μ^- and μ^+, respectively. Then the functions φ^+ and φ^- are continuous and strictly increasing on I. Moreover, $\varphi^+(x) > \varphi^-(x)$ for any $x \in I$ and $\lim\limits_{x\to 0^+} \varphi^\pm(x) = 0$, $\lim\limits_{x\to\infty} \varphi^\pm(x) = \infty$.

Let us formulate the crucial equivalence statement for inequality (2) with arbitrary positive operator T proved in [7].

LEMMA B. *Let $1 < p, q < \infty$. Inequality (2) for all functions $f \in \overset{\circ}{W}{}^1_p(\rho,\upsilon)$ is equivalent to the inequality*

$$\left(\int_0^\infty \left(\omega(x) T\left(\int_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(t)dt\right)(x)\right)^q dx\right)^{\frac{1}{q}} \leqslant C_1 \left(\int_0^\infty \rho^p(t) f^p(t)dt\right)^{\frac{1}{p}} \tag{8}$$

for all nonnegative functions $f \in L_p(\rho, I)$. Moreover, $C \approx C_1$, where $C > 0$ and $C_1 > 0$ are the best constants in (2) and (8), respectively.

Let $\alpha(x)$ and $\beta(x)$ be locally absolutely continuous and strictly increasing functions on I such that $\alpha(x) < \beta(x)$ for any $x \in I$ and $\lim\limits_{x\to 0^+} \alpha(x) = \lim\limits_{x\to 0^+} \beta(x) = 0$, $\lim\limits_{x\to\infty} \alpha(x) = \lim\limits_{x\to\infty} \beta(x) = \infty$.

Consider the integral operator

$$\mathscr{K} f(x) = \int_{\alpha(x)}^{\beta(x)} K(x,s) f(s)ds,\ \ x \in I. \tag{9}$$

Let $\Omega = \{(x,s) : 0 < x < \infty,\ \alpha(x) \leqslant s \leqslant \beta(x)\}$. Let the function $K(\cdot,\cdot) \geqslant 0$ be defined and measurable on Ω. Moreover, $K(\cdot,\cdot)$ does not decrease in the first argument. Let us define the class $\mathscr{O}_1^+(\Omega)$ of kernels of operator (9). The function $K(\cdot,\cdot)$ belongs to the class $\mathscr{O}_1^+(\Omega)$ if and only if for $K(\cdot,\cdot)$ there exist functions $K_{1,0}(x,s)$ and $V(s)$ defined and measurable on Ω and the relation

$$K(x,s) \approx K_{1,0}(x,t)V(s) + K(t,s) \tag{10}$$

holds for $0 < t \leqslant x < \infty$ and $\alpha(x) \leqslant s \leqslant \beta(t)$, where the constants of equivalency in (10) do not depend on x, t and s. Note that the classes $\mathscr{O}_n^-$ and $\mathscr{O}_n^+$, $n \geqslant 0$, are modifications of the classes O_n^- and O_n^+, $n \geqslant 0$, for variable limits of integration introduced in [15].

The following statement is proved in [2]. For its formulation, we need to construct a sequence $\{t_k\}_{k\in\mathbb{Z}}$ such that $t_0 \in I$ is a fixed point and $t_{k+1} = \alpha^{-1}(\beta(t_k))$, $k \in \mathbb{Z}$.

THEOREM A. *Let $1 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and the kernel $K(x,s)$ of the operator $\mathscr{K}$ belongs to class $\mathscr{O}_1^+(\Omega)$. Then the operator $\mathscr{K}$ is bounded from $L_{p,\rho}$ to $L_{q,\omega}$ if and only if*

$$B^+ = \left(\sum_k \left[(B_{k,1}^+)^r + (B_{k,2}^+)^r + (B_{k,3}^+)^r + (B_{k,4}^+)^r\right]\right)^{\frac{1}{r}} < \infty,$$

where

$$B_{k,1}^+ = \left(\int\limits_{\alpha(t_k)}^{\alpha(t_{k+1})} \left(\int\limits_{t_k}^{\alpha^{-1}(t)} K_{1,0}^q(x,t_k)\omega^q(x)dx\right)^{\frac{r}{q}} \times \left(\int\limits_t^{\alpha(t_{k+1})} V^{p'}(s)\rho^{-p'}(s)ds\right)^{\frac{r}{q'}} V^{p'}(t)\rho^{-p'}(t)dt\right)^{\frac{1}{r}},$$

