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A FAMILY OF HOLOMORPHIC FUNCTIONS DEFINED BY DIFFERENTIAL INEQUALITY

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Dedicated to the memory of Professor Rajagopalan Parvatham (1945 – 2021)

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Abstract. The aim of the present paper is to introduce and study a subfamily of holomorphic and normalized functions defined by a differential inequality. Some geometric properties of this family of holomorphic functions and different problems of a family of such functions are presented.

1. Introduction and preliminaries

Let $\mathscr{A}$ denote the family of functions f *holomorphic* in the open unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$ with the power expansion series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{U}, \tag{1}$$

and by $\mathscr{S}$ we denote the subfamily of $\mathscr{A}$ consisting of univalent functions

A function $\omega : \mathbb{U} \to \mathbb{C}$ is called a *Schwarz function* if ω is analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, and we denote by Ω the set of all Schwarz functions.

For two functions f and F analytic in $\mathbb{U}$ we say that the function f *is subordinate to* the function F, and we write $f(z) \prec F(z)$, if there exists a Schwarz function ω such that $f(z) = F(\omega(z))$ for all $z \in \mathbb{U}$. In particular, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

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The logarithmic coefficients γ_n of $f \in \mathscr{S}$ are defined by the following series expansion

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n,\ z \in \mathbb{U}, \tag{2}$$

where $\log 1 = 0$. Note that we use γ_n instead of $\gamma_n(f)$.

These coefficients play an important role for various estimates in the theory of univalent functions. Thus, the idea of studying the logarithmic coefficients helped Kayumov [12] to solve Brennan's conjecture for conformal mappings. The importance of the logarithmic coefficients follows from Lebedev-Milin inequalities [15, Chapter 2], see also [16, 17], where estimates of the logarithmic coefficients were used to find bounds on the coefficients of f. Milin [15] conjectured the inequality

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leqslant 0,\ n = 1,2,3,\ldots,$$

that implies Robertson's conjecture [23], and hence Bieberbach's conjecture [5], which is the famous coefficient problem of Univalent Functions Theory. L. de Branges [6] established the Bieberbach's conjecture by proving Milin's conjecture.

Recall that we can rewrite (2) in the power series form for $z \in \mathbb{U}$ as follows:

$$2\sum_{n=1}^{\infty} \gamma_n z^n = a_2 z + a_3 z^2 + a_4 z^3 + \ldots - \frac{1}{2}\left(a_2 z + a_3 z^2 + a_4 z^3 + \ldots\right)^2$$
$$+ \frac{1}{3}\left(a_2 z + a_3 z^2 + a_4 z^3 + \cdots\right)^3 + \ldots,$$

and equating the coefficients of z^n for $n = 1,2,3$, it follows that

$$\begin{cases} 2\gamma_1 = a_2, \\ 2\gamma_2 = a_3 - \dfrac{1}{2} a_2^2, \\ 2\gamma_3 = a_4 - a_2 a_3 + \dfrac{1}{3} a_2^3. \end{cases} \tag{3}$$

For $\alpha \in [0,1)$, we denote by $\mathscr{S}^*(\alpha)$ the subfamily of $\mathscr{A}$ consisting of all $f \in \mathscr{A}$ for which f is a *starlike function of order* α in $\mathbb{U}$, that is,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U}.$$

Also, for $\gamma \in (0,1]$, we denote by $\widetilde{\mathscr{S}^*}(\gamma)$ the subfamily of $\mathscr{A}$ consisting of all $f \in \mathscr{A}$ for which f is a *strongly starlike function of order* γ in $\mathbb{U}$, that is,

$$\left| \operatorname{Arg} \frac{zf'(z)}{f(z)} \right| < \frac{\gamma\pi}{2},\ z \in \mathbb{U}.$$

Note that $\widetilde{\mathscr{S}^*}(1) = \mathscr{S}^*(0) =: \mathscr{S}^*$ represents the family of *starlike functions* in $\mathbb{U}$.

