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ON THE NUMBER OF REAL ZEROS OF REAL ENTIRE FUNCTIONS WITH A NON–DECREASING SEQUENCE OF THE SECOND QUOTIENTS OF TAYLOR COEFFICIENTS

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Abstract. For an entire function $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $a_k>0$, we define the sequence of the second quotients of Taylor coefficients $Q:=\left(\frac{a_k^2}{a_{k-1}a_{k+1}}\right)_{k=1}^{\infty}$. We find new necessary conditions for a function with a non-decreasing sequence Q to belong to the Laguerre-Pólya class of type I. We also estimate the possible number of non-real zeros for a function with a non-decreasing sequence Q.

1. Introduction

The topic of zero distribution of entire functions has been the subject of study and discussion of mathematicians for many years (see, for example, [19]). In the present paper, we consider a class of entire functions with positive Taylor coefficients and investigate the condition for them to belong to the Laguerre-Pólya class of type I. We give the definitions of the Laguerre-Pólya class and the Laguerre-Pólya class of type I.

DEFINITION 1. A real entire function f is said to be in the *Laguerre-Pólya class*, written $f \in \mathscr{L}-\mathscr{P}$, if it can be expressed in the form

$$f(z)=cz^n e^{-\alpha z^2+\beta z}\prod_{k=1}^{\infty}\left(1-\frac{z}{x_k}\right)e^{zx_k^{-1}}, \tag{1}$$

where $c,\alpha,\beta,x_k\in\mathbb{R}$, $x_k\neq 0$, $\alpha\geqslant 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty}x_k^{-2}<\infty$.

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DEFINITION 2. A real entire function f is said to be in the *Laguerre-Pólya class of type I*, written $f \in \mathscr{L}-\mathscr{P}I$, if it can be expressed in the following form

$$f(z) = cz^n e^{\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right), \tag{2}$$

where $c \in \mathbb{R}, \beta \geqslant 0, x_k > 0$, n is a nonnegative integer, and $\sum_{k=1}^{\infty} x_k^{-1} < \infty$.

As usual, the product on the right-hand sides in both definitions can be finite or empty (in the latter case the product equals 1).

These classes are important for the theory of entire functions since the hyperbolic polynomials (i.e. real polynomials with only real zeros), or hyperbolic polynomials with nonnegative coefficients converge locally uniformly to these and only these functions. The following prominent theorem states even a stronger fact.

THEOREM A. (E. Laguerre and G. Pólya, see, for example, [5, p. 42–46]) and [12, chapter VIII, §3]).

(i) Let $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, be a sequence of real polynomials having only real zeros which converges uniformly on the disc $|z| \leqslant A, A > 0$. Then this sequence converges locally uniformly in $\mathbb{C}$ to an entire function from the $\mathscr{L}-\mathscr{P}$ class.

(ii) For any $f \in \mathscr{L}-\mathscr{P}$ there exists a sequence of real polynomials with only real zeros which converges locally uniformly to f.

(iii) Let $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, be a sequence of real polynomials having only real negative zeros which converges uniformly on the disc $|z| \leqslant A, A > 0$. Then this sequence converges locally uniformly in $\mathbb{C}$ to an entire function from the class $\mathscr{L}-\mathscr{P}I$.

(iv) For any $f \in \mathscr{L}-\mathscr{P}I$ there is a sequence of real polynomials with only real nonpositive zeros which converges locally uniformly to f.

Numerous properties and features of the Laguerre-Pólya class and the Laguerre-Pólya class of type I can be found in the works [20, p. 100], [21] and [18, Kapitel II] (also see the survey [19] on the zero distribution of entire functions, its sections and tails). Note that for a real entire function (not identically zero) of the order less than 2 the property of having only real zeros is equivalent to belonging to the Laguerre-Pólya class. Also, for a real entire function with positive coefficients of the order less than 1 having only real negative zeros is equivalent to belonging to the Laguerre-Pólya class of type I. In particular, the same property is valid for polynomials.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with real nonzero coefficients. We define the quotients p_n and q_n:

$$p_n = p_n(f) := \frac{a_{n-1}}{a_n}, \quad n \geqslant 1,$$

$$q_n = q_n(f) := \frac{p_n}{p_{n-1}} = \frac{a_{n-1}^2}{a_{n-2}a_n}, \quad n \geqslant 2.$$

From these definitions it follows straightforwardly that

$$a_n = \frac{a_0}{p_1 p_2 \cdots p_n}, \quad n \geqslant 1,$$
$$a_n = a_1 \left(\frac{a_1}{a_0}\right)^{n-1} \frac{1}{q_2^{n-1} q_3^{n-2} \cdots q_{n-1}^2 q_n}, \quad n \geqslant 2.$$

It is rather a complicated problem to understand whether a given entire function has only real zeros. However, in 1926, J. I. Hutchinson found quite a simple sufficient condition for an entire function with positive coefficients to have only real zeros.

THEOREM B. (J. I. Hutchinson, [6]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k. Then $q_n(f) \geqslant 4$, for all $n \geqslant 2$, if and only if the following two conditions are fulfilled:*

(i) The zeros of f are all real, simple and negative;

(ii) The zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, $m < n$, formed by taking any number of consecutive terms of f, are all real and non-positive.

For some extensions of Hutchinson's results see, for example, [3, §4].

A special entire function $g_a(z) = \sum_{k=0}^{\infty} a^{-k^2} z^k$, $a > 1$, known as a *partial theta function* (the classical Jacobi theta function is defined by the series $\theta(z) := \sum_{k=-\infty}^{\infty} a^{-k^2} z^k$), was investigated by many mathematicians and has an important role. Note that $q_n(g_a) = a^2$ for all n. The survey [23] by S. O. Warnaar contains the history of investigation of the partial theta function and some of its main properties.

In particular, in the paper [7] it was explained that for every $n \geqslant 2$, there exists a constant $c_n > 1$ such that for each $n \in \mathbb{N}$, $S_n(z, g_a) := \sum_{j=0}^{n} a^{-j^2} z^j \in \mathcal{L} - \mathcal{P}$ if and only if $a^2 \geqslant c_n$. The notation of the constants c_n having this property will be further used.

THEOREM C. (O. Katkova, T. Lobova, A. Vishnyakova, [7]). *There exists a constant q_∞ ($q_\infty \approx 3.23363666$) such that:*

1. $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow a^2 \geqslant q_\infty$;
2. $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow$ *there exists* $z_0 \in (-a^3, -a)$ *such that* $g_a(z_0) \leqslant 0$
3. *if there exists* $z_0 \in (-a^3, -a)$ *such that* $g_a(z_0) < 0$, *then* $a^2 > q_\infty$;
4. *for a given* $n \geqslant 2$ *we have* $S_n(z, g_a) \in \mathcal{L} - \mathcal{P} \Leftrightarrow$ *there exists* $z_n \in (-a^3, -a)$ *such that* $S_n(z_n, g_a) \leqslant 0$;
5. *if there exists* $z_n \in (-a^3, -a)$ *such that* $S_n(z_n, g_a) < 0$, *then* $a^2 > c_n$;
6. $4 = c_2 > c_4 > c_6 > \cdots$ *and* $\lim_{n\to\infty} c_{2n} = q_\infty$;
7. $3 = c_3 < c_5 < c_7 < \cdots$ *and* $\lim_{n\to\infty} c_{2n+1} = q_\infty$.

