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ON STEVIĆ–SHARMA OPERATOR FROM WEIGHTED BERGMAN–ORLICZ SPACES TO BLOCH–TYPE SPACES

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Abstract. In this paper, we are devoted to investigating the metrical boundedness and metrical compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from the weighted Bergman-Orlicz space $\mathscr{A}_\alpha^{\Phi_p}$ to Bloch-type space $\mathscr{B}^\mu$ and little Bloch-type space $\mathscr{B}_0^\mu$.

1. Introduction

Denote by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, and $\mathbb{N}$ the set of all positive integers.

Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$, then φ and ψ induce a composition operator $C_\varphi f = f \circ \varphi$ and a multiplication operator $M_\psi f = \psi \cdot f$, respectively, where $f \in H(\mathbb{D})$. The product $W_{\psi,\varphi} := M_\psi C_\varphi$ of these two operators is known as the weighted composition operator, i.e.,

$$(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)), \quad f \in H(\mathbb{D}),$$

which has been extensively studied. For more research about the (weighted) composition operators acting on several spaces of analytic functions, we refer to [3].

The differentiation operator D, which is defined by $(Df)(z) = f'(z)$, $f \in H(\mathbb{D})$, plays an important role in dynamical system and operator theory. Note that the product DM_u is a special case of the first-order differential operator

$$(T_{\psi_1,\psi_2} f)(z) = \psi_1(z) f(z) + \psi_2(z) f'(z), \quad f \in H(\mathbb{D}),$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$. Products of composition and differentiation operators have attracted some attention in the last fifteen years (see, e.g., [11, 14, 15, 19, 26] and the

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references therein). Moreover, six kinds of product-type operators can be defined as follows (see [22]):

$$
\begin{aligned}
(M_\psi C_\varphi D f)(z) &= \psi(z) f'(\varphi(z)),\\
(M_\psi D C_\varphi f)(z) &= \psi(z)\varphi'(z) f'(\varphi(z)),\\
(C_\varphi M_\psi D f)(z) &= \psi(\varphi(z)) f'(\varphi(z)),\\
(D M_\psi C_\varphi f)(z) &= \psi'(z) f(\varphi(z)) + \psi(z)\varphi'(z) f'(\varphi(z)),\\
(C_\varphi D M_\psi f)(z) &= \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),\\
(D C_\varphi M_\psi f)(z) &= \psi'(\varphi(z))\varphi'(z) f(\varphi(z)) + \psi(\varphi(z))\varphi'(z) f'(\varphi(z))
\end{aligned}
$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. During recent years, there has been a great interest in studying these operators between various analytic function spaces (see, for example, [1, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 20, 22, 26, 29, 31, 32, 33, 34, 35, 36, 38, 39] and also related references therein). In order to treat these operators above in a unified manner, Stević et al. [32, 33] introduced the following so-called Stević-Sharma operator:

$$
(T_{\psi_1,\psi_2,\varphi} f)(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}),
$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$.

It is clear that $T_{\psi_1,\psi_2} = T_{\psi_1,\psi_2,id}$, where id denotes the identity map. Furthermore, we can also easily obtain the six product-type operators by taking some specific choices of the involving symbols:

$$
\begin{aligned}
&M_\psi C_\varphi D = T_{0,\psi,\varphi}, \quad M_\psi D C_\varphi = T_{0,\psi\varphi',\varphi}, \quad C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi},\\
&D M_\psi C_\varphi = T_{\psi',\psi\varphi',\varphi}, \quad C_\varphi D M_\psi = T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi}, \quad D C_\varphi M_\psi = T_{\varphi'\psi'\circ\varphi,\varphi'\psi\circ\varphi,\varphi}.
\end{aligned}
$$

Recently, the study of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ has aroused the interest of experts. For instance, Stević et al. in [33] characterized the boundedness of $T_{\psi_1,\psi_2,\varphi}$ on weighted Bergman spaces $\mathscr{A}^p_\alpha$, where the conditions for boundedness were stated in terms of various suprema and pull-back measures, while the upper and lower bounds for the essential norm of $T_{\psi_1,\psi_2,\varphi}$ on $\mathscr{A}^p_\alpha$ under some assumptions were obtained in [32]. Zhang and Liu in [39] investigated the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from Hardy space to Zygmund-type space. In [5] Guo and Shu extended their results for the case of Stević weighted space, which was introduced by Stević in [25] (see also [29]). Guo et al. in [4] investigated the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ from the mixed-norm space to (little) Zygmund-type space. Wang et al. in [36] considered the differences of two Stević-Sharma operators and characterized its boundedness, compactness and order boundedness between Banach spaces of analytic functions. The boundedness and compactness of weighted composition operators and a class of integral-type operators introduced by Stević from Hardy-Orlicz and weighted Bergman-Orlicz spaces to a class of weighted-type spaces on the unit ball of $\mathbb{C}^n$ were characterized by Sehba and Stević in [20]. Besides, they also gave some more information on the growth functions appearing in the definition of Bergman-Orlicz spaces in [21]. Soon after that, Jiang in [6, 7] provided necessary and sufficient conditions for some special product-type operators acting on weighted Bergman-Orlicz spaces to be bounded or compact. Quite recently, Stević and Jiang in [30] characterized the metrical boundedness and metrical compactness of the weighted iterated radial composition operator from the weighted Bergman-Orlicz space to the weighted-type space. For some related results see also [1, 8, 16, 38].