In addition, we denote by $\mathscr{C}$ the subfamily of close-to-convex functions in $\mathbb{U}$ consisting of all $f \in \mathscr{A}$ for which

$$\operatorname{Re} f'(z) > 0, \; z \in \mathbb{U}.$$

Rønning [25] (see also [26]) introduced the families of uniformly starlike and uniformly convex functions and studied some geometric properties of such functions. According to the above mentioned issues, motivated essentially by the work [25] we will define a new subfamily of $\mathscr{A}$ as follows:

DEFINITION 1. A function $f \in \mathscr{A}$ belongs to the subclass $\mathscr{S}_C^*$ of $\mathscr{A}$ if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|f'(z)-1\right|, \; z \in \mathbb{U}. \tag{4}$$

The identity function on $\mathbb{U}$ belongs to $\mathscr{S}_C^*$ which implies that $\mathscr{S}_C^* \neq \emptyset$. Further, the relation (4) implies $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$, $z \in \mathbb{U}$, which is the well known family of convex $\mathscr{K}$ functions, hence $\mathscr{S}_C^* \subset \mathscr{K}$ and domain form (figure) of functions in the subclass $\mathscr{S}_C^*$ is convex. Moreover, we give the following example of a function that belongs to $\mathscr{S}_C^*$:

EXAMPLE 1. The function $f(z) = z + a_2 z^2$, $z \in \mathbb{U}$, with $a_2 \in \mathbb{C}$, belongs to the family $\mathscr{S}_C^*$ if

$$|a_2| \leqslant \frac{3-\sqrt{5}}{4}.$$

Proof. For $f(z) = z + a_2 z^2 \in \mathscr{A}$, letting $a_2 = \frac{r_0}{2} e^{i\theta_0}$, $r_0 \geqslant 0$, $\theta_0 \in [0, 2\pi]$, for an arbitrary $z = \rho e^{it}$, with $\rho \in [0,1)$ and $t \in [0, 2\pi]$ we have

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = 1 + \operatorname{Re}\frac{2a_2 z}{1+2a_2 z} = 1 + \frac{r_0\rho\left(r_0\rho+\cos(\theta_0+t)\right)}{1+2r_0\rho\cos(\theta+t)+r_0^2\rho^2}$$

and

$$\left|f'(z)-1\right| = |2a_2 z| = r_0\rho.$$

Thus, the inequality (4) is equivalent to

$$1 + \frac{r_0\rho\left(r_0\rho+\cos(\theta_0+t)\right)}{1+2r_0\rho\cos(\theta_0+\rho)+r_0^2\rho^2} > r_0\rho,$$

and setting $x := \cos(\theta_0+t)$ we find the values of r_0 such that

$$l(x,\rho) := 1 + \frac{r_0\rho(r_0\rho+x)}{1+2r_0\rho x+r_0^2\rho^2} - r_0\rho > 0, \text{ for all } -1 \leqslant x \leqslant 1, \; 0 \leqslant \rho < 1.$$

Assuming that $0 \leqslant r_0 \leqslant 1$, it follows that l is an increasing function with respect to x, hence

$$l(-1,\rho) > 0,\ \rho \in [0,1) \Leftrightarrow \frac{1-3r_0\rho + r_0^2\rho^2}{1-r_0\rho} > 0,\ \rho \in [0,1)$$
$$\Leftrightarrow r_0\rho \in \left(-\infty, \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}, \infty\right),\ \rho \in [0,1) \Leftrightarrow r_0 \leqslant \frac{3-\sqrt{5}}{2},$$

therefore $|a_2| \leqslant \dfrac{3-\sqrt{5}}{4}$ and this completes the proof. $\square$

In proving our results, we shall need the following theorem and lemmas.