Calculations show that $c_4 = 1+\sqrt{5} \approx 3.23607,\ c_6 \approx 3.23364$ and $c_5 \approx 3.23362,\ c_7 \approx 3.23364$.

The partial theta function is of interest to many areas such as statistical physics and combinatorics [22], Ramanujan type q-series [24], asymptotic analysis and the theory of (mock) modular forms, etc. There is a series of works by V. P. Kostov dedicated to various properties of zeros of the partial theta function and its derivative (see [9, 10] and the references therein). The paper [11] among the other results explains the role of the constant q_∞ in the study of the set of entire functions with positive coefficients having all Taylor truncations with only real zeros. In [8], the following questions are investigated: whether the Taylor sections of the function $\prod_{k=1}^{\infty}\left(1+\frac{z}{a^k}\right),\ a>1,$ and $\sum_{k=0}^{\infty}\frac{z^k}{k!a^{k^2}},\ a \geqslant 1,$ belong to the Laguerre-Pólya class of type I. In [2] and [1], some important special functions with non-decreasing sequence of the second quotients of Taylor coefficients are studied.

The first author studied a special function related to the partial theta function and the Euler function

$$f_a(z) = \sum_{k=0}^{\infty}\frac{z^k}{(a^k+1)(a^{k-1}+1)\cdots(a+1)}, \quad a>1,$$

which is also known as the q-Kummer function ${}_1\phi_1(q;-q;q,-z)$, where $q=1/a$ (see [4], formula (1.2.22)). Note that its second quotients of Taylor coefficients are

$$q_n(f_a) = \frac{a^n+1}{a^{n-1}+1},$$

which is an increasing sequence in n for $a>1$, with the limit value given by a. In [17], the conditions were found for this function to belong to the Laguerre-Pólya class.

It turns out that for many important entire functions with positive coefficients $f(z)=\sum_{k=0}^{\infty}a_k z^k$ (for example, the partial theta function from [7], functions from [2] and [1], the q-Kummer function ${}_1\phi_1(q;-q;q,-z)$ and others) the following two conditions are equivalent:

1. f belongs to the Laguerre-Pólya class of type I, and
2. There exists $x_0 \in [-\frac{a_1}{a_2}, 0]$ such that $f(x_0) \leqslant 0$.

In our previous work we proved the following necessary condition for a function to belong to the Laguerre-Pólya class.

THEOREM D. (T. H. Nguyen, A. Vishnyakova, [15]). *Let $f(z)=\sum_{k=0}^{\infty}a_k z^k$, $a_k>0$ for all k, be an entire function. Suppose that $q_2(f) \leqslant q_3(f)$. If the function f belongs to the Laguerre-Pólya class, then there exists $x_0 \in [-\frac{a_1}{a_2}, 0]$ such that $f(x_0) \leqslant 0$.*

In [16] we have obtained a criterion for belonging to the Laguerre-Pólya class of type I for real entire functions with the regularly non-decreasing sequence of second quotients of Taylor coefficients in terms of the existence of a point x_0 as in Theorem D. It was previously shown in [14] that if $f(z)=\sum_{k=0}^{\infty}a_k z^k$, $a_k>0$ for all k, is an entire function such that $q_2 \leqslant q_3 \leqslant q_4 \leqslant \cdots$, and the function f belongs to the Laguerre-Pólya class, then $\lim_{n\to\infty} q_n = c \geqslant q_\infty$, where q_∞ is a constant from Theorem C.

In the present paper we prove that the following conditions on the second quotients q_k are necessary for the function to belong to the Laguerre-Pólya I class:

THEOREM 1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $k = 0,1,2,\dots$, *be an entire function such that* $q_2(f) \leqslant q_3(f) \leqslant q_4(f) \leqslant \cdots$. *If* $f \in \mathcal{L}-\mathcal{P}I$, *then for any* $k = 1,2,3,\dots$, *the following inequality holds:* $q_{2n+1} > c_{2k+1}$ *(*c_{2k+1} *defined as in Theorem C).*

COROLLARY 1. *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, $k = 0,1,2,\dots$, *be an entire function such that* $q_2(f) \leqslant q_3(f) \leqslant q_4(f) \leqslant \cdots$. *If* $f \in \mathcal{L}-\mathcal{P}$, *then* $q_2(f) > 3$.

In [16] we obtained the following result.

THEOREM E. (T. H. Nguyen, A. Vishnyakova, [16]). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $k = 0,1,2,\dots$, *be an entire function such that* $2\sqrt[3]{2} \approx 2.51984 \leqslant q_2(f) \leqslant q_3(f) \leqslant q_4(f) \leqslant \cdots$. *Then all but a finite number of zeros of* f *are real and simple.*

Our next theorem estimates the possible number of nonreal zeros for such functions.

THEOREM 2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $k = 0,1,2,\dots$, *be an entire function such that* $2\sqrt[3]{2} \approx 2.51984 \leqslant q_2(f) \leqslant q_3(f) \leqslant q_4(f) \leqslant \cdots$. *If there exist* $j_0 = 2,3,4,\dots$ *and* $m_0 \in \mathbb{N}$, *such that* $q_{j_0} \geqslant c_{2m_0}$, *then the number of nonreal zeros of* f *does not exceed* $j_0 + 2m_0 - 2$ *(*c_{2k} *defined as in Theorem C).*

2. Proof of Theorem 1 and Corollary 1

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider a function $g(x) = a_0^{-1} f(a_0 a_1^{-1} x)$ instead of $f(x)$, due to the fact that such rescaling of f preserves its property of having real zeros as well as the second quotients: $q_n(g) = q_n(f)$ for all $n \in \mathbb{N}$. During the proof instead of $p_n(f)$ and $q_n(f)$ we use notation p_n and q_n. It is more convenient to consider a function

$$\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$$

instead of f.

Theorem D states that if φ belongs to the Laguerre-Pólya class then there exists a point $x_0 \in [0, \frac{a_1}{a_2}] = [0, q_2]$ such that $\varphi(x_0) \leqslant 0$. Let us introduce some more notation. For an entire function φ, by $S_n(x,\varphi)$ and $R_n(x,\varphi)$ we denote the nth partial sum and the nth remainder of the series, i.e.

$$S_n(x,\varphi) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k},$$

and

$$R_n(x,\varphi)=\sum_{k=n}^{\infty}\frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}.$$

First, we need the following Lemma.