Let $T : X \to Y$ be a linear operator, where X and Y are topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively. It is said that the operator $T : X \to Y$ is metrically bounded if there exists a positive constant M such that

$$d_Y(Tf,0) \leqslant M d_X(f,0)$$

for all $f \in X$. The operator $T : X \to Y$ is metrically compact if it maps bounded sets into relatively compact sets.

Inspired by the above results, this paper is devoted to investigating the metrical boundedness and metrical compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from weighted Bergman-Orlicz spaces $\mathscr{A}_\alpha^{\Phi_p}$ to Bloch-type space $\mathscr{B}^\mu$ and little Bloch-type space $\mathscr{B}_0^\mu$.

Now we are ready to present the weighted Bergman-Orlicz space and some related facts in [20]. A function $\Phi \not\equiv 0$ is called a growth function, if it is a continuous and nondecreasing function from the interval $[0,\infty)$ onto itself. We can easily conclude that $\Phi(0) = 0$ by these conditions. It is said that Φ is of positive upper type (respectively, negative upper type) if there are $q > 0$ (respectively, $q < 0$) and $C > 0$ such that

$$\Phi(st) \leqslant Ct^q\Phi(s) \quad \text{for every} \quad s > 0 \quad \text{and} \quad t \geqslant 1.$$

Denote by $\mathfrak{U}^q$ the family of all growth functions Φ of positive upper type q $(q \geqslant 1)$, such that the function $t \to \Phi(t)/t$ is nondecreasing on $(0,\infty)$. It is said that function Φ is of positive lower type (respectively, negative lower type), if there are $r > 0$ (respectively, $r < 0$) and $C > 0$ such that

$$\Phi(st) \leqslant Ct^r\Phi(s) \quad \text{for every} \quad s > 0 \quad \text{and} \quad 0 < t \leqslant 1.$$

By $\mathfrak{L}_r$ we denote the family of all growth functions Φ of positive lower type r $(0 < r \leqslant 1)$, such that the function $t \to \Phi(t)/t$ is nonincreasing on $(0,\infty)$. Moreover, if $\Phi \in \mathfrak{U}^q$ (respectively, $\mathfrak{L}_r$), we will also assume that it is convex (respectively, concave).

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. For a growth function Φ, the weighted Bergman-Orlicz space $\mathscr{A}_\alpha^\Phi = \mathscr{A}_\alpha^\Phi(\mathbb{D})$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{A}_\alpha^\Phi} = \int_{\mathbb{D}} \Phi(|f(z)|) dA_\alpha(z) < \infty.$$

The space $\mathscr{A}_\alpha^\Phi$ is endowed with the following quasi-norm

$$\|f\|_{\mathscr{A}_\alpha^\Phi}^{lux} = \inf\left\{\lambda > 0 : \int_{\mathbb{D}} \Phi\left(\frac{|f(z)|}{\lambda}\right) dA_\alpha(z) \leqslant 1\right\}.$$

If $\Phi \in \mathfrak{U}^q$ or $\Phi \in \mathfrak{L}_r$, then the quasi-norm on $\mathscr{A}_\alpha^\Phi$ is finite (see, [20, Remark 1.4]).

The classical weighted Bergman space $\mathscr{A}_\alpha^p = \mathscr{A}_\alpha^p(\mathbb{D})$ $(p > 0, \alpha > -1)$ corresponds to $\Phi(t) = t^p$, consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{A}_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

It is well known that for $p \geqslant 1$, $\mathscr{A}_\alpha^p$ is a Banach space, while for $0 < p < 1$ it is a translation-invariant metric space with $d(f,g) = \|f - g\|_{\mathscr{A}_\alpha^p}^p$. Furthermore, if $\Phi \in \mathfrak{U}^s$, then $\mathscr{A}_\alpha^{\Phi_p}$ is a subspace of $\mathscr{A}_\alpha^p$, where $\Phi_p(t) := \Phi(t^p)$. Besides, $\|f\|_{\mathscr{A}_\alpha^p} \leqslant (\Phi^{-1}(1))^{1/p} \|f\|_{\mathscr{A}_\alpha^\Phi}^{lux}$ (see, [20, Lemma 2.2]). We will always assume that $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Under this assumption, $\mathscr{A}_\alpha^{\Phi_p}$ is a complete metric space (see, [20, Lemma 2.6]). For related investigations of operators on weighted Bergman-Orlicz spaces, see [2, 6, 7, 9, 20, 23, 30].