LEMMA 1. [10, Theorem 2.9] *If $f \in \mathscr{A}$ with $f'(z) \neq 0$ for all $z \in \mathbb{U}$ and* $\operatorname{Re}\sqrt{f'(z)} > \alpha$, $z \in \mathbb{U}$, *for some $\alpha \in [1/2, 1)$, then*

$$\operatorname{Re}\frac{f(z)}{z} > \frac{2\alpha^2+1}{3},\ z \in \mathbb{U}.$$

From Theorem 2.1 and Remark 2.21 of [14], for $\gamma = 1$ we have:

LEMMA 2. [14, Theorem 2.1] *Let $p(z) = 1 + \sum_{k=n}^{\infty} a_k z^k$, $z \in \mathbb{U}$, with $a_n \neq 0$, be an analytic function in $\mathbb{U}$. If*

$$\left|\operatorname{Arg}\left(\delta p(z) + \beta\frac{zp'(z)}{p(z)}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left(\frac{n\alpha\beta}{\delta}\right)\right),\ z \in \mathbb{U},$$

then

$$|\operatorname{Arg} p(z)| < \frac{\alpha\pi}{2},\ z \in \mathbb{U},$$

for some $0 < \alpha < 1$, $\beta > 0$, and $\delta > 0$.

LEMMA 3. [19, p. 172] *If $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$, then $|w_1| \leqslant 1$ and*

$$|w_n| \leqslant 1 - |w_1|^2,\ n = 2,3,4,\cdots.$$

LEMMA 4. [13, Inequality 7, p. 10] *If $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$, $z \in \mathbb{U}$, then*

$$\left|w_2 - t w_1^2\right| \leqslant \max\{1; |t|\},\ t \in \mathbb{C}.$$

The result is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.

LEMMA 5. [7, Theorem 6.4, p. 195] [24, Theorem X, p. 70, Theorem XI, p. 72] *Let* $f(z)=\sum_{n=1}^{\infty}a_nz^n$ *and* $g(z)=\sum_{n=1}^{\infty}b_nz^n$, $z\in\mathbb{U}$, *be two analytic functions in* $\mathbb{U}$, *and suppose that* $f(z)\prec g(z)$ *where* g *is a univalent function in* $\mathbb{U}$.

(i) If g *is a convex function in* $\mathbb{U}$, *then* $|a_n|\leqslant|b_1|$, $n=1,2,3,\cdots$.

(ii) If g *is a starlike function in* $\mathbb{U}$ *(i.e. starlike with respect to* 0*), then* $|a_n|\leqslant n|b_1|$, $n=1,2,3,\cdots$.

Taking $q\to1^-$ in the first part of Theorems 2.1 of [22] we get the next result:

LEMMA 6. [22] *Suppose that* $f\in\mathscr{A}$ *with* $f'(z)\neq0$ *for all* $z\in\mathbb{U}$, *has the power series expansion of the form* (1), *and*

$$\sqrt{f'(z)}\prec\phi(z),$$

where $\phi(z)=1+\sum_{n=1}^{\infty}A_nz^n$, $z\in\mathbb{U}$, *with* $A_1>0$, *is an analytic function in* $\mathbb{U}$. *If* A_1, A_2, *and* A_3 *satisfy the conditions*

$$\left|A_1^2+2A_2\right|\leqslant7A_1\quad\text{and}\quad\left|A_1^2A_2+9A_1A_3-2A_1^4-8A_2^2\right|\leqslant8A_1^2,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4-a_3^2\right|\leqslant\frac{4A_1^2}{9}.$$

The aim of the present paper is to introduce and study a subfamily of holomorphic and normalized functions defined by a differential inequality. Some geometric properties and different problems for a family of such functions are presented.

2. Properties of the family $\mathscr{S}_C^*$

In this section we obtain some geometric properties of the class $\mathscr{S}_C^*$ like: subordination properties, radius of starlikeness of order α, bounded rotation result and distortion and covering theorems.