LEMMA 1. *Let* $\varphi(x)=1-x+\sum_{k=2}^{\infty}\frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}$ *be an entire function. Suppose that* q_k *are non-decreasing in* $k:$ $1<q_2\leqslant q_3\leqslant q_4\leqslant\cdots.$ *If there exists* $x_0\in[0,q_2]$ *such that* $\varphi(x_0)\leqslant 0,$ *then* $x_0\in(1,q_2].$

Proof. For $x\in[0,1]$ we have:

$$1\geqslant x>\frac{x^2}{q_2}>\frac{x^3}{q_2^2q_3}>\frac{x^4}{q_2^3q_3^2q_4}>\cdots,$$

whence

$$\varphi(x)>0 \quad \text{for all} \quad x\in[0,1]. \quad \square \tag{3}$$

LEMMA 2. *Let* $\varphi(x)=1-x+\sum_{k=2}^{\infty}\frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}$ *be an entire function. Suppose that* q_k *are non-decreasing in* $k:$ $1<q_2\leqslant q_3\leqslant q_4\leqslant\cdots.$ *If there exists* $x_0\in(1,q_2]$ *such that* $\varphi(x_0)\leqslant 0,$ *then for any* $n\in\mathbb{N},$ $S_{2n+1}(x_0)<0.$

Proof. Suppose that $x\in(1,q_2].$ Then we obtain

$$1<x\geqslant\frac{x^2}{q_2}>\frac{x^3}{q_2^2q_3}>\cdots>\frac{x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}>\cdots \tag{4}$$

For an arbitrary $n\in\mathbb{N}$ we have:

$$\varphi(x)=S_{2n+1}(x,\varphi)+R_{2n+2}(x,\varphi).$$

By (4) and the Leibniz criterion for alternating series, we conclude that $R_{2n+2}(x,\varphi)>0$ for all $x\in(1,q_2],$ or

$$\varphi(x)>S_{2n+1}(x,\varphi) \quad \text{for all} \quad x\in(1,q_2],n\in\mathbb{N}. \tag{5}$$

Consequently, if there exists a point $x_0\in(1,q_2]$ such that $\varphi(x_0)\leqslant 0,$ then for any $n\in\mathbb{N}$ we have $S_{2n+1}(x_0)<0.$ $\square$

Thus, we proved that if $\varphi\in\mathscr{L}-\mathscr{P},$ then there exists $x_0\in(1,q_2]$ such that the inequalities $S_{2n+1}(x_0)<0$ hold for any $n\in\mathbb{N}.$

In [14] it was proved that if an entire function $\varphi(x)=1-x+\sum_{k=2}^{\infty}\frac{(-1)^k x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}$ belongs to the Laguerre-Pólya class, where $0<q_2\leqslant q_3\leqslant q_4\leqslant\cdots,$ then $q_2\geqslant 3$ (see[14, Lemma 2.1]). So we assume that $q_2\geqslant 3.$

LEMMA 3. *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ *be an entire function. Suppose that* $3 \leqslant q_2 \leqslant q_3 \leqslant q_4 \cdots$*. Then the inequality* $S_{2n+1}(x,\varphi) \geqslant S_{2n+1}(\sqrt{q_{2n+1}}x, g_{\sqrt{q_{2n+1}}})$ *holds for any* $n \in \mathbb{N}$ *and any* $x \in (1, q_2]$ *(here* g_a *is the partial theta function and* $S_{2n+1}(y, g_a)$ *is its* $(2n+1)$*-th partial sum at the point* y*).*

Proof. We have

$$S_{2n+1}(x,\varphi) = (1-x) + \left(\frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3}\right) + \left(\frac{x^4}{q_2^3 q_3^2 q_4} - \frac{x^5}{q_2^4 q_3^3 q_4^2 q_5}\right) + \cdots + \left(\frac{x^{2n}}{q_2^{2n-1} q_3^{2n-2} \cdots q_{2n-1}^2 q_{2n}} - \frac{x^{2n+1}}{q_2^{2n} q_3^{2n-1} \cdots q_{2n}^2 q_{2n+1}}\right). \tag{6}$$

Under our assumptions, q_k are non-decreasing in k. We prove that for any fixed $k = 1, 2, \ldots, n$ and $x \in (1, q_2]$, the following inequality holds:

$$\begin{aligned}
&\frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdots q_{2k-1}^2 q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdots q_{2k-1}^3 q_{2k}^2 q_{2k+1}} \\
&\geqslant \frac{x^{2k}}{q_{2k+1}^{2k-1} q_{2k+1}^{2k-2} \cdots q_{2k+1}^2 q_{2k+1}} - \frac{x^{2k+1}}{q_{2k+1}^{2k} q_{2k+1}^{2k-1} \cdots q_{2k+1}^2 q_{2k+1}} \\
&= \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}} = \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} \cdot \left(1 - \frac{x}{q_{2k+1}^{2k}}\right).
\end{aligned}$$

For $x \in (1, q_2]$ and any fixed $k = 1, 2, \ldots, n,$ we define the following function:

$$F(q_2, q_3, \ldots, q_{2k}, q_{2k+1}) := \frac{x^{2k}}{q_2^{2k-1} q_3^{2k-2} \cdots q_{2k-1}^2 q_{2k}} - \frac{x^{2k+1}}{q_2^{2k} q_3^{2k-1} \cdots q_{2k-1}^3 q_{2k}^2 q_{2k+1}}.$$

We can observe that

$$\frac{\partial F(q_2, q_3, \ldots, q_{2k}, q_{2k+1})}{\partial q_2} = -\frac{(2k-1) \cdot x^{2k}}{q_2^{2k} q_3^{2k-2} \cdots q_{2k-1}^2 q_{2k}} + \frac{2k \cdot x^{2k+1}}{q_2^{2k+1} q_3^{2k-1} \cdots q_{2k-1}^3 q_{2k}^2 q_{2k+1}} < 0 \Leftrightarrow x < \left(1 - \frac{1}{2k}\right) \cdot q_2 q_3 \ldots q_{2k} q_{2k+1}.$$