A positive continuous function ϕ on [0,1) is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0,1)$ such that (see[24])

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \ \lim_{r\to 1} \frac{\mu(r)}{(1-r)^s} = 0;$$
$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \ \lim_{r\to 1} \frac{\mu(r)}{(1-r)^t} = \infty.$$

Let $\mu : \mathbb{D} \to (0,+\infty)$ be a function that is normal and radial, i.e., $\mu(z) = \mu(|z|)$. A function $f \in H(\mathbb{D})$ belongs to Bloch-type space $\mathscr{B}^\mu$ if

$$b_{\mathscr{B}^\mu}(f) := \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

The quantity $b_{\mathscr{B}^\mu}(f)$ is a seminorm on $\mathscr{B}^\mu$ and a norm on $\mathscr{B}^\mu/\mathbb{P}_0$, where $\mathbb{P}_0$ is the set of constant complex polynomials. $\mathscr{B}^\mu$ becomes a Banach space normed by

$$\|f\|_{\mathscr{B}^\mu} = |f(0)| + b_{\mathscr{B}^\mu}(f).$$

The little Bloch-type space, which is denoted by $\mathscr{B}_0^\mu$, consists of the functions f in $\mathscr{B}^\mu$ satisfying

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0,$$

and it is easily seen that $\mathscr{B}_0^\mu$ is a closed subspace of $\mathscr{B}^\mu$. When $\mu(z) = 1 - |z|^2$, the induced spaces $\mathscr{B}^\mu$ and $\mathscr{B}_0^\mu$ become the classical Bloch space and little Bloch space, respectively. For some results on the Bloch-type spaces and operators on them see, for example, [10, 12, 13, 14, 16, 17, 18, 22, 31, 34, 37].

Throughout this paper, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y whenever there is a positive constant C whose value may change at each occurrence such that $X \leqslant CY$. If both $X \lesssim Y$ and $Y \lesssim X$ hold, we write $X \simeq Y$.

2. Preliminaries

In this section, we state several auxiliary results which will be used in the proofs of the main results. Firstly, we quote the following two point evaluation estimates. The first one can be found in [20, Lemma 2.4].

LEMMA 1. *Let $p \geqslant 1$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $f \in \mathscr{A}_\alpha^{\Phi_p}$ and $z \in \mathbb{D}$ we have*

$$|f(z)| \leqslant \Phi_p^{-1}\left(\left(\frac{4}{1-|z|^2}\right)^{\alpha+2}\right)\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux}.$$

The second one can be found in [7, Lemma 2.3] (for a generalization see [30, Lemma 2]).

LEMMA 2. *Let $p \geqslant 1$, $\alpha > -1$, $\Phi \in \mathfrak{U}^s$ and $n \in \mathbb{N}$. Then there are two positive constants $C_n = C(\alpha, p, n)$ and $D_n = D(\alpha, p, n)$ independent of $f \in \mathscr{A}_\alpha^{\Phi_p}$ and $z \in \mathbb{D}$ such that*

$$|f^{(n)}(z)| \leqslant \frac{C_n}{(1-|z|^2)^n}\Phi_p^{-1}\left(\left(\frac{D_n}{1-|z|^2}\right)^{\alpha+2}\right)\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux}.$$

The following lemma, which was essentially proved in [20] (see also [7, Lemma 2.4]), provides a class of test functions in $\mathscr{A}_\alpha^{\Phi_p}$.

LEMMA 3. *Let $p > 0$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $t \geqslant 0$ and $w \in \mathbb{D}$ the following function is in $\mathscr{A}_\alpha^{\Phi_p}$*

$$f_{w,t}(z) = \Phi_p^{-1}\left(\left(\frac{C}{1-|w|^2}\right)^{\alpha+2}\right)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{\frac{2\alpha+4}{p}+t},$$

where C is an arbitrary positive constant. Moreover,

$$\sup_{w\in\mathbb{D}}\|f_{w,t}\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux} \lesssim 1.$$

The following lemma characterizes the metrical compactness in terms of sequential convergence, whose proof can be shown by a similar argument as [3, Proposition 3.11], so we omit the details.

LEMMA 4. *Let $p \geqslant 1$, $\alpha > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically compact if and only if $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded and for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathscr{A}_\alpha^{\Phi_p}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu} \to 0$ as $n \to \infty$.*

The following lemma can be proved similar to [18, Lemma 1].