$$B_{k,2}^+ = \left(\int\limits_{t_k}^{t_{k+1}} \left(\int\limits_{t_k}^{t} \omega^q(x)dx\right)^{\frac{r}{p}} \left(\int\limits_{\alpha(t)}^{\alpha(t_{k+1})} K^{p'}(t_k,s)\rho^{-p'}(s)ds\right)^{\frac{r}{p'}} \omega^q(t)dt\right)^{\frac{1}{r}},$$

$$B_{k,3}^+ = \left(\int\limits_{\beta(t_k)}^{\beta(t_{k+1})} \left(\int\limits_{\beta^{-1}(t)}^{t_{k+1}} K_{1,0}^q(x,\beta^{-1}(t))\omega^q(x)dx\right)^{\frac{r}{q}} \times \left(\int\limits_{\beta(t_k)}^{t} V^{p'}(s)\rho^{-p'}(s)ds\right)^{\frac{r}{q'}} V^{p'}(t)\rho^{-p'}(t)dt\right)^{\frac{1}{r}},$$

and

$$B_{k,4}^{+}=\left(\int_{t_k}^{t_{k+1}}\left(\int_{t}^{t_{k+1}}\omega^{q}(x)dx\right)^{\frac{r}{p}}\left(\int_{\beta(t_k)}^{\beta(t)}K^{p'}(t,s)\rho^{-p'}(s)ds\right)^{\frac{r}{p'}}\omega^{q}(t)dt\right)^{\frac{1}{r}}.$$

Moreover, for the norm $\|\mathscr{K}\|$ *of the operator* $\mathscr{K}$ *from* $L_{p,\rho}$ *to* $L_{q,\omega}$ *the relation* $\|\mathscr{K}\|\approx B^{+}$ *holds.*

3. Criterion for validity of (1)

Here and further we assume that condition (7) holds.

Using the functions $\mu^{-}(x)$ and $\mu^{+}(x)$, similarly to above we construct the sequence $\{t_k\}_{k\in\mathbb{Z}}$ such that $t_0\in I$ is a fixed point and $t_{k+1}=\varphi^{+}(\mu^{+}(t_k))$, $k\in\mathbb{Z}$.

Let

$$B_1=\left(\int_0^{\infty}\left(\int_t^{\infty}\omega^{q}(s)s^{q(\alpha-1)}ds\right)^{\frac{r}{p}}\left(\int_0^{\mu^{-}(t)}U(x)dx\right)^{\frac{r}{p'}}\omega^{q}(t)t^{q(\alpha-1)}dt\right)^{\frac{1}{r}},$$

$$B_{k,1}=\left(\int_{\mu^{-}(t_k)}^{\mu^{-}(t_{k+1})}\left(\int_{t_k}^{\varphi^{+}(t)}(x-t_k)^{q\alpha}\omega^{q}(x)dx\right)^{\frac{r}{q}}\left(\int_{t}^{\mu^{-}(t_{k+1})}\rho^{-p'}(s)ds\right)^{\frac{r}{q'}}\rho^{-p'}(t)dt\right)^{\frac{1}{r}},$$

$$B_{k,2}=\left(\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{t}\omega^{q}(x)dx\right)^{\frac{r}{p}}\left(\int_{\mu^{-}(t)}^{\mu^{-}(t_{k+1})}(t_k-\varphi^{-}(s))^{p'\alpha}\rho^{-p'}(s)ds\right)^{\frac{r}{p'}}\omega^{q}(t)dt\right)^{\frac{1}{r}},$$

$$B_{k,3}=\left(\int_{\mu^{+}(t_k)}^{\mu^{+}(t_{k+1})}\left(\int_{\varphi^{-}(t)}^{t_{k+1}}(x-\varphi^{-}(t))^{q\alpha}\omega^{q}(x)dx\right)^{\frac{r}{q}}\right.$$
$$\left.\times\left(\int_{\mu^{+}(t_k)}^{t}\rho^{-p'}(s)ds\right)^{\frac{r}{q'}}\rho^{-p'}(t)dt\right)^{\frac{1}{r}},$$

$$B_{k,4}=\left(\int_{t_k}^{t_{k+1}}\left(\int_{t}^{t_{k+1}}\omega^{q}(x)dx\right)^{\frac{r}{p}}\left(\int_{\mu^{+}(t_k)}^{\mu^{+}(t)}(t-\varphi^{-}(s))^{p'\alpha}\rho^{-p'}(s)ds\right)^{\frac{r}{p'}}\omega^{q}(t)dt\right)^{\frac{1}{r}}.$$