THEOREM 1. *If the function* $f\in\mathscr{S}_C^*$, *then*

$$f'(z)\prec\frac{1}{1-z}=:q_1(z),$$

or

$$f'(z)\prec\frac{1}{{}_2F_1\left(2,1,2;\dfrac{z}{z-1}\right)}=q_1(z),$$

where ${}_2F_1(a,b,c;z)$ *is the* Gaussian hypergeometric function, *and* q_1 *is the best dominant of the subordination.*

Proof. Let the function $f \in \mathscr{S}_C^*$, and define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = f'(z),\ z \in \mathbb{U}.$$

Then, p is analytic in $\mathbb{U}$, $p(0) = 1$, and

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zp'(z)}{p(z)},\ z \in \mathbb{U}.$$

Since $f \in \mathscr{S}_C^*$, we get

$$\operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) = \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|f'(z) - 1\right|$$
$$= |p(z) - 1| \geqslant \operatorname{Re}\left(1 - p(z)\right),\ z \in \mathbb{U},$$

and the last inequality implies

$$\operatorname{Re}\left(p(z) + \frac{zp'(z)}{p(z)}\right) > 0,\ z \in \mathbb{U},$$

that is equivalent to

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z}. \tag{5}$$

Now, if we take in Theorem 3.3d. of [18, p. 109] the values

$$\beta := 1, \quad \gamma := 0, \quad A := 1, \quad B := -1,$$

the subordination (3.3-11) of [18, p. 109] reduces to (5), and the assumptions of Theorem 3.3d., i.e. $\operatorname{Re}(\beta + \gamma) = 1 > 0$ and (3.3-10) of [18, p. 108] are satisfied. According to this theorem, combined with the relations (3.3-1) of [18, p. 103] and (3.3-5) of [18, p. 104], we conclude that

$$p(z) \prec \frac{1}{1-z} =: q_1(z) \prec \frac{1+z}{1-z},$$

that is,

$$f'(z) \prec \frac{1}{1-z},$$

and q_1 is the best dominant of the subordination.

Also, according to this theorem, combined with the relations (3.3-13) and (3.3-15) of [18, p. 110], we can rewrite the above subordination and conclude that

$$f'(z) \prec \frac{1}{{}_2F_1\left(2, 1, 2; \dfrac{z}{z-1}\right)} = q_1(z),$$

and q_1 is the best dominant of the subordination. $\square$

REMARK 1. Theorem 1 could be reformulated as follows:
If the function $f \in \mathscr{S}_C^*$, then $\operatorname{Re} f'(z) > 1/2$, $z \in \mathbb{U}$, and the right-hand side bound $1/2$ cannot be enlarged.

The next result gives us the radius of starlikeness of order α for the family $\mathscr{S}_C^*$:

THEOREM 2. *If the function* $f \in \mathscr{S}_C^*$ *and* $0 \leqslant \alpha < 1$, *then*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \ |z| < \frac{3-2\alpha-\sqrt{(\alpha-3)^2-3}}{1-\alpha}.$$

Proof. If the function $f \in \mathscr{S}_C^*$, then by Remark 1 we have

$$\operatorname{Re} \sqrt{f'(z)} > \frac{1}{\sqrt{2}}, \ z \in \mathbb{U},$$

and using Lemma 1 for $\alpha = 1/\sqrt{2}$ we get

$$\operatorname{Re} \frac{f(z)}{z} > \frac{2}{3}, \ z \in \mathbb{U},$$

that is equivalent to

$$\frac{f(z)}{z} \prec \frac{1-\frac{1}{3}z}{1-z} =: h(z). \tag{6}$$

Therefore, from the definition of the subordination there exists a function $\omega \in \Omega$, such that

$$\frac{f(z)}{z} = \frac{1-\frac{1}{3}\omega(z)}{1-\omega(z)}, \ z \in \mathbb{U},$$

and by logarithmical differentiation we get

$$\frac{zf'(z)}{f(z)} - 1 = \frac{\frac{2}{3}z\omega'(z)}{(1-\omega(z))\left(1-\frac{1}{3}\omega(z)\right)}, \ z \in \mathbb{U}.$$