LEMMA 5. *A closed set K in $\mathscr{B}_0^\mu$ is metrically compact if and only if it is metrically bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f'(z)|=0.$$

3. The operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu$

In this section, we characterize the metrical boundedness and metrical compactness of $T_{\psi_1,\psi_2,\varphi}:\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu$.

THEOREM 1. *Let $p\geqslant 1$, $\alpha>-1$, $\psi_1,\psi_2\in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, and $\Phi\in\mathfrak{U}^s$ such that $\Phi_p\in\mathfrak{L}_r$. Then the following conditions are equivalent:*
(i) *The operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu$ is metrically bounded.*
(ii)

$$M_1:=\sup_{z\in\mathbb{D}}\mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)<\infty,$$
$$M_2:=\sup_{z\in\mathbb{D}}\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)<\infty,$$
$$M_3:=\sup_{z\in\mathbb{D}}\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)<\infty,$$

where the constants D_1 and D_2 are the ones in Lemma 2. Moreover, if the operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu/\mathbb{P}_0$ is nonzero and metrically bounded, then

$$\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu/\mathbb{P}_0}\simeq M_1+M_2+M_3.$$

Proof. (i)$\Rightarrow$(ii). Suppose that (i) holds. We first consider the functions $f(z)=1$, $f(z)=z$ and $f(z)=\frac{z^2}{2}\in\mathscr{A}_\alpha^{\Phi_p}$, respectively. Since the operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu$ is metrically bounded, we have

$$L_0:=\sup_{z\in\mathbb{D}}\mu(z)|\psi_1'(z)|\leqslant\|T_{\psi_1,\psi_2,\varphi}1\|_{\mathscr{B}^\mu}\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu},\tag{1}$$

$$\sup_{z\in\mathbb{D}}\mu(z)|\psi_1'(z)\varphi(z)+\psi_1(z)\varphi'(z)+\psi_2'(z)|\leqslant\|T_{\psi_1,\psi_2,\varphi}z\|_{\mathscr{B}^\mu}\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu},\tag{2}$$

$$\sup_{z\in\mathbb{D}}\mu(z)\left|\psi_1'(z)\frac{\varphi(z)^2}{2}+(\psi_1(z)\varphi'(z)+\psi_2'(z))\varphi(z)+\psi_2(z)\varphi'(z)\right|$$
$$\leqslant\left\|T_{\psi_1,\psi_2,\varphi}\frac{z^2}{2}\right\|_{\mathscr{B}^\mu}\lesssim\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu}.\tag{3}$$

Employing (1), (2), the boundedness of φ and the triangle inequality, we can obtain

$$L_1 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)| \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu}. \tag{4}$$

By using (1), (3), (4), in the same manner, we have

$$L_2 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)\varphi'(z)| \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu}. \tag{5}$$

Choose the function

$$\begin{aligned} f_{\varphi(w)}(z) =& \Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)\left[-\frac{2\alpha+4+2p}{2\alpha+4}\left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}}\right. \\ &\left.+\frac{4\alpha+8+4p}{2\alpha+4+p}\left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}+1} - \left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}+2}\right], \end{aligned} \tag{6}$$

where $w\in\mathbb{D}$, then $f_{\varphi(w)}\in\mathscr{A}_\alpha^{\Phi_p}$ by Lemma 3. We can calculate that

$$f'_{\varphi(w)}(\varphi(w)) = f''_{\varphi(w)}(\varphi(w)) = 0, \quad f_{\varphi(w)}(\varphi(w)) = \Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)E_1, \tag{7}$$

where

$$E_1 = -\frac{2\alpha+4+2p}{2\alpha+4} + \frac{4\alpha+8+4p}{2\alpha+4+p} - 1 \neq 0.$$

By (7) and the metrical boundedness of $T_{\psi_1,\psi_2,\varphi}:\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu$, we have

$$\begin{aligned} \infty > \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi}f_{\varphi(w)}\|_{\mathscr{B}^\mu} \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)\left|(T_{\psi_1,\psi_2,\varphi}f_{\varphi(w)})'(z)\right| \\ &\geqslant \mu(w)|\psi_1'(w)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)E_1. \end{aligned}$$

Consequently,

$$M_1 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty. \tag{8}$$

For $w\in\mathbb{D}$, set

$$\begin{aligned} g_{\varphi(w)}(z) =& \Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)\left[-\frac{2\alpha+4+p}{2\alpha+4}\left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}}\right. \\ &\left.+\frac{4\alpha+8+p}{2\alpha+4}\left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}+1} - \left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}+2}\right]. \end{aligned} \tag{9}$$