Therefore, since $\left(1 - \frac{1}{2k}\right) q_2 q_3 \cdots q_{2k} q_{2k+1} \geqslant \frac{1}{2} q_2 q_3 \cdots q_{2k} q_{2k+1} \geqslant \frac{1}{2} q_2 q_3 > q_2$ (under our assumptions $q_3 \geqslant q_2 \geqslant 3$), we conclude that the function $F(q_2, q_3, \ldots, q_{2k}, q_{2k+1})$ is decreasing in q_2 for each fixed $x \in (1, q_2]$. Since $q_2 \leqslant q_3,$ for $k = 1$ we get:

$$F(q_2, q_3) = \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3} \geqslant \frac{x^2}{q_3} - \frac{x^3}{q_3^2 q_3} = \frac{x^2}{q_3} - \frac{x^3}{q_3^3},$$

and the desired inequality is proved for $k=1$. For $k\geqslant 2$ we have:

$$\begin{aligned}F(q_2,q_3,q_4,\dots,q_{2k},q_{2k+1})&\geqslant F(q_3,q_3,q_4,\dots,q_{2k},q_{2k+1})\\&=\frac{x^{2k}}{q_3^{4k-3}q_4^{2k-3}\cdots q_{2k-1}^2q_{2k}}-\frac{x^{2k+1}}{q_3^{4k-1}q_4^{2k-2}\cdots q_{2k-1}^3q_{2k}^2q_{2k+1}}.\end{aligned}$$

Further, we consider its derivative with respect to q_3:

$$\begin{aligned}\frac{\partial F(q_3,q_3,q_4,\dots,q_{2k},q_{2k+1})}{\partial q_3}&=-\frac{(4k-3)\cdot x^{2k}}{q_3^{4k-2}q_4^{2k-3}\cdots q_{2k-1}^2q_{2k}}\\&+\frac{(4k-1)\cdot x^{2k+1}}{q_3^{4k}q_4^{2k-2}\cdots q_{2k+1}}<0\Leftrightarrow x<\frac{4k-3}{4k-1}q_3^2q_4\dots q_{2k-1}^3q_{2k}^2q_{2k+1}.\end{aligned}$$

Under our assumptions,

$$\frac{4k-3}{4k-1}\cdot q_3^2q_4\dots q_{2k+1}\geqslant\frac{5}{7}\cdot q_3^2q_4q_5>q_2,$$

we obtain that $F(q_3,q_3,q_4,\dots,q_{2k},q_{2k+1})$ is decreasing in q_3 for each fixed $x\in(1,q_2]$ and, since $q_3\leqslant q_4$, we receive:

$$F(q_3,q_3,q_4\dots,q_{2k},q_{2k+1})\geqslant F(q_4,q_4,q_4,q_5,\dots,q_{2k},q_{2k+1}).$$

Thus, for the lth step we have:

$$\begin{aligned}&F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1})\\&=\frac{x^{2k}}{q_{l-1}^{(4k-l+1)(l-2)/2}q_l^{2k-l+1}q_{l+1}^{2k-l}\cdots q_{2k-1}^2q_{2k}}\\&\quad-\frac{x^{2k+1}}{q_{l-1}^{(4k-l+3)(l-2)/2}q_l^{2k-l+2}q_{l+1}^{2k-l+1}\cdots q_{2k-1}^3q_{2k}^2q_{2k+1}}.\end{aligned}$$

We consider its partial derivative with respect to q_{l-1}:

$$\begin{aligned}&\frac{\partial F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1})}{\partial q_{l-1}}\\&=-\frac{\frac{1}{2}(4k-l+1)(l-2)\cdot x^{2k}}{q_{l-1}^{1+(4k-l+1)(l-2)/2}q_l^{2k-l+1}q_{l+1}^{2k-l}\cdots q_{2k-1}^2q_{2k}}\\&\quad+\frac{\frac{1}{2}(4k-l+3)(l-2)\cdot x^{2k+1}}{q_{l-1}^{1+(4k-l+3)(l-2)/2}q_l^{2k-l+2}q_{l+1}^{2k-l+1}\cdots q_{2k-1}^3q_{2k}^2q_{2k+1}}<0,\end{aligned}$$

which is equivalent to the inequality:

$$x<\frac{4k-l+1}{4k-l+3}\cdot q_{l-1}^{l-2}q_lq_{l+1}\cdots q_{2k-1}q_{2k}q_{2k+1}.$$

The inequality above is valid, since

$$\frac{4k-l+1}{4k-l+3}\cdot q_{l-1}^{l-2}q_lq_{l+1}\cdots q_{2k-1}q_{2k}q_{2k+1} \geqslant \frac{9-l}{11-l}\cdot q_{l-1}^{l-2}q_lq_{l+1}\cdots q_{2k-1}q_{2k}q_{2k+1} > q_2.$$

Hence, the function $F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1})$ is decreasing in q_{l-1}. Since, under our assumptions, $q_{l-1}\leqslant q_l$, we obtain:

$$F(q_{l-1},q_{l-1},\dots,q_{l-1},q_l,q_{l+1},\dots,q_{2k},q_{2k+1}) \geqslant F(q_l,q_l,\dots,q_l,q_{l+1},\dots,q_{2k},q_{2k+1}).$$

Analogously, by the same computation, at the $(2k+1)$-th step we get:

$$F(q_{2k},q_{2k}\dots,q_{2k},q_{2k+1}) = \frac{x^{2k}}{q_{2k}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k}^{(k+1)(2k-1)}\cdot q_{2k+1}}.$$

Its derivative with respect to q_{2k} is:

$$\frac{\partial F(q_{2k},q_{2k}\dots,q_{2k},q_{2k+1})}{\partial q_{2k}} = -\frac{k(2k-1)\cdot x^{2k}}{q_{2k}^{2k^2-k+1}} + \frac{(2k^2+k-1)\cdot x^{2k+1}}{q_{2k}^{2k^2+k}q_{2k+1}} < 0 \Leftrightarrow x < \frac{2k^2-k}{2k^2+k-1}\cdot q_{2k}^{2k-1}q_{2k+1}.$$

Since we assume that

$$\frac{2k^2-k}{2k^2+k-1}\cdot q_{2k}^{2k-1}q_{2k+1} \geqslant \frac{2}{3}\cdot q_{2k}^{2k-1}q_{2k+1} > q_2,$$

we conclude that the function $F(q_{2k},q_{2k}\dots,q_{2k},q_{2k+1})$ is decreasing in q_{2k}. While $q_{2k}\leqslant q_{2k+1}$, we get:

$$F(q_{2k},q_{2k}\dots,q_{2k},q_{2k+1}) \geqslant F(q_{2k+1},q_{2k+1},\dots,q_{2k+1},q_{2k+1}).$$

Thus, we obtain the following chain of inequalities:

$$\begin{aligned} F(q_2,q_3,q_4,\dots,q_{2k},q_{2k+1}) &\geqslant F(q_3,q_3,q_4,\dots,q_{2k},q_{2k+1}) \\ &\geqslant F(q_4,q_4,q_4,q_5,\dots,q_{2k},q_{2k+1}) \geqslant \cdots \geqslant F(q_{2k},q_{2k},\dots,q_{2k},q_{2k+1}) \\ &\geqslant F(q_{2k+1},q_{2k+1},\dots,q_{2k+1},q_{2k+1}). \end{aligned}$$