Since $\omega \in \Omega$ it follows that $|\omega(z)| \leqslant |z| = r \in [0,1)$ for all $z \in \mathbb{U}$, and also the well-known inequality $|\omega'(z)| \leqslant \left(1-|w(z)|^2\right)/\left(1-|z|^2\right)$, $z \in \mathbb{U}$, for Schwarz functions holds (see also the inequality (28) of [19, p. 168]). Using these inequalities, the above relation implies

$$\begin{aligned}
\operatorname{Re} \frac{zf'(z)}{f(z)} &= 1 + \operatorname{Re} \frac{\frac{2}{3}z\omega'(z)}{(1-\omega(z))\left(1-\frac{1}{3}\omega(z)\right)} \geqslant 1 - \frac{\frac{2}{3}|z||\omega'(z)|}{|1-\omega(z)|\left|1-\frac{1}{3}\omega(z)\right|} \\
&\geqslant 1 - \frac{\frac{2}{3}|z|\left(1-|w(z)|^2\right)}{|1-\omega(z)|\left|1-\frac{1}{3}\omega(z)\right|(1-|z|^2)} \geqslant 1 - \frac{\frac{2}{3}|z|\left(1-|\omega(z)|^2\right)}{(1-|\omega(z)|)\left(1-\frac{1}{3}|z|\right)(1-|z|^2)} \\
&= 1 - \frac{\frac{2}{3}|z|(1+|\omega(z)|)}{\left(1-\frac{1}{3}|z|\right)(1-|z|^2)} \geqslant 1 - \frac{\frac{2}{3}|z|(1+|z|)}{\left(1-\frac{1}{3}|z|\right)(1-|z|^2)} \\
&\geqslant 1 - \frac{\frac{2}{3}|z|}{\left(1-\frac{1}{3}|z|\right)(1-|z|)} \geqslant 1 - \frac{\frac{2}{3}r}{(1-r)(1-\frac{1}{3}r)}, \ |z| = r < 1.
\end{aligned}$$

Now, to obtain our result we should have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqslant 1-\frac{\frac{2}{3}r}{(1-r)\left(1-\frac{1}{3}r\right)} > \alpha,\ |z|=r<1,$$

and the right-hand side inequality is equivalent to

$$T(r) := (\alpha-1)r^2+2(3-2\alpha)r+3(\alpha-1)<0.$$

Since the discriminant of the above quadratic form is $\Delta(\alpha) := 4\left[(\alpha-3)^2-3\right]>0$ for all $\alpha\in[0,1)$, and the coefficient of r^2 is $\alpha-1<0$ for all $\alpha\in[0,1)$, it follows that $T(r)<0$ whenever

$$0\leqslant r<\frac{3-2\alpha-\sqrt{(\alpha-3)^2-3}}{1-\alpha},$$

which proves our result. □

REMARK 2. The above theorem could be reformulated as follows:
If the function $f\in\mathscr{S}_C^*$ and $0\leqslant\alpha<1$, then $F(z) := f(\rho z)\in\mathscr{S}^*(\alpha)$, where

$$\rho := \frac{3-2\alpha-\sqrt{(\alpha-3)^2-3}}{1-\alpha}.$$

The following bounded rotation property for the functions in $\mathscr{S}_C^*$ holds:

THEOREM 3. *If $f\in\mathscr{S}_C^*$ with $f''(0)\neq 0$, then*

$$\left|\operatorname{Arg} f'(z)\right| < \frac{\widehat{\alpha}\pi}{2},\ z\in\mathbb{U},$$

where the value $\widehat{\alpha} :\simeq 0.6383222623$ is the (unique) solution of the equation

$$\alpha+\frac{2}{\pi}\arctan\alpha=1.$$

Proof. For $f\in\mathscr{S}_C^*$ let us define the function $p:\mathbb{U}\to\mathbb{C}$ by

$$p(z)=f'(z)=1+\sum_{k=1}^{\infty}c_kz^k,\ z\in\mathbb{U}.$$

Then, p is analytic in $\mathbb{U}$, with $p(0)=1$, and from Remark 1 it follows that $p(z)\neq 0$ for all $z\in\mathbb{U}$. Like to the proof of Theorem 1 we get that the subordination (5) holds, which is equivalent to

$$\left|\operatorname{Arg}\left(p(z)+\frac{zp'(z)}{p(z)}\right)\right| < \frac{\pi}{2},\ z\in\mathbb{U}.$$