Then we have that $g_{\varphi(w)} \in \mathscr{A}_\alpha^{\Phi_p}$ by using Lemma 3. We can also calculate that

$$g_{\varphi(w)}(\varphi(w)) = g''_{\varphi(w)}(\varphi(w)) = 0,$$
$$g'_{\varphi(w)}(\varphi(w)) = \frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)E_2, \tag{10}$$

where

$$E_2 = -\frac{2\alpha+4+p}{p} + \frac{(4\alpha+8+p)(2\alpha+4+p)}{(2\alpha+4)p} - 1 \neq 0.$$

From (10) and the metrical boundedness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ it follows that

$$\begin{aligned}\infty > \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi} g_{\varphi(w)}\|_{\mathscr{B}^\mu} \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)\left|(T_{\psi_1,\psi_2,\varphi} g_{\varphi(w)})'(z)\right| \\ &\geqslant \frac{\mu(w)|\psi_1(w)\varphi'(w)+\psi_2'(w)||\varphi(w)|}{1-|\varphi(w)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)E_2,\end{aligned}$$

which means that

$$\begin{aligned}K_1(w) &:= \frac{\mu(w)|\psi_1(w)\varphi'(w)+\psi_2'(w)||\varphi(w)|}{1-|\varphi(w)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right) \\ &\lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu}.\end{aligned} \tag{11}$$

For a fixed $\delta \in (0,1)$, by (11), we have

$$\begin{aligned}&\sup_{\{z\in\mathbb{D}:|\varphi(z)|>\delta\}} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) \\ &\leqslant \frac{1}{\delta}\sup_{z\in\mathbb{D}} K_1(z) \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty.\end{aligned} \tag{12}$$

On the other hand, by (4), we can obtain

$$\begin{aligned}&\sup_{\{z\in\mathbb{D}:|\varphi(z)|\leqslant\delta\}} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) \\ &\leqslant \frac{L_1}{1-\delta^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-\delta^2}\right)^{\alpha+2}\right) \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty,\end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$M_2 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty. \tag{14}$$

For $w \in \mathbb{D}$, take the function

$$\begin{aligned}h_{\varphi(w)}(z) =& \Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)\left[-\left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}}\right. \\ &\left. +2\left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}+1} - \left(\frac{1-|\varphi(w)|^2}{1-\overline{\varphi(w)}z}\right)^{\frac{2\alpha+4}{p}+2}\right],\end{aligned} \tag{15}$$

then $h_{\varphi(w)} \in \mathscr{A}_\alpha^{\Phi_p}$. We also have

$$
\begin{aligned}
&h_{\varphi(w)}(\varphi(w)) = h'_{\varphi(w)}(\varphi(w)) = 0,\\
&h''_{\varphi(w)}(\varphi(w)) = \frac{-2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right),
\end{aligned} \tag{16}
$$

which along with the metrical boundedness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ implies that

$$
\begin{aligned}
\infty > \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} &\gtrsim \|T_{\psi_1,\psi_2,\varphi}h_{\varphi(w)}\|_{\mathscr{B}^\mu}\\
&\geqslant \sup_{z\in\mathbb{D}} \mu(z)\left|(T_{\psi_1,\psi_2,\varphi}h_{\varphi(w)})'(z)\right|\\
&\geqslant \frac{2\mu(w)|\psi_2(w)\varphi'(w)||\varphi(w)|^2}{(1-|\varphi(w)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
K_2(w) := \frac{\mu(w)|\psi_2(w)\varphi'(w)||\varphi(w)|^2}{(1-|\varphi(w)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(w)|^2}\right)^{\alpha+2}\right)\\
\lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu}.
\end{aligned} \tag{17}
$$

For a fixed $\delta \in (0,1)$, by (17), we have

$$
\begin{aligned}
&\sup_{\{z\in\mathbb{D}:|\varphi(z)|>\delta\}} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\\
&\leqslant \frac{1}{\delta^2}\sup_{z\in\mathbb{D}} K_2(z) \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty.
\end{aligned} \tag{18}
$$

On the other hand, by (5), we get

$$
\begin{aligned}
&\sup_{\{z\in\mathbb{D}:|\varphi(z)|\leqslant\delta\}} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\\
&\leqslant \frac{L_2}{(1-\delta^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-\delta^2}\right)^{\alpha+2}\right) \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty,
\end{aligned} \tag{19}
$$

From (18) and (19) it follows that

$$
M_3 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} < \infty. \tag{20}
$$

Moreover, by using (8), (14), (20), we can get

$$
M_1 + M_2 + M_3 \lesssim \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu}. \tag{21}
$$

(ii)$\Rightarrow$(i). By Lemmas 1 and 2, for every $f \in \mathscr{A}_\alpha^{\Phi_p}$, we have

$$\begin{aligned}
&\mu(z)|(T_{\psi_1,\psi_2,\varphi} f)'(z)| \\
\leqslant &\mu(z)|\psi_1'(z)||f(\varphi(z))| + |\psi_1(z)\varphi'(z)+\psi_2(z)||f'(\varphi(z))| + \mu(z)|\psi_2(z)\varphi'(z)||f''(\varphi(z))| \\
\leqslant &\mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \\
&+\frac{C_1\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \\
&+\frac{C_2\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \\
\lesssim &(M_1+M_2+M_3)\|f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}}. \qquad (22)
\end{aligned}$$