Consequently,

$$F(q_2,q_3,q_4,\dots,q_{2k},q_{2k+1}) \geqslant F(q_{2k+1},q_{2k+1},\dots,q_{2k+1},q_{2k+1}) = \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}}.$$

Finally, we note that under our assumptions, the expression $\frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}}$ is decreasing in q_{2k+1} for each fixed $x \in (1, q_2]$, so we obtain

$$F(q_2, q_3, q_4, \dots, q_{2k}, q_{2k+1}) \geqslant \frac{x^{2k}}{q_{2k+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2k+1}^{k(2k+1)}} \geqslant \frac{x^{2k}}{q_{2n+1}^{k(2k-1)}} - \frac{x^{2k+1}}{q_{2n+1}^{k(2k+1)}}.$$

Substituting the last inequality in (6) for every $x \in (1, q_2]$ and $k = 1, 2, \dots, n,$ we get:

$$\begin{aligned} S_{2n+1}(x,\varphi) &\geqslant (1-x) + \left(\frac{x^2}{q_{2n+1}} - \frac{x^3}{q_{2n+1}^3}\right) + \left(\frac{x^4}{q_{2n+1}^6} - \frac{x^5}{q_{2n+1}^{10}}\right) + \\ &\dots + \left(\frac{x^{2n}}{q_{2n+1}^{n(2n-1)}} - \frac{x^{2n+1}}{q_{2n+1}^{n(2n+1)}}\right) = \sum_{k=0}^{2n+1} \frac{(-1)^k x^k}{\sqrt{q_{2n+1}}^{k(k-1)}} \\ &= S_{2n+1}(-\sqrt{q_{2n+1}}x, g_{\sqrt{q_{2n+1}}}), \end{aligned} \tag{7}$$

where g_a is the partial theta function and $S_{2n+1}(y, g_a)$ is its $(2n+1)$-th partial sum at the point y. $\square$

Since we have $S_{2n+1}(x,\varphi) \geqslant S_{2n+1}(-\sqrt{q_{2n+1}}x, g_{\sqrt{q_{2n+1}}})$ for any $n \in \mathbb{N}$, if there exists a point $x_0 \in (1, q_2]$ such that $S_{2n+1}(x_0, \varphi) \leqslant 0$, then $S_{2n+1}(-\sqrt{q_{2n+1}}x_0, g_{\sqrt{q_{2n+1}}}) < 0$. Therefore for $y_0 = \sqrt{q_{2n+1}}x_0$, we have $\sqrt{q_{2n+1}} \leqslant y_0 \leqslant \sqrt{q_{2n+1}}q_2 \leqslant (\sqrt{q_{2n+1}})^3$. Using the statement (5) of Theorem C, we obtain that $q_{2n+1} > c_{2n+1}$, which completes the proof of Theorem 1.

Proof of Corollary 1. As we have proved in the previous theorem, if $f \in \mathcal{L} - \mathcal{P}$, then $q_3(f) > 3$. In [15] it is proved that, under the assumptions of the Corollary, if $q_2(f) < 4$, then

$$q_3(f) \leqslant \frac{-q_2(f)(2q_2(f) - 9) + 2(q_2(f) - 3)\sqrt{q_2(f)(q_2(f) - 3)}}{q_2(f)(4 - q_2(f))}$$

(see [15, Theorem 1.4]). We have mentioned that if $f \in \mathcal{L} - \mathcal{P}$, then $q_2(f) \geqslant 3$. If $q_2(f) = 3$, then the inequality above states $q_3(f) \leqslant 3$. This contradiction proves the Corollary 1. $\square$

3. Proof of Theorem 2

As in the proof of Theorem 1 we assume that $a_0 = a_1 = 1$, and we consider the function $\varphi(x) = f(-x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ instead of f. We need the following lemma.

LEMMA 4. *Let* $\varphi(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ *be an entire function. Suppose that* $1 < q_2 \leqslant q_3 \leqslant q_4 \leqslant \cdots$. *If there exist* $j_0 = 3, 4, \dots$ *and* $m_0 \in \mathbb{N}$, *such that*

$q_{j_0} \geqslant c_{2m_0}$, *then for all* $j \geqslant j_0 + 2m_0 - 3$, *there exists* $x_j \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$ *such that the following inequality holds:*

$$(-1)^j \varphi(x_j) \geqslant 0.$$

The proof of this lemma is similar to the one of [13, Lemma 2.1].

Proof. Choose an arbitrary $j \geqslant j_0+2m_0-3$ and fix this j. For every $x \in (q_2q_3\cdots q_j,$ $q_2q_3\cdots q_jq_{j+1})$ we have

$$1 < x < \frac{x^2}{q_2} < \frac{x^3}{q_2^2q_3} < \cdots < \frac{x^j}{q_2^{j-1}q_3^{j-2}\cdots q_{j-1}^2q_j},$$

and

$$\frac{x^j}{q_2^{j-1}q_3^{j-2}\cdots q_{j-1}^2q_j} > \frac{x^{j+1}}{q_2^jq_3^{j-1}\cdots q_{j-1}^3q_j^2q_{j+1}} > \frac{x^{j+2}}{q_2^{j+1}q_3^j\cdots q_{j-1}^4q_j^3q_{j+1}^2q_{j+2}} > \cdots.$$

We observe that

$$(-1)^j\varphi(x) = \sum_{k=0}^{j-2m_0} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2q_k} + \sum_{k=j-2m_0+1}^{j+1} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2q_k} + \sum_{k=j+2}^{\infty} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2q_k} =: \Sigma_1(x) + h(x) + \Sigma_2(x).$$

Summands in $\Sigma_1(x)$ are increasing in modulus and the sign of the last (biggest) summand is positive. So, for all $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, we have $\Sigma_1(x) > 0$. Summands in $\Sigma_2(x)$ are decreasing in modulus and the sign of the first (biggest) summand is positive. Consequently, for all $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, we get $\Sigma_2(x) > 0$. Thus, we obtain

$$(-1)^j\varphi(x) > h(x) = \sum_{k=j-2m_0+1}^{j+1} \frac{(-1)^{k+j}x^k}{q_2^{k-1}q_3^{k-2}\ldots q_{k-1}^2q_k} = -\frac{x^{j+1}}{q_2^jq_3^{j-1}\ldots q_j^2q_{j+1}} + \frac{x^j}{q_2^{j-1}q_3^{j-2}\ldots q_{j-1}^2q_j} - \frac{x^{j-1}}{q_2^{j-2}q_3^{j-3}\ldots q_{j-2}^2q_{j-1}} + \ldots + \frac{x^{j-2m_0+2}}{q_2^{j-2m_0+1}q_3^{j-2m_0}\ldots q_{j-2m_0+1}^2q_{j-2m_0+2}} - \frac{x^{j-2m_0+1}}{q_2^{j-2m_0}q_3^{j-2m_0-1}\ldots q_{j-2m_0}^2q_{j-2m_0+1}} \tag{8}$$