Assume

$$B_2 = \left(\sum_k \left(B_{k,1}^r + B_{k,2}^r + B_{k,3}^r + B_{k,4}^r \right) \right)^{\frac{1}{r}},$$

$$U(t) = \frac{d}{dt} \int_0^{\mu^-(t)} |\varphi^+(x) - \varphi^-(x)|^{p'} \rho^{-p'}(x) dx.$$

THEOREM 1. *Let $1 < q < p < \infty$ and $\alpha > \frac{1}{p}$. Let the function $U(t)$ be non-increasing for $t > 0$. Then the Riemann-Liouville operator I_α is bounded from $\overset{\circ}{W}{}_p^1(\rho, \upsilon)$ to $L_q(\omega, I)$ if and only if $B = \max\{B_1, B_2\} < \infty$. Moreover, for the norm $\|I_\alpha\|_{W \to q}$ of the operator I_α from $\overset{\circ}{W}{}_p^1(\rho, \upsilon)$ to $L_q(\omega, I)$ the relation $\|I_\alpha\|_{W \to q} \approx B$ holds.*

Proof. In order not to repeat the steps in the proof of this theorem, similar to those in the proofs of Theorems 3.1, 3.2, and 3.3 in the work [8], we omit them. Therefore, for more details, we refer to [8].

By Lemma B inequality (1) holds if and only if the operator

$$\widetilde{\mathscr{I}}_\alpha f(s) \equiv I_\alpha \left(\int_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(x) dx \right)(s)$$

is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$. Moreover, $\|I_\alpha\|_{W \to q} \approx \|\widetilde{\mathscr{I}}_\alpha\|_{p \to q}$, where $\|\widetilde{\mathscr{I}}_\alpha\|_{p \to q}$ is the norm of the operator $\widetilde{\mathscr{I}}_\alpha$ from $L_p(\rho, I)$ to $L_q(\omega, I)$. Arguing similarly as in [8], we get that the operator $\widetilde{\mathscr{I}}_\alpha$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the operators

$$\widetilde{\mathscr{I}}_{1,\alpha} f(s) \equiv \int_0^{\mu^-(s)} f(x) \int_{\varphi^-(x)}^{\varphi^+(x)} (s-t)^{\alpha-1} dt dx$$

and

$$\widetilde{\mathscr{I}}_{2,\alpha} f(s) \equiv \int_{\mu^-(s)}^{\mu^+(s)} (s - \varphi^-(x))^\alpha f(x) dx$$

are bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ with, in addition, $\|\widetilde{\mathscr{I}}_\alpha\|_{p \to q} \approx \|\widetilde{\mathscr{I}}_{1,\alpha}\|_{p \to q} + \|\widetilde{\mathscr{I}}_{2,\alpha}\|_{p \to q}$, where $\|\widetilde{\mathscr{I}}_{1,\alpha}\|_{p \to q}$ and $\|\widetilde{\mathscr{I}}_{2,\alpha}\|_{p \to q}$ are the norms of the operators $\widetilde{\mathscr{I}}_{1,\alpha}$ and $\widetilde{\mathscr{I}}_{2,\alpha}$ from $L_p(\rho, I)$ to $L_q(\omega, I)$.

If the operator $\widetilde{\mathscr{I}}_{1,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$, then arguing as in [8], from the known characterization of inequality (5) for the Hardy operator with variable upper bound (see [20, Theorem 4.1]), for $f \geqslant 0$ we obtain the estimate

$$\|\widetilde{\mathscr{I}}_{1,\alpha}\|_{p \to q} \gg B_1. \tag{11}$$

Now, we suppose that $B_1 < \infty$. Then we have

$$\|\omega \widetilde{\mathscr{I}}_{1,\alpha} f\|_q \leqslant \left(\int_0^\infty \omega^q(s) \left| \int_{\mu^-(\frac{s}{2})}^{\mu^-(s)} (s-\varphi^-(x))^{\alpha-1} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty \omega^q(s) \left| \int_0^{\mu^-(\frac{s}{2})} (s-\varphi^-(x))^{\alpha-1} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$= F_1 + F_2. \tag{12}$$