We also have

$$\begin{aligned}
&|(T_{\psi_1,\psi_2,\varphi} f)(0)| \\
\leqslant &|\psi_1(0) f(\varphi(0))| + |\psi_2(0) f'(\varphi(0))| \\
\leqslant &\Bigg[|\psi_1(0)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(0)|^2}\right)^{\alpha+2}\right) \\
&+\frac{C_1|\psi_2(0)|}{(1-|\varphi(0)|^2)^n}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(0)|^2}\right)^{\alpha+2}\right)\Bigg]\|f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}}.
\end{aligned}$$

From the above inequalities and the conditions in (ii), we conclude that $T_{\psi_1,\psi_2,\varphi}$: $\mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded. If we consider the space $\mathscr{B}^\mu/\mathbb{P}_0$, we have that (see, for example, [27, 28])

$$\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu/\mathbb{P}_0} \lesssim M_1+M_2+M_3. \qquad (23)$$

Hence we obtain the asymptotic expression of $\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu/\mathbb{P}_0}$ by using (21) and (23). $\square$

THEOREM 2. *Let $p \geqslant 1$, $\alpha > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:*

(i) *The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically compact.*

(ii) *The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded and*

$$\lim_{|\varphi(z)|\to 1}\mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)=0, \qquad (24)$$

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)=0, \qquad (25)$$

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)=0, \qquad (26)$$

where the constants D_1 and D_2 are the ones in Lemma 2.

Proof. (i)⇒(ii). Suppose that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically compact, and consequently metrically bounded. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Now take the following sequence of functions from the family in (6)

$$f_n(z) = f_{\varphi(z_n)}(z),$$

which is a bounded sequence in $\mathscr{A}_\alpha^{\Phi_p}$. Moreover, from the proof of [20, Theorem 3.6] it follows that $\{f_n\}_{n\in\mathbb{N}}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ as $n \to \infty$. By Lemma 4, we have that

$$\lim_{n\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu} = 0,$$

We also have

$$f_n'(\varphi(z_n)) = f_n''(\varphi(z_n)) = 0, \quad f_n(\varphi(z_n)) = \Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z_n)|^2}\right)^{\alpha+2}\right)E_1.$$

Consequently,

$$\mu(z_n)|\psi_1'(z_n)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z_n)|^2}\right)^{\alpha+2}\right) \lesssim \|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu},$$

which along with $|\varphi(z_n)| \to 1$ as $n \to \infty$ implies that

$$\lim_{|\varphi(z_n)|\to 1} \mu(z_n)|\psi_1'(z_n)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z_n)|^2}\right)^{\alpha+2}\right) = 0,$$

from which we can see that (24) holds.

By using the two sequences of functions

$$g_n(z) = g_{\varphi(z_n)}(z) \quad \text{and} \quad h_n(z) = h_{\varphi(z_n)}(z),$$

where $g_{\varphi(z_n)}(z)$ and $h_{\varphi(z_n)}(z)$ are defined in (9) and (15), respectively. By a similar argument, we can obtain (25) and (26).

(ii)⇒(i). Suppose that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded and (24), (25), (26) hold. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{A}_\alpha^{\Phi_p}$ such that $\sup_{n\in\mathbb{N}} \|f_n\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \leqslant L$ and $f_n \to 0$ uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$. For the metrical compactness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$, it is sufficient to show that $\lim_{n\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu} = 0$ by Lemma 4.

For every $\varepsilon > 0$, there exists $\eta \in (0,1)$ such that

$$\mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon, \tag{27}$$

$$\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon, \tag{28}$$

$$\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon, \tag{29}$$

whenever $\eta < |\varphi(z)| < 1$. Then by Lemmas 1, 2, (1), (4), (5) and (27), (28), (29) we have