(we rewrite the sum from the end to the beginning). After factoring out the term $\frac{x^{j+1}}{q_2^j q_3^{j-1}\dots q_j^2 q_{j+1}}$, we get

$$
\begin{aligned}
(-1)^j\varphi(x) > h(x) = {} & \frac{x^{j+1}}{q_2^j q_3^{j-1}\cdots q_j^2 q_{j+1}} \cdot \Bigg(-1 + \frac{q_2 q_3 \cdots q_j q_{j+1}}{x} \\
& - \frac{(q_2 q_3 \cdots q_j q_{j+1})^2}{x^2 q_{j+1}} + \frac{(q_2 q_3 \cdots q_j q_{j+1})^3}{x^3 q_{j+1}^2 q_j} - \dots \\
& + \frac{(q_2 q_3 \cdots q_j q_{j+1})^{2m_0-1}}{x^{2m_0-1} q_{j+1}^{2m_0-2} q_j^{2m_0-3} \cdots q_{j-2m_0+5}^2 q_{j-2m_0+4}} \\
& - \frac{(q_2 q_3 \cdots q_j q_{j+1})^{2m_0}}{x^{2m_0} q_{j+1}^{2m_0-1} q_j^{2m_0-2} \cdots q_{j-2m_0+5}^3 q_{j-2m_0+4}^2 q_{j-2m_0+3}} \Bigg) \\
=: {} & \frac{x^{j+1}}{q_2^j q_3^{j-1}\cdots q_j^2 q_{j+1}} \cdot \psi(x).
\end{aligned} \tag{9}
$$

Now we introduce some more notation. Set $y := \frac{q_2 q_3 \dots q_j q_{j+1}}{x}$, and observe that $x \in (q_2 q_3 \cdots q_j, q_2 q_3 \cdots q_j q_{j+1}) \Leftrightarrow y \in (1, q_{j+1})$. Further we change the numeration of the second quotients:

$$
s_2 := q_{j+1},\ s_3 := q_j,\ s_4 := q_{j-1},\ \dots, s_{2m_0-1} := q_{j-2m_0+4},\ s_{2m_0} := q_{j-2m_0+3}.
$$

By our assumptions, $q_2 \leqslant q_3 \leqslant q_4 \leqslant \cdots$, thus, we get $s_2 \geqslant s_3 \geqslant s_4 \geqslant \cdots \geqslant s_{2m_0} > 1$, and $y \in (1, s_2)$. In new notation we have

$$
\psi(y) = -1 + y - \sum_{k=2}^{2m_0} \frac{(-1)^k y^k}{s_2^{k-1} s_3^{k-2} \cdots s_{k-1}^2 s_k}. \tag{10}
$$

We want to prove that there exists a point $y_j \in (1, q_{j+1}) = (1, s_2)$ such that $h(y_j) \geqslant 0$. To do this we compare the expression in brackets with the corresponding partial sum of the partial theta function. We have

$$
\begin{aligned}
\psi(y) = {} & (-1 + y) \\
& + \left(-\frac{y^2}{s_2} + \frac{y^3}{s_2^2 s_3} \right) + \left(-\frac{y^4}{s_2^3 s_3^2 s_4} + \frac{y^5}{s_2^4 s_3^3 s_4^2 s_5} \right) \\
& + \cdots + \left(-\frac{y^{2m_0-2}}{s_2^{2m_0-3} s_3^{2m_0-4} \cdots s_{2m_0-3}^2 s_{2m_0-2}} + \frac{y^{2m_0-1}}{s_2^{2m_0-2} s_3^{2m_0-3} \cdots s_{2m_0-2}^2 s_{2m_0-1}} \right) \\
& - \frac{y^{2m_0}}{s_2^{2m_0-1} s_3^{2m_0-2} \cdots s_{2m_0-2}^3 s_{2m_0-1}^2 s_{2m_0}}.
\end{aligned} \tag{11}
$$

We provide estimations similar to those in the proof of Lemma 3. Firstly, under our assumptions, one can see that

$$-\frac{y^{2m_0}}{s_2^{2m_0-1}s_3^{2m_0-2}\cdots s_{2m_0-1}^2 s_{2m_0}} \geqslant -\frac{y^{2m_0}}{s_{2m_0}^{2m_0-1}s_{2m_0}^{2m_0-2}\cdots s_{2m_0}^2 s_{2m_0}} = -\frac{y^{2m_0}}{s_{2m_0}^{m_0(2m_0-1)}}. \tag{12}$$

We prove that for any fixed $k=1,2,\dots,m_0-1,$ the following inequality holds:

$$\begin{aligned} &-\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k}} + \frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k}^2 s_{2k+1}} \\ &\geqslant -\frac{y^{2k}}{s_{2m_0}^{2k-1}s_{2m_0}^{2k-2}\cdots s_{2m_0}} + \frac{y^{2k+1}}{s_{2m_0}^{2k}s_{2m_0}^{2k-1}\cdots s_{2m_0}^2 s_{2m_0}} \\ &= -\frac{y^{2k}}{s_{2m_0}^{k(2k-1)}} + \frac{y^{2k+1}}{s_{2m_0}^{k(2k+1)}}. \end{aligned} \tag{13}$$

Firstly, we consider (13) for $k=1.$ Since $s_2 \geqslant s_3$, we have

$$-\frac{y^2}{s_2} + \frac{y^3}{s_2^2 s_3} \geqslant -\frac{y^2}{s_2} + \frac{y^3}{s_2^3}.$$

We observe that

$$\frac{\partial}{\partial s_2}\left(-\frac{y^2}{s_2} + \frac{y^3}{s_2^3}\right) = \frac{y^2}{s_2^2} - \frac{3y^3}{s_2^4} > 0 \Leftrightarrow y < \frac{s_2^2}{3}.$$

The inequality above is valid since $y < q_{j+1} = s_2,$ and we suppose that if there exist $j_0 = 2,3,4,\dots$ and $m_0 \in \mathbb{N},$ such that $q_{j_0} \geqslant c_{2m_0},$ we fix an arbitrary $j \geqslant j_0 + 2m_0 - 3$ and get $s_2 \geqslant s_{2m_0} = q_{j-2m_0+3} \geqslant q_{j_0} \geqslant c_{2m_0} > 3.$ Therefore, the function $\left(-\frac{y^2}{s_2} + \frac{y^3}{s_2^3}\right)$ is increasing in $s_2,$ whence

$$-\frac{y^2}{s_2} + \frac{y^3}{s_2^2 s_3} \geqslant -\frac{y^2}{s_2} + \frac{y^3}{s_2^3} \geqslant -\frac{y^2}{s_{2m_0}} + \frac{y^3}{s_{2m_0}^3}. \tag{14}$$