Since $c_1 = f''(0)/2 \neq 0$, using Lemma 2 for the special case $\delta = \beta = \gamma = 1$, we obtain

$$|\operatorname{Arg} p(z)| < \frac{\alpha\pi}{2}, \ z \in \mathbb{U},$$

where $\alpha + \dfrac{2}{\pi}\arctan\alpha = 1$. Since the function $\varphi(\alpha) := \alpha + \dfrac{2}{\pi}\arctan\alpha - 1$, $\alpha \in \mathbb{R}$ is increasing on $\mathbb{R}$, $\varphi(0) = -1$ and $\varphi(1) = 1/2$, it follows that φ has a unique zero $\widehat{\alpha} :\simeq 0.6383222623 \in (0,1)$. $\square$

The final result of this section represents a distortion and a covering theorem for the family $\mathscr{S}_C^*$, respectively:

THEOREM 4. *If the function $f \in \mathscr{S}_C^*$, then*

$$\frac{1}{1+r} \leqslant |f'(z)| \leqslant \frac{1}{1-r}, \ |z| \leqslant r < 1,$$

and

$$\frac{r(3+r)}{3(1+r)} \leqslant |f(z)| \leqslant \frac{r(3-r)}{3(1-r)}, \ |z| \leqslant r < 1.$$

Proof. Let the function $f \in \mathscr{S}_C^*$. For the proof of the first inequality, using Theorem 1 we have

$$f'(z) \prec \frac{1}{1-z} =: q_1(z).$$

If we let $\mathbb{U}_r := \{z \in \mathbb{C} : |z| < r\}$, by using the subordination principle, the above subordination implies

$$f'\left(\overline{\mathbb{U}}_r\right) \subset q_1\left(\overline{\mathbb{U}}_r\right), \tag{7}$$

for all $r \in (0,1)$. Since q_1 is a circular transformation symmetric respecting the real axis, it follows that

$$\frac{1}{1+r} = q_1(-r) \leqslant |q_1(z)| \leqslant q_1(r) = \frac{1}{1-r}, \ |z| = r < 1, \tag{8}$$

for all $r \in (0,1)$. Thus, from (7) and (8) we get the required result.

For the proof of the second inequality we will use the fact that the subordination (6) holds, and by using the same method as the above we obtain our result. $\square$

3. Coefficient bounds

In this section we present some problems for the coefficients of functions that belong to the family $\mathscr{S}_C^*$. Also, the bounds of the logarithmic coefficients, of the Fekete-Szegő functional, and of the second Hankel determinant (see [1, 2, 3, 8, 9, 11, 20, 27, 28]) of the functions belonging to this family are determined.

THEOREM 5. *If the function* $f \in \mathscr{S}_C^*$ *has the form* (1)*, then*

$$|a_n| \leqslant \frac{1}{n}, \; n = 1, 2, 3, \cdots.$$

Proof. If $f \in \mathscr{S}_C^*$, from Theorem 1 it follows that

$$f'(z) \prec q_1(z).$$

Since q_1 is a univalent and convex function in $\mathbb{U}$, and $f'(z) = 1 + \sum_{n=1}^{\infty} n a_n z^{n-1}$, $z \in \mathbb{U}$, using Lemma 5(i) we get $n|a_n| \leqslant |q_1'(0)| = 1$ for $n = 1, 2, 3, \cdots$. □

THEOREM 6. *Let the function* $f \in \mathscr{S}_C^*$ *be of the form* (1)*. Then, the logarithmic coefficients of* f *satisfy the inequalities*

$$|\gamma_1| \leqslant \frac{1}{4}, \quad |\gamma_2| \leqslant \frac{1}{6}, \quad |\gamma_3| \leqslant \frac{1}{8}.$$

Proof. If $f \in \mathscr{S}_C^*$, according to Theorem 1 we have

$$f'(z) \prec q_1(z) = \frac{1}{1-z}.$$

Therefore, by the definition of the subordination, there exists a function $\omega \in \Omega$, with $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$, $z \in \mathbb{U}$, such that

$$f'(z) = q_1(\omega(z)) = 1 + c_1 z + \left(c_2 + c_1^2\right) z^2 + \left(c_3 + 2c_1 c_2 + c_1^3\right) z^3 + \cdots, \; z \in \mathbb{U},$$

and equating the coefficients of the above relation it follows that

$$\begin{cases} 2a_2 = c_1, \\ 3a_3 = c_2 + c_1^2, \\ 4a_4 = c_3 + 2c_1 c_2 + c_1^3. \end{cases} \tag{9}$$