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu} \\
=&|(T_{\psi_1,\psi_2,\varphi} f_n)(0)| + \sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f_n)'(z)| \\
\leqslant&|\psi_1(0) f_n(\varphi(0))| + |\psi_2(0) f_n'(\varphi(0))| \\
&+ \sup_{|\varphi(z)|\leqslant\eta} \mu(z)|\psi_1'(z)||f_n(\varphi(z))| + \sup_{\eta<|\varphi(z)|<1} \mu(z)|\psi_1'(z)||f_n(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\leqslant\eta} \mu(z)|\psi_1(z)\varphi'(z) + \psi_2(z)||f_n'(\varphi(z))| \\
&+ \sup_{\eta<|\varphi(z)|<1} \mu(z)|\psi_1(z)\varphi'(z) + \psi_2(z)||f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\leqslant\eta} \mu(z)|\psi_2(z)\varphi'(z)||f_n''(\varphi(z))| + \sup_{\eta<|\varphi(z)|<1} \mu(z)|\psi_2(z)\varphi'(z)||f_n''(\varphi(z))| \\
\leqslant&|\psi_1(0) f_n(\varphi(0))| + |\psi_2(0) f_n'(\varphi(0))| \\
&+ L_0 \sup_{|\varphi(z)|\leqslant\eta} |f_n(\varphi(z))| + L_1 \sup_{|\varphi(z)|\leqslant\eta} |f_n'(\varphi(z))| + L_2 \sup_{|\varphi(z)|\leqslant\eta} |f_n''(\varphi(z))| \\
&+ \sup_{\eta<|\varphi(z)|<1} \mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f_n\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \\
&+ \sup_{\eta<|\varphi(z)|<1} \frac{C_1\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f_n\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \\
&+ \sup_{\eta<|\varphi(z)|<1} \frac{C_2\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f_n\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \\
\leqslant&|\psi_1(0) f_n(\varphi(0))| + |\psi_2(0) f_n'(\varphi(0))| \\
&+ L_0 \sup_{|w|\leqslant\eta} |f_n(w)| + L_1 \sup_{|w|\leqslant\eta} |f_n'(w)| + L_2 \sup_{|w|\leqslant\eta} |f_n''(w)| + 3L\varepsilon.
\end{aligned} \tag{30}$$

Since f_n converges to zero uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$, Cauchy's estimate shows that f_n' and f_n'' also do as $n \to \infty$. In particular, $\{\varphi(0)\}$ and $\{w : |w| \leqslant \eta\}$ are compact subsets of $\mathbb{D}$, so letting $n \to \infty$ in (30) yields

$$\limsup_{n\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu} \leqslant 3L\varepsilon.$$

Since ε is an arbitrary positive number, from the last inequality we obtain

$$\lim_{n\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}^\mu} = 0,$$

from which by Lemma 4 we conclude that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically compact. $\square$

4. The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$

In this section, we characterize the metrical boundedness and metrical compactness of $T_{\psi_1,\psi_2,\varphi}$ from $\mathscr{A}_\alpha^{\Phi_p}$ to the little Bloch space $\mathscr{B}_0^\mu$.

THEOREM 3. *Let $p \geqslant 1$, $\alpha > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:*
(i) *The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically bounded.*
(ii) *The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded and*

$$\lim_{|z|\to 1} \mu(z)|\psi_1'(z)| = 0, \tag{31}$$

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)| = 0, \tag{32}$$

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)\varphi'(z)| = 0. \tag{33}$$

Proof. (i)$\Rightarrow$(ii). Assume that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically bounded, then it is evident that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded and for every $f \in \mathscr{A}_\alpha^{\Phi_p}$, $T_{\psi_1,\psi_2,\varphi} f \in \mathscr{B}_0^\mu$. Taking $f(z) = 1 \in \mathscr{A}_\alpha^{\Phi_p}$, we have

$$\lim_{|z|\to 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}1)'(z)| = \lim_{|z|\to 1} \mu(z)|\psi_1'(z)| = 0.$$

That is, (31) follows. Taking the functions $f(z) = z$ and $f(z) = \frac{z^2}{2} \in \mathscr{A}_\alpha^{\Phi_p}$, we obtain

$$\lim_{|z|\to 1} \mu(z)|\psi_1'(z)\varphi(z) + \psi_1(z)\varphi'(z) + \psi_2'(z)| = 0, \tag{34}$$

$$\lim_{|z|\to 1} \mu(z)\left|\psi_1'(z)\frac{\varphi(z)^2}{2} + (\psi_1(z)\varphi'(z) + \psi_2'(z))\varphi(z) + \psi_2(z)\varphi'(z)\right| = 0, \tag{35}$$

respectively. By (31), (34), the triangle inequality and the boundedness of φ we obtain (32). The proof of (33) can be handled in much the same way by using (35), and the details are omitted.