We apply analogous reasoning to prove (13) for every $k=1,2,\dots,m_0-1$. Let us define the following function:

$$H(s_2,s_3,\dots,s_{2k},s_{2k+1}) := -\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k-1}^2 s_{2k}} + \frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k-1}^3 s_{2k}^2 s_{2k+1}}$$

for $s_2 \geqslant s_3 \geqslant \cdots \geqslant s_{2k+1}$. Obviously,

$$\begin{aligned} &H(s_2,s_3,\dots,s_{2k},s_{2k+1}) \geqslant H(s_2,s_3,\dots,s_{2k},s_{2k}) \\ &\qquad = -\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k-1}^2 s_{2k}} + \frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k-1}^3 s_{2k}^3}. \end{aligned}$$

We have

$$\frac{\partial H(s_2,s_3,\dots,s_{2k},s_{2k})}{\partial s_{2k}}=\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k-1}^2 s_{2k}^2}-\frac{3y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k-1}^3 s_{2k}^4}.$$

Thus,

$$\frac{\partial H(s_2,s_3,\dots,s_{2k},s_{2k})}{\partial s_{2k}}>0 \Leftrightarrow y<\frac{s_2s_3\cdots s_{2k-1}s_{2k}^2}{3}.$$

Since $y\in(1,s_2)\Leftrightarrow y<s_2,$ we obtain that the function $H(s_2,s_3,\dots,s_{2k},s_{2k})$ is increasing in $s_{2k},$ whence

$$\begin{aligned}
&H(s_2,s_3,\dots,s_{2k-1},s_{2k},s_{2k+1})\geqslant H(s_2,s_3,\dots,s_{2k-1},s_{2k},s_{2k})\\
&\geqslant H(s_2,s_3,\dots,s_{2k-1},s_{2m_0},s_{2m_0})=-\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k-1}^2 s_{2m_0}}\\
&\quad+\frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k-1}^3 s_{2m_0}^3}.
\end{aligned}$$

Now we consider the derivative of the latter function:

$$\frac{\partial H(s_2,s_3,\dots,s_{2k-1},s_{2m_0},s_{2m_0})}{\partial s_{2k-1}}\ \frac{2y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k-1}^3 s_{2m_0}}-\frac{3y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k-1}^4 s_{2m_0}^3}.$$

Hence,

$$\frac{\partial H(s_2,s_3,\dots,s_{2k-1},s_{2m_0},s_{2m_0})}{\partial s_{2k-1}}>0 \Leftrightarrow y<\frac{2s_2s_3\cdots s_{2k-1}s_{2k-1}s_{2m_0}^2}{3}.$$

The inequality above is valid since $y<s_2$ and $s_{2m_0}>3,$ therefore, we obtain that the function $H(s_2,s_3,\dots,s_{2k-1},s_{2m_0},s_{2m_0})$ is increasing in $s_{2k-1},$ whence

$$\begin{aligned}
H(s_2,s_3,\dots,s_{2k-2},s_{2k-1},s_{2m_0},s_{2m_0})&\geqslant H(s_2,s_3,\dots,s_{2k-2},s_{2m_0},s_{2m_0},s_{2m_0})\\
&=-\frac{y^{2k}}{s_2^{2k-1}s_3^{2k-2}\cdots s_{2k-2}^3 s_{2m_0}^3}+\frac{y^{2k+1}}{s_2^{2k}s_3^{2k-1}\cdots s_{2k-2}^4 s_{2m_0}^6}.
\end{aligned}$$

Applying similar arguments we get the following chain of inequalities.

$$H(s_2,s_3,\dots,s_{2k},s_{2k+1})\geqslant H(s_2,s_3,\dots,s_{2k-1},s_{2m_0},s_{2m_0})\geqslant$$

$$H(s_2,s_3,\dots,s_{2k-2},s_{2m_0},s_{2m_0},s_{2m_0})\geqslant\dots\geqslant H(s_{2m_0},s_{2m_0},\dots,s_{2m_0},s_{2m_0}).$$

Thus, we have proved (13).

We substitute the inequality (12) and (13) into (11) to get the following

$$\psi(y)\geqslant-\sum_{k=0}^{2m_0}\frac{(-1)^k y^k}{s_{2m_0}^{\frac{k(k-1)}{2}}}=-S_{2m_0}(-\sqrt{s_{2m_0}}y,g_{\sqrt{s_{2m_0}}}), \tag{15}$$

where g_a is a partial theta function and $S_n(x,g_a) := \sum_{j=0}^{n} x^j a^{-j^2}$ is its partial sum. By our assumption $(\sqrt{s_{2m_0}})^2 = s_{2m_0} = q_{j-2m_0+3}$ and $j \geqslant j_0 + 2m_0 - 3$, so $s_{2m_0} = q_{j-2m_0+3} \geqslant q_{j_0} \geqslant c_{2m_0}$, and we conclude that $S_{2m_0}(x, g_{s_{2m_0}}) \in \mathscr{L} - \mathscr{P}$ (see Theorem C). Whence, by part (4) of Theorem C, there exists $x_0 \in (-(\sqrt{s_{2m_0}})^3, -\sqrt{s_{2m_0}})$ such that $S_{2m_0}(x_0, g_{s_{2m_0}}) \leqslant 0$. We put $-\sqrt{s_{2m_0}}y_0 := x_0$, i.e. $y_0 := -\frac{x_0}{\sqrt{s_{2m_0}}} \in (1, s_{2m_0}) \subset (1, s_2)$, and we have

$$S_{2m_0}(-\sqrt{s_{2m_0}}y_0, g_{\sqrt{s_{2m_0}}}) \leqslant 0.$$

Substituting the last inequality in (15) we obtain:

$$\psi(y_0) \geqslant -S_{2m_0}(-\sqrt{s_{2m_0}}y_0, g_{\sqrt{s_{2m_0}}}) \geqslant 0. \tag{16}$$

Using (16) and substituting (15) into (9), we get:

$$(-1)^j \psi(x) > h(x) = \frac{x^{j+1}}{q_2^j q_3^{j-1} \cdots q_j^2 q_{j+1}} \cdot \psi(y_0) \geqslant 0,$$

which is the desired inequality. It remains to recall that $x_j := \frac{q_2 q_3 \ldots q_j q_{j+1}}{y_0}$, and, since $y_0 \in (1, s_2) = (1, q_{j+1})$, we have $x_j \in (q_2 q_3 \ldots q_j, q_2 q_3 \ldots q_j q_{j+1})$. $\square$

Now we apply the following lemma.