Then, by substituting values of a_n, $n = 1, 2, 3$, from (9) in (3), we obtain

$$\begin{cases} 2\gamma_1 = \dfrac{c_1}{2}, \\[2ex] 2\gamma_2 = \dfrac{8c_2 + 5c_1^2}{24}, \\[2ex] 2\gamma_3 = \dfrac{1}{4}\left(c_3 + \dfrac{4}{3} c_1 c_2 + \dfrac{1}{2} c_1^3\right). \end{cases}$$

From the first of the above relations, by applying Lemma 3 we get $|\gamma_1| \leqslant \dfrac{1}{4}$.

Next, applying Lemma 4 for $t=\frac{5}{8}$ we have

$$|\gamma_2| \leqslant \frac{8\left|c_2+\frac{5}{8}c_1^2\right|}{48} \leqslant \frac{1}{6}.$$

According to [21], using the notations of [4, Lemma 5] we have

$$|c_3+q_1c_1c_2+q_2c_1^3| \leqslant H(q_1,q_2),$$

where $H(q_1,q_2)$ is given by [4, Lemma 5]. Since for the case $(q_1,q_2)=\left(\frac{4}{3},\frac{1}{2}\right)\in D_2$ we have $H\left(\frac{4}{3},\frac{1}{2}\right)=1$, and we obtain the estimate

$$|\gamma_3| = \frac{1}{8}\left|c_3+\frac{4}{3}c_1c_2+\frac{1}{2}c_1^3\right| \leqslant \frac{1}{8}. \quad \square$$

THEOREM 7. *If the function* $f\in\mathscr{S}_C^*$ *has the form* (1)*, then the second Hankel determinant satisfies the inequality*

$$\left|a_2a_4-a_3^2\right| \leqslant \frac{1}{9}.$$

Proof. If $f\in\mathscr{S}_C^*$ from Theorem 1 it follows that

$$\sqrt{f'(z)} \prec \frac{1}{\sqrt{1-z}} = \mathrm{e}^{\frac{1}{2}\log(1-z)} =: \phi(z),$$

where the branch of the logarithm is the main one, that is, $\phi(0)=1$, and

$$\phi(z) = 1+\sum_{n=1}^{\infty}A_nz^n = 1+\frac{1}{2}z+\frac{3}{8}z^2+\frac{5}{16}z^3+\cdots,\ z\in\mathbb{U}.$$

Therefore, using Lemma 6 we obtain our result. $\square$

THEOREM 8. *Let the function* $f\in\mathscr{S}_C^*$ *be of the form* (1)*. Then, the next inequalities hold for the parameter* $\mu\in\mathbb{R}$*:*

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} \frac{1}{3}-\frac{\mu}{4}, & \textit{for } \mu\leqslant 0,\\ \frac{1}{3}, & \textit{for } 0\leqslant\mu\leqslant\frac{8}{3},\\ \frac{\mu}{4}-\frac{1}{3}, & \textit{for } \mu\geqslant\frac{8}{3}. \end{cases}$$

Proof. If $f\in\mathscr{S}_C^*$ has the form (1), from (9) and $\nu=\frac{3\mu}{4}-1$ we get

$$\left|a_3-\mu a_2^2\right| = \frac{1}{3}\left|c_2-\nu c_1^2\right|,$$

and our result follows by using Lemma 4. $\square$

4. Conclusion

In this research we studied the family $\mathscr{S}_C^*$ of holomorphic and normalized functions. Further, we obtained some geometric properties of this family like: subordination properties, radius of starlikeness of order α, bounded rotation result, and distortion and covering theorems. In addition, we have presented some related problems for the coefficients of functions that belong to the mentioned family.

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