(ii)$\Rightarrow$(i). If $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically bounded and (31), (32), (33) hold, then for each polynomial p, we have

$$\begin{aligned}
&\mu(z)|(T_{\psi_1,\psi_2,\varphi}p)'(z)| \\
\leqslant &\mu(z)|\psi_1'(z)||p(\varphi(z))| + \mu(z)|\psi_1(z)\varphi'(z) + \psi_2(z)||p'(\varphi(z))| \\
&+ \mu(z)|\psi_2(z)\varphi'(z)||p''(\varphi(z))| \\
\leqslant &\mu(z)|\psi_1'(z)|\|p(\varphi(z))\|_\infty + \mu(z)|\psi_1(z)\varphi'(z) \\
&+ \psi_2(z)|\|p'(\varphi(z))\|_\infty + \mu(z)|\psi_2(z)\varphi'(z)|\|p''(\varphi(z))\|_\infty,
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Letting $|z| \to 1$ in the above inequality and using (31), (32), (33) yields

$$\lim_{|z|\to 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}p)'(z)| = 0,$$

from which it follows that $T_{\psi_1,\psi_2,\varphi}p \in \mathscr{B}_0^\mu$. Since the set of all polynomials is dense in $\mathscr{A}_\alpha^{\Phi_p}$, and hence for each $f \in \mathscr{A}_\alpha^{\Phi_p}$, there is a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|p_n - f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} = 0,$$

which along with the metrical boundedness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ implies that

$$\|T_{\psi_1,\psi_2,\varphi}p_n - T_{\psi_1,\psi_2,\varphi}f\|_{\mathscr{B}^\mu} \leqslant \|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{A}_\alpha^{\Phi_p}\to\mathscr{B}^\mu} \cdot \|p_n - f\|^{lux}_{\mathscr{A}_\alpha^{\Phi_p}} \to 0$$

as $n \to \infty$. Since $\mathscr{B}_0^\mu$ is a closed subspace of $\mathscr{B}^\mu$, we have that $T_{\psi_1,\psi_2,\varphi}f \in \mathscr{B}_0^\mu$, and consequently $T_{\psi_1,\psi_2,\varphi}(\mathscr{A}_\alpha^{\Phi_p}) \subset \mathscr{B}_0^\mu$. As a consequence, the metrical boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ implies that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically bounded. □

THEOREM 4. *Let $p \geqslant 1$, $\alpha > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:*

(i) *The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically compact.*

(ii)

$$\lim_{|z|\to 1} \mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) = 0, \tag{36}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) = 0, \tag{37}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) = 0, \tag{38}$$

where the constants D_1 and D_2 are the ones in Lemma 2.

Proof. (i)⇒(ii). Assume that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically compact, then $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}^\mu$ is metrically compact. Moreover, for every $\varepsilon > 0$, there exists $\eta \in (0,1)$ such that (27), (28), (29) hold for $\eta < |\varphi(z)| < 1$ by the proof of Theorem 2. On the other hand, the metrical compactness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ implies that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically bounded, by Theorem 3 we know that (31), (32), (33) hold. Thus for every $\varepsilon > 0$, there exists $\gamma \in (0,1)$ such that

$$\mu(z)|\psi_1'(z)| \leqslant \Phi_p\left(\left(\frac{4}{1-\eta^2}\right)^{\alpha+2}\right)\varepsilon, \tag{39}$$

$$\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)| \leqslant (1-\eta^2)\Phi_p\left(\left(\frac{D_1}{1-\eta^2}\right)^{\alpha+2}\right)\varepsilon, \tag{40}$$

$$\mu(z)|\psi_2(z)\varphi'(z)| \leqslant (1-\eta^2)^2\Phi_p\left(\left(\frac{D_2}{1-\eta^2}\right)^{\alpha+2}\right)\varepsilon, \tag{41}$$

for $\gamma < |z| < 1$. By (29), when $\gamma < |z| < 1$ and $\eta < |\varphi(z)| < 1$,

$$\mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon. \tag{42}$$

When $\gamma < |z| < 1$ and $|\varphi(z)| \leqslant \eta$, by using (39), we obtain

$$\mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) \leqslant \mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-\eta^2}\right)^{\alpha+2}\right) < \varepsilon. \tag{43}$$

From (42) and (43) it follows that (36) holds. Employing (28) and (40), (29) and (41), with the similar argument, we can get (37) and (38), respectively.

(ii)$\Rightarrow$(i). Suppose that (36), (37), (38) hold. It is evident that $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically bounded by Theorems 1 and 3. Analysis similar to (22) in the proof of Theorem 1 shows that

$$\begin{aligned}\mu(z)|(T_{\psi_1,\psi_2,\varphi}f)'(z)| \leqslant & \mu(z)|\psi_1'(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux}\\ &+\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2}\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux}\\ &+\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right)\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux}.\end{aligned}$$

Taking the supremum in the above inequality over all $f \in \mathscr{A}_\alpha^{\Phi_p}$ such that $\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux} \leqslant 1$ and letting $|z| \to 1$, we can obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathscr{A}_\alpha^{\Phi_p}}^{lux} \leqslant 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}f)'(z)| = 0.$$

Therefore, the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{A}_\alpha^{\Phi_p} \to \mathscr{B}_0^\mu$ is metrically compact by Lemma 5. □

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