LEMMA 5. ([16, Lemma 2.1]). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $k = 0, 1, 2, \ldots$, *be an entire function such that* $2\sqrt[3]{2} \leqslant q_2(f) \leqslant q_3(f) \leqslant q_4(f) \leqslant \cdots$. *For an arbitrary integer* $k \geqslant 2$ *we define*

$$\rho_k(f) := q_2(f) q_3(f) \cdots q_k(f) \sqrt{q_{k+1}(f)}.$$

Then, for all sufficiently large k, *the function* f *has exactly* k *zeros on the disk* $\{z : |z| < \rho_k(f)\}$ *counting multiplicities.*

Let us choose an arbitrary $k \geqslant 2$, being large enough to get the statement of the previous lemma, and $k \geqslant j_0 + 2m_0 - 2$. Then the number of zeros of φ (counting multiplicities) in the disk $\{z : |z| < q_2 q_3 \cdots q_k \sqrt{q_{k+1}}\}$ is equal to k. By Lemma 4 we have

$$\begin{aligned} &\operatorname{sgn}\varphi(x_{j_0+2m_0-3}) = -\operatorname{sgn}\varphi(x_{j_0+2m_0-2});\ \operatorname{sgn}\varphi(x_{j_0+2m_0-2}) \\ &= -\operatorname{sgn}\varphi(x_{j_0+2m_0-1}); \ldots;\ \operatorname{sgn}\varphi(x_{k-2}) = -\operatorname{sgn}\varphi(x_{k-1}), \end{aligned}$$

and

$$0 < x_{j_0+2m_0-3} < x_{j_0+2m_0-2} < \cdots < x_{k-1} < q_2 q_3 \cdots q_k < q_2 q_3 \cdots q_k \sqrt{q_{k+1}}.$$

Hence, the function φ has $k - j_0 - 2m_0 + 3$ sign changes in the interval $(0, q_2 q_3 \cdots q_k \sqrt{q_{k+1}})$, whence the number of real zeros of φ in the disk $\{z : |z| < q_2 q_3 \cdots q_k \sqrt{q_{k+1}}\}$ is at least $k - j_0 - 2m_0 + 2$. Therefore, the number of nonreal zeros of φ in this disk is less than or equal to $j_0 + 2m_0 - 2$. Since k is an arbitrary large enough integer, we get that φ has not more than $j_0 + 2m_0 - 2$ nonreal zeros.

REFERENCES

[1] A. BOHDANOV, *Determining bounds on the values of parameters for a function* $\varphi_a(z,m) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}(k!)^m,\ m \in (0,1),$ *to Belong to the Laguerre-Pólya Class*, Comput. Methods Funct. Theory, **18** (2018), 35–51, doi:10.1007/s40315-017-0210-6.

[2] A. BOHDANOV AND A. VISHNYAKOVA, *On the conditions for entire functions related to the partial theta-function to belong to the Laguerre-Pólya class*, J. Math. Anal. Appl., **434**, 2 (2016), 1740–1752, doi:10.1016/j.jmaa.2015.09.084.

[3] T. CRAVEN AND G. CSORDAS, *Complex zero decreasing sequences*, Methods Appl. Anal., **2** (1995), 420–44.

[4] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Series*, Encyclopedia of mathematics and its applications, Cambridge University Press, United Kingdom, Cambridge, 2004.

[5] I. I. HIRSCHMAN AND D. V. WIDDER, *The Convolution Transform*, Princeton University Press, Princeton, New Jersey, 1955.

[6] J. I. HUTCHINSON, *On a remarkable class of entire functions*, Trans. Amer. Math. Soc., **25** (1923), 325–332.

[7] O. KATKOVA, T. LOBOVA AND A. VISHNYAKOVA, *On power series having sections with only real zeros*, Comput. Methods Funct. Theory, **3**, 2 (2003), 425–441.

[8] O. KATKOVA, T. LOBOVA AND A. VISHNYAKOVA, *On entire functions having Taylor sections with only real zeros*, J. Math. Phys., Anal., Geom., **11**, 4 (2004), 449–469.

[9] V. P. KOSTOV, *On a partial theta function and its spectrum*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **146**, 3 (2016), 609–623.

[10] V. P. KOSTOV, *The closest to 0 spectral number of the partial theta function*, C. R. Acad. Bulgare Sci., **69** (2016), 1105–1112.

[11] V. P. KOSTOV, B. SHAPIRO, *Hardy-Petrovitch-Hutchinson's problem and partial theta function*, Duke Math. J., **162**, 5 (2013), 825–861.

[12] B. JA. LEVIN, *Distribution of Zeros of Entire Functions*, Transl. Math. Mono., **5**, Amer. Math. Soc., Providence, RI, 1964; revised ed. 1980.

[13] T. H. NGUYEN AND A. VISHNYAKOVA, *On the entire functions from the Laguerre-Pólya class having the decreasing second quotients of Taylor coefficients*, Journal of Mathematical Analysis and Applications, **465**, 1 (2018), 348–359, `https://doi.org/10.1016/j.jmaa.2018.05.018`.

[14] T. H. NGUYEN AND A. VISHNYAKOVA, *On the necessary condition for entire function with the increasing second quotients of Taylor coefficients to belong to the Laguerre-Pólya class*, Journal of Mathematical Analysis and Applications, **480**, 2 (2019), `https://doi.org/10.1016/j.jmaa.2019.123433`.

[15] T. H. NGUYEN AND A. VISHNYAKOVA, *On the closest to zero roots and the second quotients of Taylor coefficients of entire functions from the Laguerre-Pólya I class*, Results in Mathematics, **75**, 115 (2020), `https://doi.org/10.1007/s00025-020-01245-w`.

[16] T. H. NGUYEN AND A. VISHNYAKOVA, *On the entire functions from the Laguerre-Pólya I class having the increasing second quotients of Taylor coefficients*, Journal of Mathematical Analysis and Applications, **498**, 1 (2021), `https://doi.org/10.1016/j.jmaa.2021.124955`.

[17] T. H. NGUYEN, *On the conditions for a special entire function related to the partial theta-function and the Euler function to belong to the Laguerre-Pólya class*, Computational Methods and Function Theory (2021), `https://doi.org/10.1007/s40315-021-00361-0`.

[18] N. OBRESCHKOV, *Verteilung und Berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[19] I. V. OSTROVSKII, *On zero distribution of sections and tails of power series*, Israel Math. Conference Proceedings, **15** (2001), 297–310.

[20] G. PÓLYA, *Collected Papers, Vol. II Location of Zeros*, (R. P. Boas ed.) MIT Press, Cambridge, MA, 1974.

[21] G. PÓLYA AND J. SCHUR, *Uber zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math., **144** (1914), 89–113.

[22] A. D. SOKAL, *The leading root of the partial theta function*, Advances in Mathematics, **229**, 5 (2012), 2063–2621.

[23] S. O. WARNAAR, *Partial theta functions*,
`https://www.researchgate.net publication327791878_Partial_theta_functions`.
[24] S. O. WARNAAR, *Partial Theta Functions. I. Beyond the Lost Notebook*, Proc. London Math. Soc., **87**, 3 (2003), 363–395.

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