In F_1 we change the variables $x = \mu^-(t)$ in the inner integral and get

$$F_1 = \left(\int_0^\infty \omega^q(s) \left| \int_{\frac{s}{2}}^{s} (s-t)^{\alpha-1} U^{\frac{1}{p'}}(t) \widetilde{\rho}(t) \widetilde{f}(t) dt \right|^q ds \right)^{\frac{1}{q}}, \tag{13}$$

where $\widetilde{\rho}(t) = \rho(\mu^-(t)) \left(\frac{d\mu^-(t)}{dt}\right)^{\frac{1}{p}}$ and $\widetilde{f}(t) = f(\mu^-(t))$. Using (13) and the non-increase of the function U, we have

$$F_1^q = \sum_k \int_{2^k}^{2^{k+1}} \omega^q(s) \left| \int_{\frac{s}{2}}^{s} (s-t)^{\alpha-1} U^{\frac{1}{p'}}(t) \widetilde{\rho}(t) \widetilde{f}(t) dt \right|^q ds$$

$$\ll \sum_k \left(U(2^{k-1}) 2^{k-1} \right)^{\frac{q}{p'}} \int_{2^k}^{2^{k+1}} \omega^q(s) s^{q(\alpha-1)} ds \left(\int_{2^{k-1}}^{2^{k+1}} |\widetilde{\rho}(t) \widetilde{f}(t)|^p dt \right)^{\frac{q}{p}}.$$

In the last expression using the discrete Hölder's inequality for $\frac{q}{r}$ and $\frac{q}{p}$ and taking into account that

$$U(2^{k-1}) 2^{k-1} \leqslant \int_0^{2^{k-1}} U(x) dx = \int_0^{\mu^-(2^{k-1})} u^{p'}(x) \rho^{-p'}(x) dx,$$

$$\left(\int_{2^k}^{2^{k+1}} \omega^q(s) s^{q(\alpha-1)} ds \right)^{\frac{r}{q}} \leqslant \int_{2^k}^{2^{k+1}} \left(\int_t^\infty \omega^q(s) s^{q(\alpha-1)} ds \right)^{\frac{r}{p}} \omega^q(t) t^{q(\alpha-1)} dt,$$

we obtain

$$F_1^q \leqslant \left(\sum_k \int_{2^k}^{2^{k+1}} \left(\int_t^{\infty} \omega^q(s) s^{q(\alpha-1)} ds \right)^{\frac{r}{p}} \left(\int_0^{\mu^-(t)} u^{p'}(x) \rho^{-p'}(x) dx \right)^{\frac{r}{p'}} \omega^q(t) t^{q(\alpha-1)} dt \right)^{\frac{q}{r}}$$

$$\times \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} |\widetilde{\rho}(t) \widetilde{f}(t)|^p dt \right)^{\frac{q}{p}} \leqslant B_1^q \|\rho f\|_p^q. \quad (14)$$

Arguing again as in [8], from the known characterization of inequality (5) for the Hardy operator with variable upper bound, we obtain $F_2 \ll B_1 \|\rho f\|_p$. The latter, together with (12) and (14), yields that the operator $\widetilde{\mathscr{I}}_{1,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ and the estimate $\|\widetilde{\mathscr{I}}_{1,\alpha}\|_{p\to q} \ll B_1$ holds. This estimate and (11) give that the operator $\widetilde{\mathscr{I}}_{1,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if $B_1 < \infty$. Moreover, $\|\widetilde{\mathscr{I}}_{1,\alpha}\|_{p\to q} \approx B_1$.

Since the kernel $K(t,x) = (t - \varphi^-(x))^\alpha$ of the operator $\widetilde{\mathscr{I}}_{2,\alpha}$ belongs to the class $\mathscr{O}_1^+(\Omega)$ (see the proof of Theorem 3.2 in [8]), by Theorem A the operator $\widetilde{\mathscr{I}}_{2,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if $B_2 < \infty$. Moreover, $\|\widetilde{\mathscr{I}}_{2,\alpha}\|_{p\to q} \approx B_2$. The proof of Theorem 1 is complete. □

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